On locally dually flat general \((\alpha, \beta)\)-metrics

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Abstract  Locally flat Finsler metrics arise from information geometry. Some special locally dually flat Finsler metrics had been studied in Cheng et al. [3] and Xia [4] respectively. As we know, a new class of Finsler metrics called general \((\alpha, \beta)\)-metrics are introduced, which are defined by a Riemannian metrics \(\alpha\) and 1-form \(\beta\). These metrics generalize \((\alpha, \beta)\)-metrics naturally. In this paper, we give a characterization of locally dually flat general \((\alpha, \beta)\)-metrics on an \(n\)-dimensional manifold \(M^n (n \geq 3)\), which generalizes some results in Cheng et al. [3] and Xia [4].

Keywords  Finsler metric, general \((\alpha, \beta)\)-metrics, locally dually flat Finsler metric.

MR(2000) Subject Classification  53B40; 53C60; 58B20

1. Introduction

The notion of dually flat metrics was first introduced by Amari and Nagaoka [1] when they study information geometry on Riemannian spaces. Later on, Shen extends the notion of dually flatness to Finsler metrics [2]. A Finsler metric \(F = F(x, y)\) on an \(n\)-dimensional manifold \(M^n\) is called locally dually flat if at every point there is a coordinate system \((x^i)\) in which the spray coefficients are in following form

\[ G^i = \frac{1}{2} g^{ij} H_{ij}, \]  

where \(H = H(x, y)\) is a local scalar function on the tangent bundle \(TM\) of \(M\). Such a coordinate system is called an adapted coordinate system. In [2], the author proved that a Finsler metric \(F = F(x, y)\) on an open subset \(U \subset \mathbb{R}^n\) is dually flat if and only if it satisfies the following PDE

\[ [F^2]_{x^k y^l y^m} - 2[F^2]_{x^l} = 0. \]  

In this case, \(H = -\frac{1}{2} \left[ F^2 \right]_{x^m} y^m\).

It is known that a Riemannian metric \(F = \sqrt{g_{ij}(x)y^i y^j}\) is locally dually flat if and only if in an adapted system,

\[ g_{ij} = \frac{\partial \psi}{\partial x^i \partial x^j}(x), \]

where \(\psi = \psi(x)\) is a \(C^\infty\) function [1, 2].

A special type Finsler metrics on a manifold \(M\) are Randers metrics defined by \(F = \alpha + \beta\), where \(\alpha\) is a Riemannian metric and \(\beta\) is a 1-form on \(M\) with \(b := \|\beta(x)\|_\alpha < 1\),


Received ; Accepted
Supported by the National Natural Science Foundation of China (Grant No. 11071005).
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which were first introduced by physicist G. Randers in 1941 from the standpoint of general relativity. Randers metrics, in particular, the curvature properties of Randers metrics, are extensively studied by some authors ([2, 3, 9] and the references therein). Recently, the classification of locally dually flat Randers metrics with almost isotropic flag curvature is given in [3].

The first example of non-Riemannian dually flat metrics is Funk metric given as follows (4, 7, 11):

$$ F = \frac{\sqrt{(1-|s|^2)|y|^2 + \langle x, y \rangle^2}}{1-|x|^2} \pm \frac{\langle x, y \rangle}{1-|x|^2}. $$

(1.3)

This metric is defined on the unit ball $\mathbb{B}^n \subset \mathcal{R}^n$ and is a Randers metric with constant flag curvature $K = -\frac{1}{4}$. This is the only known example of locally dually flat metrics with non-zero constant flag curvature up to now.

Let $(\alpha, \beta)$-metrics be a broad class of Finsler metrics defined by $F = \alpha \phi(s)$, where $\alpha$ is a Riemannian metric, $\beta$ is a 1-form, and $s = \frac{\beta}{\alpha}$. When $\phi(s) = 1 + s$, we get Randers metrics $F = \alpha + \beta$. When $\phi(s) = 1 + \epsilon s + ks^2$, we obtain metrics $F = \alpha + \epsilon \beta + k \frac{d^2}{\alpha}$. In recent years, $(\alpha, \beta)$-metrics have been studied extensively ([2, 4, 9, 10], and the references therein). In 2009, On locally dually flat $(\alpha, \beta)$-metrics are given in [4], On a class of locally dually flat Randers metrics of scalar flag curvatures is given in [5].

These facts inspire us to study dually flat general $(\alpha, \beta)$-metrics $F = \alpha \phi(b^2, s)$, where $\alpha$ is a Riemannian metric, $\beta$ is a 1-form, and $s = \frac{\beta}{\alpha}$. If $\phi_1(b^2, s) = 0$, it is obvious that $F = \alpha \phi(s)$ is an $(\alpha, \beta)$-metrics. However, if $\phi_1(b^2, s) \neq 0$, we can’t assure that $F = \alpha \phi(b^2, s)$ isn’t an $(\alpha, \beta)$-metrics. For instance, in [6], the general $(\alpha, \beta)$-metric $F = \frac{\sqrt{(1+\frac{b^2}{\alpha} + \frac{b^2}{\beta})}}{2}$ is actually an $(\alpha, \beta)$-metric.

General $(\alpha, \beta)$-metrics are a more broad class of Finsler metrics in Finsler geometry. In 2009, C. Yu and H. Zhu provide a sufficient condition for the general $(\alpha, \beta)$ metrics $F = \alpha \phi(b^2, \frac{\beta}{\alpha})$ to be projectively flat [6]. In this paper, we obtain an equivalent characterization for locally dually flat general $(\alpha, \beta)$-metrics, which are not Riemannian. We obtain the following theorem.

**Theorem 1.1.** Let $F = \alpha \phi(b^2, s)$ be an general $(\alpha, \beta)$-metric on an $n$-dimensional manifold $M^n (n \geq 3)$, where $\alpha = \sqrt{\sigma(x)}y^i$, $\beta = b_i(x)y^i \neq 0$, $s = \frac{\beta}{\alpha}$, and $b = ||\beta||_\alpha$ is the norm of $\beta$. Suppose $F$ is not Riemannian, $\phi(b^2, s) \neq \sqrt{f(b^2)s + g(s)}$ and $\phi_2(b^2, 0) \neq 0$. Then $F$ is locally dually flat on $M^n$ if and only if in an adapted coordinate system $\alpha, \beta$ and $\phi = \phi(b^2, s)$ satisfy

$$ s = \frac{1}{3}(\beta b_i - \alpha b_i), $$

(1.4)

$$ r = \frac{2}{3} \theta \beta + \left\{ \tau + \frac{2}{3}(\tau - \theta b_i) \right\} \alpha^2 - \frac{1}{b_i^2} \tau \beta^2, $$

(1.5)

$$ G_{\alpha} = \frac{1}{3}[2\theta + (3k_1 - 2)\tau b_i]y^j + \frac{1}{3}(\theta' - \tau b_i)\alpha^2 + \frac{1}{2}k_2 \tau \beta^2 b_i, $$

(1.6)

$$ \tau \{s(\phi_2 - s\phi_2^2 - s\phi_2^2)k_2 b_i(b_i - s^2) - 1 \} - b^2(\phi_2^2 + \phi_2^2) + b^2 k_1 \phi(\phi - s\phi_2) = 0. $$

(1.7)
where $k_1$, $k_2$ and $f(s)$ are function with respect to $b^2$, $g(s)$ are function with respect to $s$, defined by (3.17) in Section 3, $\tau := \tau(x)$ is a scalar function, $\theta = \theta_i(x)y^i$ is a 1-form on $M$ and $\theta^i := a^{im}\theta_m$.

