Two different kinds of liminf results on the LIL for two-parameter Wiener processes

Li-Xin Zhang

Department of Mathematics and Information Science, Hangzhou University, Hangzhou 310028, People’s Republic of China

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Abstract

Some probability inequalities are obtained, and some liminf results are established for a two-parameter Wiener process by using these inequalities. The results obtained improve those of Lacey (1989) and get the watershed between the Chung type laws of the iterated logarithm and the Lacey type laws of the iterated logarithm.

Keywords: Two-parameter Wiener process; Laws of the iterated logarithm; Chung-type laws of the iterated logarithm

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1. Introduction and main results

Let \( \{W(x, y) : 0 \leq x, y < \infty\} \) be a two-parameter Wiener process, and let \( 0 < a_T \leq T \) and \( b_T \geq T^{1/2} \) be two non-decreasing functions of \( T \). Let \( D_T = \{(x, y) : x, y \leq T, 0 \leq x, y \leq b_T\} \) and \( D_T^* = \{(x, y) : xy = T, 0 \leq x, y \leq b_T\} \). For the rectangle \( R = [x_1, x_2] \times [y_1, y_2] \), define \( \lambda(R) = (x_2 - x_1)(y_2 - y_1) \) and \( W(R) = W(x_2, y_2) - W(x_1, y_2) - W(x_2, y_1) + W(x_1, y_1) \). Let \( L_T = \{R : R \subset D_T, \lambda(R) \leq a_T\} \) be a set of rectangles \( R = [x_1, x_2] \times [y_1, y_2] \). Then, define

\[
\alpha_T = \left\{2T(\log(1 + \log b_T T^{-1/2}) - \log \log T)\right\}^{-1/2},
\]

\[
\beta_T = T^{-1/2} \left(\frac{\log \log T}{\log b_T T^{-1/2}}\right)^{1/2} \left(\frac{\log \log T}{\log b_T T^{-1/2}}\right)^{-3/2},
\]

where \( \log x = \ln(\max\{x, e\}) \), \( \log \log x = \log(\log x) \).

Lacey (1989) established the following law of the iterated logarithm. 

\[ \text{(1.1)} \]
Theorem A. Suppose \( \lambda'_T = \{2T \log(1 + \log b_T T^{-1/2})\}^{-1/2} \) satisfy

\[
\lim_{\theta \to 1} \lim_{T \to \infty} \sup_{i \geq 1} \lambda'_T = 1 \quad (\ast)
\]

and \( b'_T = b_T T^{-1/2} \) is a non-decreasing function of \( T, b'_T \to \infty \). Assume also that for any \( 0 < \varepsilon < 1 \) and some \( 0 < a < 1 \), we have

\[
\sum_k \exp\{- (\log b'_m)^a\} < \infty,
\]

where \( m_k = \exp(k^a), k \in \mathbb{N} \). Then

\[
\lim_{T \to \infty} \sup_{(x,y) \in D_T^2} \lambda'_T W(x,y) = 1 \quad \text{a.s.} \quad (1.3)
\]

Particularly,

\[
\liminf_{T \to \infty} \sup_{0 \leq x, y \leq T} \frac{W(x,y)}{\sqrt{2T \log \log T}} = 1 \quad \text{a.s.} \quad (1.4)
\]

Recently, Talagrand (1993) obtained a Chung-type law of the iterated logarithm for the two-parameter Wiener process which reads as follows.

Theorem B. We have

\[
0 < \liminf_{T \to \infty} \frac{(\log \log T)^{1/2}}{T(\log \log \log T)^{3/2}} \sup_{0 \leq x, y \leq T} |W(x,y)| < \infty \quad \text{a.s.} \quad (1.5)
\]

Lacey (1989) asked whether (1.2) is necessary for (1.3). It is of interest to investigate whether there are any results similar to (1.3) possible in the case one were to weaken (1.2). It is also of interest to sort out the nature of having (1.4) type liminf results versus having (1.5) type liminf results. The purpose of this paper is to solve these problems. We find the watershed between these two different kinds liminfs. We also show that condition (\ast) in Theorem A is superfluous. Our results read as follows.

