Group actions, the Mattila integral and continuous sum-product problems

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Abstract

The Mattila integral,

\[ \mathcal{M}(\mu) = \int \left( \int_{S^{d-1}} |\hat{\mu}(r\omega)|^2 d\omega \right)^2 r^{d-1} dr, \]

developed by Mattila, is the main tool in the study of the Falconer distance problem. Recently this integral is interpreted by Greenleaf et al. in terms of the \( L^2 \)-norm of the natural measure on \( E - gE, \ g \in O(d) \), the orthogonal group. Following this group-theoretic viewpoint, we develop an analog of the Mattila integral associated with arbitrary groups. As an application, we prove for any \( E, F, H \subset \mathbb{R}^2 \), \( \dim_H(E) + \dim_H(F) + \dim_H(H) > 4 \), the set

\[ E \cdot (F + H) = \{ x \cdot (y + z) : x \in E, y \in F, z \in H \} \]

has positive Lebesgue. In particular, it implies that for any \( A \subset \mathbb{R} \),

\[ |A(A + A)| > 0 \]

whenever \( \dim_H(A) > \frac{2}{3} \). We also give a very simple argument to show that on \( \mathbb{R}^2 \), \( \dim_H(E) > 1 \) is sufficient for \( |E \cdot (E \pm E)| > 0 \), where the dimensional threshold is optimal. By taking \( E = A \times A \), it follows that

\[ |A(A + A) + A(A + A)| > 0 \]

whenever \( \dim_H(A) > \frac{1}{2} \), which is also sharp. We therefore conjecture that \( \frac{1}{2} \) is the best dimensional threshold for \( A \subset \mathbb{R} \) to ensure \( |A(A + A)| > 0 \).