**Remark 1.** Note that $\phi_1$ means the derivation of $\phi(b^2, s)$ with respect to the first variable $b^2$, $\phi_2$ means the derivation of $\phi(b^2, s)$ with respect to the second variable $s$.

From Theorem 1.1, we get

**Corollary 1.1.** Let $F = \alpha \phi(b^2, s)$, where $b = \|\beta\|_\alpha, s = \frac{\beta}{\alpha}$, be an general $(\alpha, \beta)$-metric on an $n$-dimensional manifold $M^n (n \geq 3)$ with the same assumptions as Theorem 1.1. If $\phi$ further satisfies

$$s(\phi_2 - s\phi_2)\gamma_2(k_2b^2(b^2 - s^2) - 1) = b^2(\phi_2^2 + \phi\phi_2) + \gamma_2k_1\phi(\phi - s\phi_2) \neq 0. \quad (1.8)$$

then $F$ is locally dually flat on $M$ if and only if $\alpha$ and $\beta$ satisfy

$$s_{10} = \frac{1}{3}(\beta\theta_l - \theta b_l), \quad (1.9)$$

$$r_{00} = \frac{2}{3}[\beta\beta - (\theta b^l)\alpha^2], \quad (1.10)$$

$$G^l_\gamma = \frac{1}{3}[2\theta y^l + \theta^l \alpha^2], \quad (1.11)$$

where $k_1, k_2$ and $\alpha$ are the same as those of Theorem 1.1.

In particular, if $F = \sqrt{(1 - |x|^2)^2 + (x_2)^2}$ be Funk metric on $M^n[7]$, in this case, $s_{10} = 0, r_{00} = 0$ and $G^l_\gamma = 0$ satisfy (1.9)-(1.11), then it is dually flat Finsler metrics.

In fact, there are many functions $\phi(b^2, s)$ with $\phi(b^2, s) \neq \sqrt{f(b^2)s + g(s)}$ and $\phi_2(b^2, 0) \neq 0$ are solutions of (1.8).

Moreover, when $\phi(b^2, s) = \sqrt{f(b^2)s + g(s)}$ and $\phi_2(b^2, 0) \neq 0$, we have

**Theorem 1.2.** Let $F = \alpha \phi(b^2, s)$ be an general $(\alpha, \beta)$-metric on an $n$-dimensional manifold $M^n (n \geq 3)$, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric, $\beta = b_i(x)y^i \neq 0$ is a 1-form, $s = \frac{\beta}{\alpha}$, and $b = \|\beta\|_\alpha$ is the norm of $\beta$. Suppose $F$ is not Riemannian, $\phi(b^2, s) = \sqrt{f(b^2)s + g(s)}$ and $\phi_2(b^2, 0) \neq 0$. The $F$ is locally dually flat on $M^n$ if and only if in an adapted coordinate system $\alpha, \beta$ and $\phi = \phi(b^2, s)$ satisfy

$$s_{10} = \frac{1}{3}(\beta\theta_l - \theta b_l), \quad (1.12)$$

$$r_{00} = \frac{2}{3}[\beta\beta + \frac{2}{3}(\beta^l - \beta b^l)]\alpha^2 + \frac{1}{3}(3k_2 - 2 - 3k_3b^2)\gamma_2\beta^2, \quad (1.13)$$

$$G^l_\gamma = \frac{1}{3}[2\theta + (3k_1 - 2)\gamma_2\gamma - \frac{1}{3}(\theta^l - \gamma_2b^l)\alpha^2 + \frac{1}{2}k_3\gamma_2\beta^2b^l, \quad (1.14)$$

$$\tau[s(k_2 - k_3s^2)(\phi\phi_2 - s\phi_2^2 - s\phi_2) - (\phi_2^2 + \phi_2\phi_2) + k_1\phi(\phi - s\phi_2)] = 0. \quad (1.15)$$

where

$$k_1 := \Phi(b^2, 0), \quad k_2 := \frac{\Phi_2(b^2, 0)}{Q(b^2, 0)};$$

$$k_3 := \frac{1}{6Q(b^2, 0)^2} [3Q_{22}(b^2, 0)\Phi_2(b^2, 0) - 6\Phi_2^2(b^2, 0) - Q(b^2, 0)^2\Phi_{222}(b^2, 0)],$$
\(\tau = \tau(x)\) is a scalar function, \(\theta := \theta_i(x)y^i\) is a 1-form on \(M\) and \(\theta^i := a^{im} \theta_m\).

Obviously, \(\phi_1(b^2, s) = 0\) satisfy \(\phi(b^2, s) = \sqrt{f(b^2)s + g(s)}\), namely, \(2R - s\Gamma = 0\). If \(\phi_1(b^2, s) = 0\) and \(\phi_2(b^2, 0) \neq 0\), then Theorem 1.2 is exactly Theorem 1.1 in \([5]\).

In particular, if \(F = \alpha + \beta\) be a Randers metric on \(M^n\), in this case, \(k_1 = 1, k_2 = -1, k_3 = 0\) and \(\phi' = 1\), then Theorem 1.2 is exactly Theorem 1.1 in \([3]\). If \(F\) is a Finsler metric in the form \(F = \alpha(1 + \varepsilon s + k s^2)\) for constants \(\varepsilon \neq 0\) and \(k \neq 0\), in this case, \(k_1 = \varepsilon^2 + 2k, k_2 = -(\varepsilon^2 - 4k), k_3 = 2k(\varepsilon^2 - 4k)\) and \(\phi(s) = 1 + \varepsilon s + k s^2\) satisfy \(\phi'(0) = \varepsilon \neq 0\) and (1.15), then then Theorem 1.2 is exactly Theorem 3.1 in \([5]\).

In the following, we shall use Einstein sum convention unless otherwise stated.

2. Preliminaries

Let \(M\) be an \(n\)-dimensional smooth manifold. We denote by \(TM\) the tangent bundle of \(M\) and by \((x, y) = (x^i, y^i)\) the local coordinates on the tangent bundle \(TM\). A Finsler manifold \((M, F)\) is a smooth manifold equipped with a function \(F : TM \to [0, \infty)\), which has the following properties

(i) Regularity: \(F\) is smooth in \(TM\setminus\{0\}\);
(ii) Positively homogeneity: \(F(x, \lambda y) = \lambda F(x, y)\), for \(\lambda > 0\);
(iii) Strong convexity: the Hessian matrix of \(F^2\), \((g_{ij}(x, y)) := \frac{1}{2}(\frac{\partial^2 F^2(x, y)}{\partial y^i \partial y^j})\), is positive definite on \(TM\setminus\{0\}\). We call \(F\) and tensor \(g_{ij}\) the Finsler metric and the fundamental tensor of \(M\) respectively.