Theorem 1.1. If

\[
\Delta_T =: \frac{\log b_T T^{-1/2}}{\log \log T} \to \infty \quad (T \to \infty),
\]

then

\[
\liminf_{T \to \infty} \lambda_T \sup_{R \subset D_T} |W(R)| = \liminf_{T \to \infty} \lambda_T \sup_{R \subset D_T^2} |W(R)|
\]

\[
= \liminf_{T \to \infty} \lambda_T \sup_{(x,y) \in D_T} |W(x,y)| = \liminf_{T \to \infty} \lambda_T \sup_{(x,y) \in D_T^2} |W(x,y)| = 1 \quad \text{a.s.} \quad (1.7)
\]

If

\[
\Delta_T =: \frac{\log b_T T^{-1/2}}{\log \log T} \to 0 \quad (T \to \infty),
\]

then

\[
\liminf_{T \to \infty} \lambda_T \sup_{R \subset D_T} |W(R)| = \liminf_{T \to \infty} \lambda_T \sup_{R \subset D_T^2} |W(R)|
\]

\[
= \liminf_{T \to \infty} \lambda_T \sup_{(x,y) \in D_T} |W(x,y)| = \liminf_{T \to \infty} \lambda_T \sup_{(x,y) \in D_T^2} |W(x,y)| = 1 \quad \text{a.s.} \quad (1.8)
\]
then for some positive numerical numbers $C_1, C_2$ we have

$$C_1 \leq \liminf_{T \to \infty} \beta_T \sup_{R \in D_T} |W(R)| \leq C_2 \quad \text{a.s.,}$$

$$C_1 \leq \liminf_{T \to \infty} \beta_T \sup_{(x,y) \in D_T} |W(x,y)| \leq C_2 \quad \text{a.s.}$$

(1.9)

(1.10)

The truncated hyperbola interpolating between these two liminf results works as a bridge. If we take $b_T = T^{1/2}$, then (1.10) is just (1.5). And, if we take $b_T = T$, then (1.7) is the Lacey’s type LIL (1.4). Also, our condition (1.6) is much weaker than Lacey’s condition (1.2). To verify this fact, it is enough to note that under the condition in Theorem A, (1.2) is equivalent to

$$\lim_{T \to \infty} \frac{\log \log b_T}{\log \log \log T} = \infty.$$  

(1.11)

In fact, if (1.11) is true, then

$$\liminf_{k \to \infty} \frac{\log \log b_m^T}{\log \log k} = \liminf_{k \to \infty} \frac{\log \log b_m^T}{\log \log \log m} > \frac{2}{\epsilon}.$$  

So, for $k$ large enough, we have

$$(\log b_m^T)^{\gamma} \geq (\log k)^2,$$

which implies (1.2).

On the other hand, if (1.2) is true, noting that $b_T$ is non-decreasing, we have

$$k \exp(-(\log b_m^T)^{\gamma}) \leq \sum_{k=1}^{\infty} \exp(-(\log b_m^T)^{\gamma}) < \infty.$$  

So,

$$\exp((\log b_m^T)^{\gamma}) \geq Ck,$$

which implies

$$\liminf_{k \to \infty} \frac{\log \log b_m^T}{\log \log \log m_k} \geq \frac{1}{\epsilon}.$$

For $m_k \leq T \leq m_{k+1}$ we have

$$\frac{\log \log b_T^T}{\log \log \log T} \geq \frac{\log \log b_m^T}{\log \log \log m_k} \frac{\log \log m_k}{\log \log \log m_{k+1}}.$$  

Then

$$\liminf_{T \to \infty} \frac{\log \log b_T^T}{\log \log \log T} \geq \frac{1}{\epsilon},$$

which implies (1.11) by letting $\epsilon \to 0$.