In Finsler geometry, general \((\alpha, \beta)\)-metric is a class of important Finsler metric. By definition, an general \((\alpha, \beta)\)-metric is expressed as the following form,

\[
F = \alpha \phi(b^2, s), \quad b = \|\beta\|, \quad s = \frac{\beta}{\alpha},
\]

where \(\alpha = \sqrt{a_{ij}(x)y^iy^j}\) is a Riemannian metric, \(\beta = b_i(x)y^i\) is a 1-form and \(b = \|\beta\|\) is the norm of \(\beta\). \(\phi = \phi(b^2, s)\) is a \(C^\infty\) positive function satisfying

\[
\phi > 0, \quad \phi - s\phi_2 + (b^2 - s^2)\phi_2 > 0, \quad |s| \leq b < b_0.
\]

It is known that \(F = \alpha \phi(b^2, s)\) is a Finsler metric if and only if \(\|\beta(x)\| < b_0\) for any \(x \in M\) \([6]\). In particular, when \(\phi(b^2, s) = (\sqrt{1 + b^2} + s)^2\), where \(\alpha = \frac{\sqrt{(1-|x|^2)|x|^2+|x|^2}}{1-|x|^2}\), \(\beta = \frac{x^i}{(1-|x|^2)^2}\), then \(F\) is L. Berwald's metric. Let \(G^i(x, y)\) and \(G^i_{\alpha}(x, y)\) denote the spray coefficients of \(F\) and \(\alpha\), respectively. To express formulae for the spray coefficients \(G^i\) of \(F\) in terms of \(\alpha\) and \(\beta\), we need to introduce some notations. Let \(b_{ij}\) denote the coefficient of the covariant derivative of \(\beta\) with respect to \(\alpha\). Denote

\[
\tau_{ij} = \frac{1}{2}(b_{ij} + b_{ji}), \quad s_{ij} = \frac{1}{2}(b_{ij} - b_{ji}), \quad r_{00} = r_{ij}y^iy^j, \quad s_{i} = \alpha^{ik}s_{kj}, \quad r_{0} = r_{ij}y^j,
\]

\[
s_{0} = s_{ij}y^j, \quad r_{i} = b^j\tau_{ij}, \quad s_{i} = b^j\tau_{ij}, \quad r_{0} = r_{ij}y^j.
\]
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\begin{align*}
  s_0 &= s_i y^i, \\
  r^i &= a^i r_j, \\
  s^i &= a^i s_j, \\
  r &= b^i r_i. 
\end{align*}

(2.3)

**Lemma 2.1.** For a general \((\alpha, \beta)\)-metric \(F = \alpha \phi(b^2, \frac{s}{a})\), its spray coefficients \(G_i\) are related to the spray coefficients \(G_i^\alpha\) of \(\alpha\) by

\begin{align*}
  G_i = & G_i^\alpha + \alpha Q s_i + \{\Theta(-2\alpha Q s_0 + r_0 + 2\alpha^2 R r) + \alpha \Omega(r_0 + s_0)\} \frac{y^i}{\alpha} \\
  & + \{\Psi(-2\alpha Q s_0 + r_0 + 2\alpha^2 R r) + \alpha \Pi(r_0 + s_0)\} b^i - \alpha^2 R(r^i + s^i), 
\end{align*}

(2.4)

where

\begin{align*}
  Q &= \frac{\phi_2}{\phi - s \phi_2}, \\
  R &= \frac{\phi_1}{\phi - s \phi_2}, \\
  \Theta &= \frac{(\phi - s \phi_2) \phi_2 - s \phi \phi_2}{2\phi (\phi - s \phi_2 + (b^2 - s^2) \phi_2)}, \\
  \Psi &= \frac{\phi_{22}}{2(\phi - s \phi_2 + (b^2 - s^2) \phi_2)} \\
  \Pi &= \frac{(\phi - s \phi_2) \phi_12 - s \phi \phi_12}{(\phi - s \phi_2)(\phi - s \phi_2 + (b^2 - s^2) \phi_2)}, \\
  \Omega &= \frac{2 \phi_1}{\phi} - s \phi + (b^2 - s^2) \phi_2^2 \Pi, 
\end{align*}

and \(b^i := a^i b_j\) and \(b^i := a^i b_j b_j = b_j b^i\).

Denote

\begin{align*}
  \Phi &= \frac{\phi_2^2 + \phi \phi_2}{\phi(\phi - s \phi_2)}, \\
  \Gamma &= \frac{\phi_1 \phi_2 + \phi \phi_12}{\phi(\phi - s \phi_2)}, 
\end{align*}

Using (1.2), we can prove the following

**Lemma 2.2.** An general \((\alpha, \beta)\)-metric \(F = \alpha \phi(b^2, s)\), where \(b = \|\beta\|\alpha, s = \frac{\beta}{\alpha}\), is dually flat on an open subset \(U \subset \mathbb{R}^n\) if and only if

\begin{align*}
  2\alpha^2 a_m G^m_\alpha + Q(3s_{t0} - r_{t0}) \alpha^3 - \alpha^2 \left( y_m \frac{\partial G^m_\alpha}{\partial y} + \alpha Q b_m \frac{\partial G^m_\alpha}{\partial y} \right) \\
  + Q(\alpha r_{t0} + 2b_m G^m_\alpha) y_i + \{2Q(y_m G^m_\alpha) + \Phi(\alpha r_{t0} + 2(b_m \alpha - s y_m) G^m_\alpha) \\
  + 2\Gamma(r_0 + s_0)\}(\alpha b_t - s_y) + 4R[(r_0 + s_0)y_i - \alpha^2(r_i + s_i)] = 0.
\end{align*}

(2.5)

where \(r_{t0} := r_{ij} y^j, s_{t0} := s_{ij} y^j\) and \(y_i := a_{ij} y^j\).

**Proof.** By direct computation, \(F\) is dually flat on \(U\) if and only if

\begin{align*}
  \alpha^2(\alpha x + y^k - 2\alpha x') + \phi_2^2 \alpha y^l(\alpha x + y^k) + \phi \phi_1(b^2 x + y^k \alpha y') + 2\alpha^2 y^k y^l \\
  + 2\alpha \phi \phi_2(s x + y^k \alpha y') + \alpha^2 \phi \phi_2(s x + y^k y^l - 2s x) \\
  + \alpha^2[b^2 x + y^k s y'](\phi_1 \phi_2 + \phi \phi_12) + \alpha^2(\phi_1 \phi_2 s y') + (\phi \phi_2 + \phi_2^2) = 0.
\end{align*}

(2.6)

On the other hand,

\begin{align*}
  \alpha x^i &= \frac{1}{\alpha} \frac{\partial G^m_\alpha}{\partial y} y_m, \\
  \alpha x^i + y^k &= \frac{2}{\alpha} G^m_\alpha y_m, \\
  \alpha y^l &= \frac{y_l}{\alpha}, \\
  [\alpha^2] y^l &= 2y_l, \\
  [b^2] x^i &= 2(r_k + s_k), \\
  [y^2] x^i &= 2(r_0 + s_0), \\
  s x^i &= \frac{1}{\alpha} b m_i y^m + \frac{1}{\alpha^2} (a b_m - s y_m) \frac{\partial G^m_\alpha}{\partial y}, \\
  s y^l &= \frac{\alpha b_l - s y_l}{\alpha^2}.
\end{align*}

(2.7)
\[ s_{x^k y^k} = \frac{r_{00}}{\alpha} + \frac{2}{\alpha^2} (ab_m - sy_m) G_{\alpha}^m, \]  
(2.10)

\[ \alpha_{x^k y^k} y^k - 2\alpha_{x^k} = \frac{2}{\alpha^3} (a_{ml} \alpha^2 - y_m y_l) G_{\alpha}^m - \frac{1}{\alpha} \frac{\partial G_{\alpha}^m}{\partial y^l} y_m, \]  
(2.11)

\[ s_{x^k y^k} y^k - 2s_{x^k} = -\frac{r_{00}}{\alpha} y_l + \frac{2}{\alpha} s_{l0} - \frac{2}{\alpha^2} (ab_m - sy_m) G_{\alpha}^m \]  
\[ + \frac{2}{\alpha^2} (\frac{y_l}{\alpha} b_m - \frac{ab_l - sy_l}{\alpha^2} y_m - sa_{ml}) G_{\alpha}^m \]  
\[ - \frac{1}{\alpha} b_{ml} y^m \frac{\partial G_{\alpha}^m}{\partial y^l}. \]  
(2.12)

Putting (2.7)-(2.12) into (2.6) and noting \( b_{ml} y^m = r_{0l} + s_{0l} \) yields

\[ 2\phi (\phi - s\phi_L) \alpha^2 a_{ml} G_{\alpha}^m + \phi \phi_2 (3s_{l0} + r_{l0}) \alpha^3 - \alpha^2 \phi \left[ (\phi - s\phi_L) y_m \frac{\partial G_{\alpha}^m}{\partial y^l} + \alpha \phi_2 b_m \frac{\partial G_{\alpha}^m}{\partial y^l} \right] \]  
\[ + \phi \phi_2 \alpha [r_{0l} + 2b_m G_{\alpha}^m] y_l + (2\phi \phi_2 (y_m G_{\alpha}^m) + (\phi_2^2 + \phi \phi_2)) \alpha [r_{0l} + 2(b_m - sy_m) G_{\alpha}^m] \]  
\[ + 2(r_0 + s_0) (\phi_1 \phi_2 + \phi \phi_2) (ab_l - sy_l) + 4\phi \phi_1 (r_0 + s_0) y_l - \alpha^2 (r_1 + s_1) = 0. \]  
(2.13)

This completes the proof.

3. Some lemmas

To prove Theorems, we need some lemmas. Firstly, we have the following trivial lemmas.

Lemma 3.1. If \( \frac{Q}{s} \) is independent of \( s \), then \( \phi (b^2, s) = f(b^2) \sqrt{1 + g(b^2)s^2} \), where \( f(b^2), g(b^2) \) is a function with respect to \( b^2 \) and \( f(b^2) > 0 \). In this case, \( Q \) is Riemannian.

Lemma 3.2. If \( Q - s\Psi = 0 \), then \( \phi (b^2, s) = \sqrt{f(b^2)} + g(b^2)s \), where \( f(b^2), g(b^2) \) is a function with respect to \( b^2 \) and \( f(b^2) \neq 0 \). In this case, \( Q = \frac{g(b^2)}{f(b^2)}s \) and \( F \) is Riemannian.

Lemma 3.3.[4] Let \( \eta_a y^a (2 \leq a \leq n) \) be a 1-form on a neighborhood \( \mathcal{U} \subset M^n (n \geq 3) \) of \( x_0 \). If \( \eta_a \phi = \eta_a y^a (2 \leq a \leq n) \), then \( \eta_a = 0 \).

Lemma 3.4.[4] Let \( \Omega_{00} := \Omega_{ac} y^a y^c (2 \leq a, c \leq n) \) be a 2-form on a neighborhood \( \mathcal{U} \subset M^n (n \geq 3) \) of \( x_0 \). If \( \Omega_{00} \phi = \Omega_{00} y^a (2 \leq a \leq n) \), then \( \Omega_{00} = \frac{tr \Omega}{\alpha} \Omega_{00} \), where \( \Omega_{00} = \Omega_{ac} y^a y^c \) and \( tr \Omega = \eta_{a=2} \Omega_{00} \).

Lemma 3.5. If \( 2R - s\Gamma = 0 \), then \( \phi (b^2, s) = \sqrt{f(b^2)s + g(s)} \), where \( f(b^2) \) is a function with respect to \( b^2 \), \( g(s) \) is a function with respect to \( s \).

To prove Theorem 1.1 in section 1, we must simplify Eq.(2.5). However, it is difficult to simplify this equation because of the complexity of \( \phi \). To overcome the difficult, a usually use technique (firstly used in [10]) is to take a local coordinate system at a fixed
point $x_0 \in M$ such that
\begin{equation}
\alpha = \sqrt{\sum_{i=1}^{n} (y^i)^2}, \quad \beta = by^1.
\end{equation}

Since $s = \frac{\beta}{\alpha}$, we have
\begin{equation}
y^1 = \frac{s}{\sqrt{b^2 - s^2} \alpha},
\end{equation}
where
\begin{equation}
\overline{\alpha} := \sqrt{\sum_{i=0}^{n} (y^a)^2}.
\end{equation}

In the following, we use the following index conventions unless otherwise stated:

\begin{align*}
1 \leq i, j, k, l, \ldots \leq n, \quad 2 \leq a, c, d, h, \ldots \leq n.
\end{align*}

Thus one obtains a coordinate transformation $(s, y^a) \rightarrow (y^i)$ given by
\begin{equation}
y^1 = \frac{s}{\sqrt{b^2 - s^2} \overline{\alpha}}, \quad y^a = y^a.
\end{equation}

Then
\begin{equation}
\alpha = \frac{b}{\sqrt{b^2 - s^2} \overline{\alpha}}, \quad \beta = \frac{bs}{\sqrt{b^2 - s^2} \overline{\alpha}}.
\end{equation}

Under this coordinate system, we have the following expressions:
\begin{align}
r_{00} &= \frac{s^2r_{11} \overline{\alpha}^2 + 2sr_{10} \overline{\alpha} + r_{00}}{\sqrt{b^2 - s^2} \overline{\alpha}} + \overline{r}_{00}, \\
r_{10} &= \frac{sr_{11} \overline{\alpha} + \overline{r}_{10}}{\sqrt{b^2 - s^2} \overline{\alpha}}, \quad r_{00} = \frac{sr_{a1} \overline{\alpha} + \overline{r}_{a0}}{\sqrt{b^2 - s^2} \overline{\alpha}}; \\
s_{11} = 0, \quad s_{10} = \overline{s}_{10}, \quad s_{a0} = \frac{ss_{a1} \overline{\alpha} + \overline{s}_{a0}}{\sqrt{b^2 - s^2} \overline{\alpha}},
\end{align}

where $\overline{r}_{00} := r_{ac}y^ay^c, \overline{r}_{a0} := r_{ai}y^a, \overline{s}_{a0} := s_{ai}y^a$.