From Theorem 1.1 we also conclude that if

$$\lim_{T \to \infty} \frac{\log \log b_T T^{-1/2}}{\log \log \log T} = r \geq 1,$$
then
\[
\liminf_{T \to \infty} \lambda_T^+ \sup_{(x, y) \in D_T} |W(x, y)| = \left( \frac{r - 1}{r} \right)^{1/2} \text{ a.s.} \quad (1.12)
\]
Hence, from (1.12) it is easy to see that (1.2) or (1.11) is very close to being necessary for (1.3).

2. Some probability inequalities

Theorem 1.1 of the introduction is based upon the probability inequalities of this section.

**Theorem 2.1.** For any \( \varepsilon > 0 \), there exist constants \( C = C(\varepsilon) > 0, \ u_0 = u_0(\varepsilon) > 0 \) and \( T_0 = T_0(\varepsilon) > 0 \) such that
\[
P \left( \sup_{R \in L_T} |W(R)| \leq uT^{1/2} \right) \geq \exp\left( -CTa_T^{-1}(1 + \log Ta_T^{-1})(1 + \log b_T a_T^{-1/2})e^{-u^2/(2+\varepsilon)} \right),
\]
(2.1)
\[
P \left( \sup_{(x, y) \in D_T^*} |W(x, y)| \leq uT^{1/2} \right) \leq \exp\left( -C(1 + \log b_T T^{-1/2})e^{-u^2/(2-\varepsilon)} \right)
\]
(2.2)
holds for any \( u \geq u_0, T \geq T_0 \).

**Proof.** Eq. (2.1) comes from Theorem 2.1 of Zhang (1995). We now show (2.2). Let \( L = L(T) \) be the largest integer for which we have
\[
T^{1/2}M^{L+1} < b_T \quad (M > 1).
\]
Define the rectangles
\[
S_i = S_i(T) = [x_1(i), x_2(i)] \times [y_1(i), y_2(i)] = [T^{1/2}M^i, T^{1/2}M^{i+1}] \times [0, T^{1/2}M^{-i-1}], \quad i = 0, 1, \ldots, L.
\]
Then \( S_i \subset D_T, \lambda(S_i) = T(1 - i/M), \ i = 0, 1, \ldots, L, \) and \( L \geq (\log b_T T^{-1/2})/\log M \).
Let
\[
\tilde{S}_i = [0, T^{1/2}M^i] \times [0, T^{1/2}M^{-i-1}], \quad i = 0, 1, \ldots, L.
\]
Then
\[
P \left( \sup_{(x, y) \in D_T^*} |W(x, y)| \leq uT^{1/2} \right) \\
\leq P \left( \sup_{0 \leq i \leq L} |W(x_2(i), y_2(i))| \leq uT^{1/2} \right) \\
= P \left( \sup_{0 \leq i \leq L} |W(\tilde{S}_i) + W(S_i)| \leq uT^{1/2} \right). \quad (2.3)
\]
We employ a conditioning argument. Let \( \sigma_i = \sigma(W(x, y); 0 \leq y \leq b_T, 0 \leq x \leq T^{1/2}M^{i+1}) \), then \( W(S_i) \in \sigma_i \), \( W(\bar{S}_i) \in \sigma_{i-1} \), and \( W(S_i) \) is independent of \( \sigma_{i-1} \). So for \( M \) large enough, we have

\[
P \left( \sup_{0 \leq i \leq L} |W(\bar{S}_i) + W(S_i)| \leq uT^{1/2} \right)
\]

\[
= E \left[ \left\{ \sup_{0 \leq i \leq L-1} |W(\bar{S}_i) + W(S_i)| \leq uT^{1/2} \right\} \right.A
\]

\[
\times P\left( |W(\bar{S}_L) + W(S_L)| \leq uT^{1/2} \right) = \left[ P\left( |W(S_L)| \leq uT^{1/2} \right) \right]
\]

\[
\leq E \left[ \left\{ \sup_{0 \leq i \leq L-1} |W(\bar{S}_i) + W(S_i)| \leq uT^{1/2} \right\} \right.A
\]

\[
\times P\left( |W(S_L)| \leq uT^{1/2} \right) = \left[ P\left( |W(S_L)| \leq uT^{1/2} \right) \right]
\]

\[
\leq \cdots \leq \prod_{i=0}^L P(|W(S_i)| \leq uT^{1/2})
\]

\[
\leq \left\{ 1 - \Phi \left( -\left( \frac{M}{M-1} \right)^{1/2} u \right) \right\}^{L+1} \leq \left\{ 1 - \exp \left( \frac{-u^2}{2} \right) \right\}^{L+1}
\]

\[
\leq \exp \left\{ -c \frac{1}{\log M} \left( \log b_T T^{-1/2} \right) e^{-u^2/(2-\varepsilon)} \right\}. \tag{2.4}
\]

Hence, by (2.3) and (2.4),

\[
P \left( \sup_{(x, y) \in D_L} |W(x, y)| \leq u \right) \leq \exp \left\{ -c \frac{1}{\log M} \left( \log b_T T^{-1/2} \right) e^{-u^2/(2-\varepsilon)} \right\}
\]

which implies (2.2).

**Theorem 2.2.** There exists a numerical constant \( C_2 > 0 \) such that for any \( 0 < u \leq \frac{1}{2} \) and \( T > 0 \),

\[
\exp \left( -C_2 \frac{1}{u} \log b_T T^{-1/2} \right) \leq P \left( \sup_{R \subset D_T} |W(R)| \leq u \left( \log \frac{1}{u} \right)^{1/2} \right)
\]

\[
\leq \exp \left( -\frac{1}{C_2 u} \log b_T T^{-1/2} \right); \tag{2.5}
\]

\[
\exp \left( -C_2 \frac{1}{u} \log b_T T^{-1/2} \right) \leq P \left( \sup_{(x, y) \in D_T} |W(x, y)| \leq u \left( \log \frac{1}{u} \right)^{1/2} \right)
\]

\[
\leq \exp \left( -\frac{1}{C_2 u} \log b_T T^{-1/2} \right). \tag{2.6}
\]

To prove Theorem 2.2, we need some lemmas.
Lemma 2.1 (Khatri–Šidák lemma). Let $T$ be a parameter set, and $\{Y(t), t \in T\}$ be a Gaussian process with mean zero. Then

$$P \left( \sup_{t \in A} \frac{|Y(t)|}{x(t)} \leq 1, |Y(t_0)| \leq x(t_0) \right) \geq P \left( |Y(t_0)| \leq x(t_0) \right) P \left( \sup_{t \in A} \frac{|Y(t)|}{x(t)} \leq 1 \right)$$

for every $A \subset T$, $x(t) > 0 (t \in A), t_0 \in T$.

If there is a countable set $T_c$ and a Gaussian process $U(t)$ with mean zero on $T_c$ and a function $u(t) > 0 (t \in T_0)$ such that

$$\left\{ \sup_{t \in T} \frac{|Y(t)|}{x(t)} \leq 1 \right\} \supset \left\{ \sup_{t \in T_c} \frac{|U(t)|}{u(t)} \leq 1 \right\} \text{ a.s.}$$

then for any mean zero Gaussian process $\{Z(s), s \in T_1\}$ and function $\{z(s) > 0, s \in T_1\}$ we have

$$P \left( \sup_{s \in T_1} \frac{|Z(s)|}{z(s)} \leq 1, \sup_{t \in T} \frac{|Y(t)|}{x(t)} \leq 1 \right) \geq P \left( \sup_{s \in T_1} \frac{|Z(s)|}{z(s)} \leq 1 \right) \prod_{i=1}^\infty P \left( \frac{|U(t)|}{u(t)} \leq 1 \right) = P \left( \sup_{s \in T_1} \frac{|Z(s)|}{z(s)} \leq 1 \right) p_i.$$

A lower bound obtained through the above method will be called a KS lower bound of $P(\sup_{t \in T} |Y(t)|/x(t) \leq 1)$, denoted by (KSLB).

The following lemma is a direct consequence of the Khatri–Šidák lemma.