Express $G^i_\alpha = \frac{1}{2} G^i_{jk} (x) y^j y^k$, where $G^i_{jk} = G^i_{kj}$. Let
\begin{align*}
G^i_{10} &:= G^i_{1a} y^a, \quad G^i_{00} := G^i_{ac} y^ay^c, \\
G^i_{11} &:= G^i_{1ayd}, \quad G^i_{10} := G^i_{1ayd} y^a, \quad G^i_{00} := G^i_{ac} y^ay^c;
\end{align*}

where $y_i := a_{ij} y^j = \delta_{ij} y^j$. Then we obtain the expressions for spray coefficients $G^i_\alpha, y_i G^i_\alpha, b_i G^i_\alpha$ and their derivatives with respect to $y^i$ under the transformation (3.4):
\begin{align}
G^i_\alpha &= \frac{s G^{11}_{11}}{2(b^2 - s^2) \overline{\alpha}^2} + \frac{G^{10}_{10} \overline{\alpha} + \frac{1}{2} G^{00}_{00}}{\sqrt{b^2 - s^2} \overline{\alpha}}; \\
y_i G^i_\alpha &= \frac{s^{2} s^{11}_{11} \overline{\alpha}^3}{2(b^2 - s^2) \overline{\alpha}^2} + \frac{s G^{10}_{10} \overline{\alpha} + G^{00}_{00}}{\sqrt{b^2 - s^2} \overline{\alpha}} \\
&\quad + \frac{2 G^{10}_{10} + s G^{00}_{00}}{2(b^2 - s^2) \overline{\alpha}} + \frac{1}{2} G^{00}_{00}; \\
b_i G^i_\alpha &= b G^i_\alpha = \frac{bs^{2} s^{11}_{11}}{2(b^2 - s^2) \overline{\alpha}^2} + \frac{bs G^{10}_{10} \overline{\alpha} + \frac{1}{2} b G^{00}_{00}}{\sqrt{b^2 - s^2} \overline{\alpha}}.
\end{align}
\[
\frac{\partial (u,G^n_\alpha)}{\partial y^l} = \frac{s^2}{2(b^2 - s^2)}(\delta_{11}G_{11}^1 + 2G_{11}^1 + \delta_{1a}G_{11}^a)\alpha^2 + \frac{s}{\sqrt{b^2 - s^2}}(\delta_{11}G_{10}^1 + G_{10}^1)
\]
\[
+ \delta_{1a}G_{10}^a + G^{(1)}_1)\alpha + \frac{1}{2}(\delta_{11}G_{00}^1 + \delta_{1a}G_{00}^a + G^{(0)}_1);
\]
\[
\frac{\partial (b_0G^n_\alpha)}{\partial y^l} = \frac{b_sG_{11}^1}{\sqrt{b^2 - s^2}}\alpha + bG_{10}^1;
\]

**Lemma 3.6.** Suppose \( b = b(x) \neq 0 \) at \( x_0 \subseteq M \) and \( \phi(b^2, s) \) satisfies \( Q - s\Phi \neq 0 \) and the following equation

\[
s(Q - s\Phi)[\lambda(b^2 - s^2) - b\tau] - b^3\tau\Phi + b^2\mu = 0.
\]

where \( \lambda = \lambda(x), \mu = \mu(x) \) and \( \tau = \tau(x) \) (which are independent of \( s \)) are functions on a neighborhood \( U \subseteq M \) of \( x_0 \). Then one of the following holds.

1. If \( Q(b^2, 0) \neq 0 \), then
   \[
   \mu = k_1 b\tau, \quad \lambda = k_2 b^3\tau;
   \]

2. If \( Q(b^2, 0) = 0 \) and \( \tau \equiv 0 \), then \( \lambda = \mu = \tau = 0 \).

3. If \( Q(b^2, 0) = 0 \) and \( \tau \neq 0 \), then

   \[
   Q_1(b^2, s)|_{s=0} = \Phi(b^2, 0) = \left. \frac{\partial^2 \phi}{\partial (b^2, s)} \right|_{s=0}, \quad \left. \frac{\partial^n Q(b^2, s)}{\partial s} \right|_{s=0} = \left. \frac{\partial^n \Phi(b^2, s)}{\partial s} \right|_{s=0} = 0.
\]

In this case, for given the second partial derivative of \( \phi(b^2, s) \) with respect to \( s \) at \( s = 0 \), the partial derivatives of \( \phi(b^2, s) \) at \( s = 0 \) are given by the following

\[
\left. \frac{\partial^{2n-1} \phi(b^2, s)}{\partial s} \right|_{s=0} = 0,
\]
\[
\left. \frac{\partial^{2n+2} \phi(b^2, s)}{\partial s} \right|_{s=0} = (-1)^{n-1}(2n-1)!!(2n-3)!!\left[ \frac{\partial^{2n+2} \phi}{\partial s} \right]_{s=0}.
\]

where \( n \geq 1, \mu = k_1 b\tau \) and \( \lambda, \tau \neq 0 \) are arbitrary function, here \( \frac{\partial^n \phi}{\partial s} \) means the \( n \)th partial derivatives of \( \phi(\rho, s) \).

**Proof.** Taking \( s = 0 \) in (3.18), we get \( \mu = b\Phi(b^2, 0)\tau \). By computing the first, second and third order partial derivative of the left side of (3.18) with respect to \( s \) at \( s = 0 \), we get

\[
Q(b^2, 0)(b\lambda + \xi - \tau) = b^2\tau \Phi_2(b^2, 0),
\]

(3.20)
4. Proof of theorems

In this section, we shall complete the proof of Theorems 1.1 and 1.2. In fact, Therorem 1.1 following from the following Therorem 4.1 directly. So, we firstly prove the following

\[ 2[Q_2(b^2,0) - \Phi(b^2,0)](b\lambda + \xi - \tau) = b^2\tau \Phi_{22}(b^2,0), \]  
\[ (3.21) \]

and
\[ 3b[Q_{22}(b^2,0) - \Phi_2(b^2,0)](b\lambda + \xi - \tau) - 6Q(b^2,0)\lambda = b^3\tau \Phi_{222}(b^2,0). \]  
\[ (3.22) \]

(1) Since \( Q(b^2,0) \neq 0 \), we can solve from (3.20)and (3.22) \( \lambda \) in terms of \( \tau \), which are exactly (3.15)_1 - (3.15)_2. Plugging (3.15) into (3.14) yields (3.16).