Lemma 2.2. Let $T_i, i = 1, 2, \ldots$, be parameter sets, and $\{Y_i(t), t \in T_i\}, i = 1, 2, \ldots$ be Gaussian processes with mean zero. Assume that

$$P \left( \sup_{t \in T_i} \frac{|Y_i(t)|}{x_i(t)} \leq 1 \right) \overset{KS}{\geq} p_i, \quad i = 1, 2, \ldots,$$

namely, the $p_i$ are KSLB of $P(\sup_{t \in T_i} |Y_i(t)|/x_i(t) \leq 1) (i = 1, 2, \ldots)$. Then

$$P \left( \sup_{t \in T} \sup_{i=1}^\infty \frac{|Y_i(t)|}{x_i(t)} \leq 1 \right) \overset{KS}{\geq} \prod_{i=1}^\infty p_i,$$

where $x_i(t) > 0 (t \in T_i) (i = 1, 2, \ldots)$.

The following lemma comes from Theorem 1.1 of Talagrand (1994) (see also Shao and Wang (1995), Corollary 1.1 and its proof).

Lemma 2.3. There exists a constant $C > 0$ such that for any $0 < u \leq \frac{1}{2}$ and $T_1, T_2 > 0$ we have

$$P \left( \sup_{(x,y) \in [0,T_1] \times [0,T_2]} |W(x,y)| \leq (T_1 T_2)^{1/2} \left( u \left( \log \frac{1}{u} \right)^3 \right)^{1/2} \right) \overset{KS}{\geq} \exp \left( -\frac{C}{u} \right).$$
Proof of Theorem 2.2. Noting that
\[ \sup_{(x, y) \in D} |W(x, y)| \leq \sup_{R \subset D} |W(R)| \leq 4 \sup_{(x, y) \in D} |W(x, y)|, \tag{2.7} \]
we need only to prove (2.5). First we establish the lower bound. Without loss of generality, we can assume $T = 1$. Write $D$ for $D_T = D_1$, and so on. Take $R_j = [0, 2^{j-1}] \times [0, 2^{j-1}]$, for $j = -1 - \log_2 b, \ldots, 1 + \log_2 b$. Notice that each $R_j$ has area 2, and
\[ R^* := \bigcup_{j = -1 - \log_2 b}^{1 + \log_2 b} R_j \supset D = \{(x, y) : xy \leq 1, 0 \leq x, y \leq b\}. \]
Then, by Lemmas 2.2 and 2.3, we have
\[
P \left( \sup_{(x, y) \in D} |W(x, y)| \leq \left( u \left( \log \frac{1}{u} \right) \right)^{1/2} \right),
\]
\[
\geq P \left( \sup_{(x, y) \in R^*} |W(x, y)| \leq \left( u \left( \log \frac{1}{u} \right) \right)^{1/2} \right),
\]
\[
\geq \prod_{j = -1 - \log_2 b}^{1 + \log_2 b} P \left( \sup_{(x, y) \in R_j} |W(x, y)| \leq \left( u \left( \log \frac{1}{u} \right) \right)^{1/2} \right),
\]
\[
\geq \exp \left( - \frac{C}{u} \log b \right). \]
Hence, we have obtained the lower bound. For the upper bound, we define
\[ S_i = [T^{1/2} M^i, T^{1/2} M^{i-1}] \times [0, T^{1/2} M^{i-1}], \quad i = 0, 1, \ldots, L, \]
where $L$ is the largest integer for which $T^{1/2} M^{L-1} < b_T$ ($M > 1$). Then, by Theorem 1.1 of Talagrand (1994), we have
\[
P \left( \sup_{R \subset D_T} |W(R)| \leq T^{1/2} \left( u \left( \log \frac{1}{u} \right) \right)^{1/2} \right),
\]
\[
\leq P \left( \sup_{i} \sup_{R \subset S_i} |W(R)| \leq T^{1/2} \left( u \left( \log \frac{1}{u} \right) \right)^{1/2} \right),
\]
\[
= \prod_{i = 0}^{L} P \left( \sup_{R \subset S_i} |W(R)| \leq T^{1/2} \left( u \left( \log \frac{1}{u} \right) \right)^{1/2} \right),
\]
\[
= \prod_{i = 0}^{L} P \left( \sup_{R \subset [0,1] \times [0,1]} |W(R)| \leq \left( \frac{M}{M-1} \right)^{1/2} \left( u \left( \log \frac{1}{u} \right) \right)^{1/2} \right),
\]
\[
\leq \prod_{i = 0}^{L} P \left( \sup_{0 \leq x, y \leq 1} |W(x, y)| \leq \left( \frac{M}{M-1} \right)^{1/2} \left( u \left( \log \frac{1}{u} \right) \right)^{1/2} \right).}
\[ \leq \exp \left( -\frac{C}{u} \frac{M - 1}{M} L \right) \leq \exp \left( -\frac{C}{u} \frac{M - 1}{\log M} \log b_T^2 T^{-1} \right) \]
\[ = \exp \left( -\frac{C}{u} \log b_T T^{-1/2} \right). \]
Hence, we have proved Theorem 2.2.

**Remark.** The author is pleased to thank a referee for his suggestions simplifying the proof of Theorem 2.2.

### 3. The proof of the main results

**Proof of (1.7).** By using Theorem 2.1 instead of Theorem 2.2, the proof of the upper bound of (1.7) is something similar to that of (3.10) below and is omitted here. For details one can refer to Zhang (1995). We now verify the lower bound. It is sufficient to show that if \( \Delta_T \to \infty \), then

\[
\liminf_{T \to \infty} \lambda_T \sup_{(x,y) \in D_T^*} |W(x,y)| \geq 1 \quad \text{a.s.} \\
\text{(3.1)}
\]

We can assume \( \lambda_T = \{2T(\log \log b_T T^{-1/2} - \log \log \log T)\}^{-1/2} \). Let

\[
A_k = \left\{ T : \frac{e^k}{T^{1/2}} \leq e^{k+1} \right\}, \quad k \geq 0, \\
A_{k,j} = \left\{ T : e^{\sqrt{j}} \leq T \leq e^{\sqrt{j+1}}, T \in A_k \right\}, \quad k \geq 0, j \geq 0, \\
b(T_{k,j}) = \inf \{ b_T : T \in A_{k,j} \}, \quad b(T_{k,j}^*) = \sup \{ b_T : T \in A_{k,j} \}, \\
D_{k,j}^x = \{(x,y) : xy = e^{\sqrt{j}}, 0 \leq x, y \leq b(T_{k,j})\}, \\
L_{k,j} = \{R \subset \{(x,y) : xy \leq e^{\sqrt{j+1}}, 0 \leq x, y \leq b(T_{k,j}^*)\} : \lambda(R) \leq e^{\sqrt{j+1} - e^{\sqrt{j}}}\}. \\
\text{(3.2)}
\]

Note that \( \Delta_T \to \infty \) and \((k+1)/\sqrt{j} \log j \geq \Delta_T \) for \( T \in A_{k,j} \). Also, we know that for any \( M > 4 \), if \( j \) is large enough, then

\[ A_{k,j} = \emptyset \quad \text{if} \quad k \leq M \log j. \]

Therefore,

\[
\liminf_{T \to \infty} \lambda_T \sup_{(x,y) \in D_T^*} |W(x,y)| \\
\geq \liminf_{j \to \infty} \inf_k \liminf_{T \in A_{k,j}} \sup_{(x,y) \in D_T^*} |W(x,y)| \\
\geq \liminf_{j \to \infty} \inf_k \liminf_{T \in A_{k,j}} \sup_{(x,y) \in D_T^*} |W(x,y)| \\
-4 \limsup_{j \to \infty} \sup_k \liminf_{T \in A_{k,j}} \sup_{R \subset L_{k,j}} |W(R)|
\]
By Theorem 2.1, we have for $j$ large enough

$$P \left( \{2e^{\sqrt{j+1}}(\log(k+1) - \log \log j^{1/2})\}^{-1/2} \sup_{(x,y) \in D^*_x} |W(x,y)| \leq 1 - 2\varepsilon \right)$$