(2) By the assumption that \( \tau = 0 \), we get \( \mu = 0 \). Since \( (Q-s\Phi) \neq 0 \), (3.14) is reduced to
\[ \lambda(b^2-s^2) = 0, \]  
\[ (3.23) \]

which implies \( \lambda = 0 \).

(3) The proof of (3) is the same as that of Lemma 3.5 (iii) in [4].

Lemma 3.7. Under the same assumption as Lemma 3.5, if \( \phi(b^2, s) \) further satisfies
\[ Q(b^2,0) \neq 0, \quad and \quad s(Q-s\Phi)[k_2b_2(b^2-s^2)-1] - b^2\Phi + b^2k_1 \neq 0. \]  
\[ (3.24) \]

Then \( \lambda = \mu = \tau = 0 \).

4. Proof of theorems

Theorem 4.1. Let \( F = \alpha \phi(b^2,s) \) be an general \( (\alpha, \beta) \)-metric on an \( n \)-dimensional manifold \( M^n(n \geq 3) \), where \( s = \frac{\beta}{\alpha}, \alpha = \sqrt{a_{ij}(x)y^iy^j}, \beta = \hat{b}_i(x)y^i \neq 0, \) and \( b = \|\beta\|_\alpha \) is the norm of \( \beta \). Suppose \( F \) is not Riemannian, \( \phi(b^2, s) \neq \sqrt{f(b^2)s+g(s)} \) and \( \phi_2(b^2,0) \neq 0 \). Then \( F \) is locally dually flat on \( M^n \) if and only if in an adapted coordinate system \( \alpha, \beta \) and \( \phi = \phi(b^2, s) \) satisfy
\[ s_{t_0} = \frac{1}{3}(\beta \theta_t - \theta b_t), \]  
\[ (4.1) \]
\[ r_{00} = \frac{2}{3}\theta \beta + \left\{ \tau + \frac{2}{3}b^2\tau - \theta(b^2) \right\} \alpha^2 - \frac{1}{b^2} \tau \beta^2, \]  
\[ (4.2) \]
\[ G_{\alpha} = \frac{1}{3}[2\theta + (3k_1 - 2)\tau \beta]y^j + \frac{1}{\beta}(\theta - \tau b)\alpha^2 + \frac{1}{2}k_2 \tau \beta^2 b^i, \]  
\[ (4.3) \]
\[ \tau \{ s(Q-s\Phi)[k_2b_2(b^2-s^2)-1] - b^2\Phi + b^2k_1 \} = 0. \]  
\[ (4.4) \]

where \( k_1 \) and \( k_2 \) are function with respect to \( b^2 \), defined by (3.17) in Section 3, \( \tau := \tau(x) \) is a scalar function, \( \theta = \theta_1(x)y^i \) is a 1-form on \( M \) and \( \theta^i := a^{lm}\theta_m \).

Proof. The sufficiency of theorem can be checked by lemma 2.2. Hence we need only prove the necessity of theorem. Let \( F \) be locally dually flat. By lemma 2.2, (2.5) holds. Fixing an arbitrary point \( x_0 \) in an open subset \( U \subset M \), we make transformation of coordinate \( (s, y^a) \rightarrow (y^i) \) as (3.4). Then we have (3.5)-(3.13). Plugging (3.5)-(3.13) into (2.5),
we get a system of equations in the following form
\[ \Xi + \Omega \sigma = 0, \quad (4.5) \]

where \( \Xi \) and \( \Omega \) are polynomials in \( y^i \). We must have
\[ \Xi = 0, \quad \Omega = 0. \quad (4.6) \]

Denote
\[ B := (b^2 - s^2)(Q - s\Phi), \quad A_{ij} := B[(b^2 - s^2)G_{ij}^1 + br_{ij}]. \quad (4.7) \]

For \( l = 1 \) in (2.4), we get
\[
sA_{11}\sigma^2 - b^2[(b^2 - s^2)(1 + b^2\Phi) + sB\tilde{G}_{00}^1 - sA_{00} - b^2(b^2 - s^2)\Phi\sigma_{00} \\
+ (b^2 - s^2)(b^2 - 2sB)\tilde{G}_{10}^0) - 2b(2R - s\Gamma)(b^2 - s^2)(r_{11} + s_{11}) = 0, \quad (4.8)\]

\[
|b^2(s + b^2Q)\tilde{G}_{10}^1 - 2A_{10} + b^3Q(3\sigma_{10} + \tau_{10}) - s(b^2 - sB)\tilde{G}_{11}^0\tau^2 + (b^2 - s^2)BG_{00}^0 \\
+ 2b[2Rs + (b^2 - s^2)\Gamma](b^2 - s^2)(\sigma_{10} + \tau_{10}) = 0. \quad (4.9)\]

where \( \tilde{A}_{00} := A_{ac}y^ay^c \) and \( \tilde{A}_{10} := A_{1a}y^a. \) For \( l = a \) in (2.11), we get
\[
|b^2s^2G_{11}^a + sb^3Q(3s_{a1} - r_{a1}) - sb^2(s + b^2Q)G_{1a}^1\sigma^4 + [b^2(b^2 - s^2)(\tilde{G}_{00}^a - \tilde{G}_{1a}^a) \\
+ 2sA_{10}y_a - s^3B\tilde{G}_{11}^a y_a - 4Rb^3(b^2 - s^2)(r_{1a} + s_{1a})\sigma^2 - s(b^2 - s^2)B\tilde{G}_{00}^0 y_a \\
+ 2b(2R - s\Gamma)(b^2 - s^2)(\sigma_{10} + \tau_{10})y_a = 0, \quad (4.10)\]

\[
\left[ \frac{s^2}{b^2 - s^2}A_{11}y_a + 2b^2s\tilde{G}_{10}^a - b^2(s + b^2Q)\tilde{G}_{a0}^1 - b^2s\tilde{G}_{1a}^0 + b^3Q(3\sigma_{a0} - \sigma_{a0}) \right] \sigma^2 \\
+ (\tilde{A}_{00} - 2s^2BG_{00}^0)y_a + 2bs(2R - s\Gamma)(b^2 - s^2)(r_{11} + s_{11})y_a = 0. \quad (4.11)\]

Note that contracting (4.10) with \( y^a \) yields (4.9). Let \( s = 0 \) in (4.10), we get
\[
b^4|G_{10}^a - \tilde{G}_{00}^0 - 4Rb(r_{1a} + s_{1a})\sigma^2 + 4Rb^3(\sigma_{10} + \tau_{10})y_a = 0. \quad (4.12)\]

Form (4.9), (4.10) and (4.12) and using \( b \neq 0 \), it follows that
\[
[sG_{11}^a + bQ(3s_{a1} - r_{a1}) - (s + b^2Q)G_{1a}^1\sigma^2 \\
= [s\tilde{G}_{11}^a + bQ(3\sigma_{01} - \tau_{01}) - (s + b^2Q)\tilde{G}_{10}^1]y_a. \quad (4.13)\]

By lemma 3.3, we have
\[
G_{1a}^a - G_{1a}^1 + bQ_s(3s_{a1} - r_{a1} - bG_{1a}^1) = 0, \quad (4.14)\]

and
\[
\tilde{G}_{1a}^0 - \tilde{G}_{10}^0 + bQ_s(3\sigma_{01} - \sigma_{01} - b\tilde{G}_{10}^1) = 0. \quad (4.15)\]
Since $F$ is not Riemannian, it follows that $G_{\alpha}$ is not independent of $s$ from Lemma 3.1. Thus (4.14) and (4.15) are reduced to
\begin{equation}
G_{11}^a = G_{1a}^0 \iff \overline{G}_{11}^0 = \overline{G}_{10}^1.
\end{equation}
\hspace{1cm} (4.16)
and
\begin{equation}
3s_{a1} - r_{a1} = bG_{1a}^1 \iff 3\overline{s}_{01} - \overline{r}_{01} = b\overline{G}_{10}^1.
\end{equation}
\hspace{1cm} (4.17)
Plugging (4.12), (4.16) and (4.17) into (4.9) yields
\begin{equation}
B\{[(2b^2 - 3s^2)\overline{G}_{11}^0 + 2b\overline{r}_{10}]\overline{\tau}_{01} - (b^2 - s^2)\overline{G}_{00}^0\}
+ 2b[2R_s + \Gamma(b^2 - s^2)](b^2 - s^2)(\overline{r}_{10} + \overline{r}_{10}) = 0.
\end{equation}
\hspace{1cm} (4.18)
Since $F$ is not Riemannian, $B \neq 0$ by Lemma 3.2. Thus, we have
\begin{equation}
[(2b^2 - 3s^2)\overline{G}_{11}^0 + 2b\overline{r}_{10}]\overline{\tau}_{01} - (b^2 - s^2)\overline{G}_{00}^0 = 0,
\end{equation}
\hspace{1cm} (4.19)
\begin{equation}
2b(2R_s + \Gamma(b^2 - s^2)](b^2 - s^2)(\overline{r}_{10} + \overline{r}_{10}) = 0.
\end{equation}
\hspace{1cm} (4.20)
We can suppose $\overline{G}_{00}^0 = 2\overline{r}_a^2$ for some 1-form $\overline{\theta} = \theta_ay^a$. Comparing the coefficient of the second power of $s$ in (4.19) yields
\begin{equation}
\overline{G}_{11}^0 = \frac{2}{3}\overline{\tau}, \quad \overline{r}_{10} = \frac{b}{3}\overline{\tau}, \quad r_{1a} = \frac{b}{3}\theta_a.
\end{equation}
\hspace{1cm} (4.21)
Combining (4.12)-(4.13) with (4.16), we obtain
\begin{equation}
\overline{G}_{10}^1 = \overline{G}_{11}^0 = \frac{2}{3}\overline{\tau}, \quad \overline{r}_{10} = \frac{b}{3}\overline{\tau},
\end{equation}
\hspace{1cm} (4.22)
which are equivalent to
\begin{equation}
G_{1a}^1 = G_{11}^0 = \frac{2}{3}\theta_a, \quad s_{1a} = -\frac{b}{3}\theta_a.
\end{equation}
\hspace{1cm} (4.23)
Differentiating $\overline{G}_{00}^0 = 2\overline{r}_a^2$ with respect to $y^a$ and using (4.12), (4.20)-(4.22), we get
\begin{equation}
\overline{G}_{00}^0 = \overline{G}_{00}^a = \frac{2}{3}(2\overline{\theta}y_a + \theta_s\overline{\tau}^2).
\end{equation}
\hspace{1cm} (4.24)
which implies
\begin{equation}
G_{aa}^0 = \frac{2}{3}(\theta^d\delta_{ac} + \theta_a\delta_{ca}^d + \theta_c\delta_{a}^d).
\end{equation}
\hspace{1cm} (4.25)
On the other hand, contracting (4.11) with $y^a$ yields
\begin{equation}
\frac{s^2A_{11}}{b^2 - s^2}\overline{\tau}^2 + s(b^2 - 2sB)\overline{G}_{10}^0 - b^2(s + b^2Q)\overline{G}_{00}^0 - b^3Q\overline{r}_{00} + \overline{A}_{00}
+ 2bs(2R_s - \Gamma)(b^2 - s^2)(r_{11} + s_{11}).
\end{equation}
\hspace{1cm} (4.26)
which is equivalent to (4.8) by using (4.7), we can use (4.11) and (4.8) to eliminate $\overline{A}_{11}, \overline{A}_{00}$ and $2bs(2R_s - \Gamma)(b^2 - s^2)(r_{11} + s_{11})$ in (4.11). Thus, we have
\begin{equation}
[2s\overline{G}_{10}^0 - (s + b^2Q)\overline{G}_{00}^0 - s\overline{G}_{11}^0 + bQ(\overline{s}_{00} - \overline{r}_{00})\overline{\tau}^2]
\overline{\tau}^2
= [s\overline{G}_{10}^0 - (s + b^2Q)\overline{G}_{00}^0 - bQ\overline{r}_{00}]y_a.
\end{equation}
\hspace{1cm} (4.27)
By Lemma 3.4, there is a function $\mu(x) := \frac{1}{n-1} \sum c(G_{1c}^c - G_{cc}^1)$ and $\tau(x) := \frac{1}{n-1} \sum c(r_{cc} + bG_{cc})$ on $\mathcal{U}$ such that

$$s\sigma_{10}^2 - (s + b^2Q)^{G_{10}^1} - bQ\tau_{10}^2 = (s\mu - b\tau)^2 \sigma_1^2. \quad (4.27)$$

Plugging $(4.27)$ into $(4.26)$ yields

$$(2\sigma_{10}^2 - \sigma_{10}^2 - G_{10}^1 - \mu a) + \frac{Q}{S}[3\mu a - \tau a - bG_{10}^1 + \tau y_a] = 0. \quad (4.28)$$