$$\leq \exp \left\{ -C(\log \beta(T_{k,j})e^{-\sqrt{j}^2} \right. \times \exp \left\{ \frac{2(1 - 2\varepsilon)^2e^{\sqrt{j+1}}(\log(k+1) - \log \log j^{1/2})}{(2 - 2\varepsilon)e^{\sqrt{j}}} \right\}$$

$$\leq \exp \left\{ -C(\log \varepsilon^k) \exp \left\{ \frac{2(1 - \varepsilon)^2(\log(k+1) - \log \log j^{1/2})}{(2 - 2\varepsilon)} \right\} \right\}$$

$$\leq \exp \left\{ -Ck \log \varepsilon \exp \left\{ -(1 - \varepsilon)(\log(k+1) - \log \log j^{1/2}) \right\} \right\}$$

$$\leq \exp \left\{ -C \left( \frac{k}{\log j} \right)^\varepsilon \log j \right\}. \quad (3.4)$$

Note that for $M$ and $j_0$ large enough we have

$$\sum_{j=j_0}^{\infty} \sum_{k \geq M \log j} \exp \left\{ -C \left( \frac{k}{\log j} \right)^\varepsilon \log j \right\}$$

$$\leq \sum_{j=j_0}^{\infty} \sum_{k \geq M \log j} \exp \left\{ -C \left( \frac{k}{\log j} \right)^{\varepsilon/2} \log j - \left( \frac{k}{\log j} \right)^{\varepsilon/2} \right\}$$

$$\leq \sum_{j=j_0}^{\infty} \exp \left\{ -CM^{\varepsilon/2} \log j \right\} \cdot \sum_{k \geq M \log j} \exp \left\{ - \left( \frac{k}{\log j} \right)^{\varepsilon/2} \right\}$$

$$\leq \sum_{j=j_0}^{\infty} (\log j) \exp \left\{ -CM^{\varepsilon/2} \log j \right\} < \infty, \quad (3.5)$$

which together with (3.4) implies

$$I_1 \geq 1 - 2\varepsilon \quad \text{a.s.} \quad (3.6)$$

For $I_2$ note that, by (2.1) or Theorem 1.12.6 of Csörgő and Révész (1981), we have

$$P \left( \sup_{R \subset L_{k,j}} |W(R)| \geq \alpha(e^{\sqrt{j+1}} - e^{\sqrt{j}})^{1/2} \right)$$

$$\leq C \frac{e^{\sqrt{j+1}}}{e^{\sqrt{j+1}} - e^{\sqrt{j}}} \left( 1 + \log \frac{e^{\sqrt{j+1}}}{e^{\sqrt{j+1}} - e^{\sqrt{j}}} \right)$$

$$\times (1 + \log b(T_{k,j})(e^{\sqrt{j+1}} - e^{\sqrt{j}})^{-1/2}) e^{-\gamma/(2 + \varepsilon)}$$

$$\leq C \sqrt{j}(1 + \log j)(1 + k + \log j) e^{-\gamma/(2 + \varepsilon)} \leq Cjke^{-\gamma/(2 + \varepsilon)}. \quad (3.7)$$
we have
\[
\sum_{j=j_0}^{\infty} \mathbb{P} \left( \sup_{k \geq M} \{2e^{\frac{1}{2}(\log k - \log(\log(j + 1)^{1/2})}\}^{1/2} \sup_{R \subset L_{k,j}} |W(R)| > \frac{\varepsilon}{4} \right) \\
\leq C \sum_{j=j_0}^{\infty} \sum_{k \geq M} j \exp \left( -\varepsilon' \sqrt{j \log \frac{k}{\log j}} \right) \\
\leq C \sum_{j=j_0}^{\infty} j \left( \log j \right)^{\varepsilon' \sqrt{j}} \sum_{k \geq M} k^{-\varepsilon' \sqrt{j + 1}} \\
\leq C \sum_{j=j_0}^{\infty} j \left( \log j \right)^{2M - \varepsilon' \sqrt{j}} < \infty, \quad (3.8)
\]
where \( \varepsilon' > 0 \) depends only on \( \varepsilon \), which implies
\[
I_2 \leq \varepsilon / 4 \quad \text{a.s.} \quad (3.9)
\]
Hence, by (3.6) and (3.9), we have proved (3.1). The proof of (1.7) is now complete.