Note that $\frac{Q}{S}$ is not independent of $s$ by Lemma 3.1. Hence $(4.28)$ is reduced to

$$2\sigma_{10}^2 - \sigma_{10}^2 - G_{10}^1 = \mu y_a, \quad (4.29)$$

and

$$3\mu a - \tau a - bG_{10}^1 = -\tau y_a. \quad (4.30)$$

Contracting $(4.29)$ and $(4.30)$ with $y^a$ respectively yield

$$\sigma_{10}^2 = \mu a^2, \quad (4.31)$$

$$\tau_{10}^2 = -bG_{10}^1 + \tau y_a. \quad (4.32)$$

Differentiating $(4.31)$ with respect to $y^a$ and then combing this with $(4.29)$ produce

$$G_{10} = G_{10}^1 = G_{10}^1 + \mu y_a. \quad (4.33)$$

Similarly, differentiating $(4.32)$ with respect to $y^a$ and then combing this with $(4.30)$ produce

$$s_{ac} = 0. \quad (4.34)$$

Plugging $(4.31)$ - $(4.32)$ into $(4.25)$ and using $(4.7)$ yield

$$[s^2 BG_{11} + s^2(a(Q - s\Phi)^r_{11} + s(b^2 - 2sB)\mu - b^3\tau Q + b\tau b\sigma^2 - 3s bG_{10}^1 + 2hs(2R - s\Gamma)(b^2 - s^2)(r_{11} + s_{11}) = 0. \quad (4.35)$$

From $(4.35)$, there is a function $\sigma(x)$ on $\mathcal{U}$ such that

$$\sigma_{00}^2 = \frac{1}{3} \sigma_{20}^2 \iff G_{ac}^1 = \frac{1}{3} \sigma_{2c}. \quad (4.36)$$

Since $\phi(b^2, s) \neq \sqrt{f(b^2)s + g(s)}$, $2R - s\Gamma \neq 0$ by Lemma 3.5, we have

$$r_{11} + s_{11} = 0, \quad (4.37)$$

Thus, by $(4.32)$ - $(4.33)$ and $(4.34)$ we have

$$G_{1a}^c = (\mu + \frac{1}{3} \sigma) \delta_a^c, \quad r_{11} = 0, \quad r_{ac} = (\tau - \frac{1}{3} b\sigma) \delta_{ac}. \quad (4.38)$$

from $(4.10)$ - $(4.11)$ and $(4.36)$ - $(4.37)$. Plugging $(4.36)$ - $(4.37)$ into $(4.35)$ again, we get

$$s(Q - \phi)(b^2 - s\Phi) - b^3\tau \Phi + b^2\mu = 0. \quad (4.39)$$

where $\lambda := G_{11} - 2\mu - \sigma$. Since $F$ is not Riemannian and $\phi_1(b^2, 0) \neq 0$ and $\phi_2(b^2, 0) \neq 0$ by Lemma 3.2 and $Q(b^2, 0) \neq 0$. Hence from Lemma 3.6, we conclude that

$$\mu = kbr, \quad \lambda = k_3 b^3 \tau; \quad (4.40)$$
On locally dually flat general \((\alpha, \beta)\)-metrics

\[ \tau \{ s(Q - s\Phi)[k_2 b^2 (b^2 - s^2) - 1] - b^2 \Phi + b^2 k_1 \} = 0, \]

(4.41)

So,

\[ G^{1}_{11} = \sigma + 2k_1 b \tau + k_2 b^3 \tau. \]

(4.42)

Finally, let us summarize what we have proved so far

\[ s_{11} = 0, \quad s_{1a} = \frac{b}{3} \theta_a, \quad s_{ac} = 0; \]

(4.43)

\[ r_{11} = 0, \quad r_{1a} = \frac{b}{3} \theta_a, \quad r_{ac} = (\tau - \frac{1}{3} b \sigma) \delta_{ac}; \]

(4.44)

\[ G^{1}_{11} = \sigma + 2k_1 b \tau + k_2 b^3 \tau, \quad G_{1a}^{1} = \frac{2}{3} \theta^a, \quad G_{ac}^{1} = \frac{1}{3} \sigma \delta_{ac}, \]

(4.45)

\[ G^{a}_{11} = \frac{2}{3} \theta_a, \quad G_{ia}^{a} = (\mu + \frac{1}{3} \sigma) \delta_{ia}, \quad G_{ac}^{d} = \frac{2}{3} (\theta^d \delta^a_c + \theta_a \delta^d_c + \theta^d \delta_{ac}). \]

(4.46)

Let \( \theta_1 := \frac{1}{4} (\sigma + 2b \tau) \). Then the above identities imply

\[ s_{ij} = \frac{1}{3} (b_i \theta_j - b_j \theta_i); \]

(4.47)

\[ r_{ij} = \frac{1}{3} (b_i \theta_j + \theta_j b_i) + \left\{ \tau + \frac{2}{3} [b^2 \tau - \theta b^1] \right\} a_{ij} - \frac{1}{b^2} \tau b_i b_j; \]

(4.48)

\[ G^{k}_{ij} = \frac{2}{3} (\theta^i \delta^k_j + \theta^j \delta^k_i + \theta^k a_{ij}) + \frac{2}{3} \left( (3k_1 - 2) b_i \delta^k_j + (3k_1 - 2) b_j \delta^k_i - 2k a_{ij} \right) + k_2 \tau b_i b_j \delta^k. \]

(4.49)

Noting the above identities and (4.40) only hold at \( x_0 \). Since \( x_0 \) is arbitrary, (4.47)-(4.49) hold on \( U \). This complete the proof of theorem. \( \square \)

From Theorem 4.1 and Lemma 3.7, we obtain the following

**Corollary 1.2.** Let \( F = \alpha \phi(b^2, s) \), where \( b = \| \beta \|_\alpha, s = \frac{\Phi}{\sigma} \), be an general \((\alpha, \beta)\)-metric on an \( n \)-dimensional manifold \( M^n (n \geq 3) \) with the same assumptions as Theorem 4.1. If \( \phi \) further satisfies

\[ s(Q - s\Phi)[k_2 b^2 (b^2 - s^2) - 1] - b^2 \Phi + b^2 k_1 \neq 0, \]

(4.50)

where function \( k_1, k_2 \) are defined by (3.17), then \( F \) is locally dually flat on \( M \) if and only if \( \alpha \) and \( \beta \) satisfy

\[ s_{10} = \frac{1}{3} (\beta \theta_l - \theta b_l), \]

(4.51)

\[ r_{00} = \frac{2}{3} [\theta \beta - (\theta b^1) \alpha^2], \]

(4.52)

\[ G'_{10} = \frac{1}{3} [2 \theta y^j + \theta^l \alpha^2], \]

(4.53)

where \( \theta := \theta_i (x) y^i \) is a 1-form on \( M \) and \( \theta^l := a^l m \theta_m \).

**Proof of Theorem 1.2.** The former part of proof of Theorem 1.2 is the same as that
of Theorem 1.4 completely before Eq. (4.36) also hold. Since $\phi(b^2, s) = \sqrt{f(b^2)s + g(s)}$,
we get

\[2R - \Gamma = 0\]

from (4.10)-(4.11) and (4.36)-(4.37). Plugging (4.36)-(4.37) into (4.35) again, we get

\[s(Q - s\Phi)[\lambda(b^2 - s^2) + b\xi - b\tau] - b^3s\Phi + b^2\mu = 0,\]

where $\lambda = G_{11}^1 - 2\mu - \sigma$ and $\xi := r_{11}$. The rest of proof is similar to that of Theorem 4.1
in [4]. □

References