**Proof of (1.9) and (1.10).** Let \( C_2 \) be as in Theorem 2.2. First we prove
\[
\liminf_{T \to \infty} \beta_T \sup_{R \subset D_T} |W(x,y)| \leq C_2^{1/2} \quad \text{a.s.} \quad (3.10)
\]
Let \( T_n = \varepsilon^p (p > 1) \), \( D'_{T_n+1} = D_{T_n+1} \cap D_{T_n}', D''_{T_n+1} = \{(x,y) : 0 \leq x, y \leq b_{T_n+1}, xy \leq 2T_n\} \). We employ an independent argument which was used by Zhang (1995). Let \( l_n \) be a polygonal line between the hyperbolas \( xy = T_n, xy = 2T_n \) with edges parallel to the coordinate axes and vertexes on \( xy = T_n, xy = 2T_n \), one of these vertexes is \((\sqrt{2T_n}, \sqrt{2T_n})\). The polygonal line \( l_n \) cut the plane into two parts. We denote the upper (resp. down) part by \( U_n \) (resp. \( V_n \)). For any \( R = [x_1, x_2] \times [y_1, y_2] \subset D'_{T_n+1} \), we have \( R \cap U_n \subset D'_{T_n+1}, R \cap V_n \subset D''_{T_n+1} \) and there exist rectangles \( R_1, \ldots, R_k \subset D'_{T_n+1}, \tilde{R}_1, \ldots, \tilde{R}_k \subset D''_{T_n+1} \) with disjoint interiors such that \( R \cap U_n = \bigcup_{i=1}^{k} R_i, R \cap V_n = \bigcup_{i=1}^{k} \tilde{R}_i \). We define \( W(R \cap U_n) = \sum_{i=1}^{k} W(R_i) \) and \( W(R \cap V_n) = \sum_{i=1}^{k} W(\tilde{R}_i) \). We also know that the value of \( W(R \cap U_n) \) (resp. \( W(R \cap V_n) \)) is the sums and differences of the values of \( W(\cdot) \) on the vertexes of \( R \cup U_n \) (resp. \( R \cup V_n \)). Let
\[
Y_{n+1} = \sup_{R \subset D'_{T_n+1}} |W(R \cap U_n)|.
\]
Obviously, \( Y_n \) (\( n = 1, 2, \ldots \)) are independent. For any \( R = [x_1, x_2] \times [y_1, y_2] \subset D'_{T_n+1} \), we have \( W(R) = W(R \cap U_n) + W(R \cap V_n) \). Let \( M_n(R) \) be the number of the vertexes of \( R \cap V_n \). It is easy to see that, if \( R \subset D'_{T_n+1} \) or \( R \subset D''_{T_n+1} \), then \( M_n(R) \leq 6 \). Suppose \( (x_1, y_1) \in D_{T_n} \) and \( (x_2, y_2) \in D_{T_n+1} \cap (D'_{T_n+1})^c \), denote \((u_1, v_1), \ldots, (u_k, v_k) \) \((u_1 < \cdots < u_k)\) all the vertexes of \( l_n \) on the hyperbola \( xy = T_n \) and contained in \( R \). Then \( v_1 = T_n/u_1, u_k = u_l 2^{k-1}, v_k = (T_n/u_1) \left( \frac{1}{2} \right)^{k-1} \). From the fact \((u_l - u_1)(v_1 - v_k) \leq \lambda(R)\), i.e., \( 2^{k-1}(1 - 2^{-k+1})T_n \leq \lambda(R) \), it follows that \( k \leq 2 \log(\lambda(R)/T_n + 2 \text{ (if } k \geq 3, \text{ which implies } M_n(R) \leq 2(k + 4) \leq 4 \log(\lambda(R)/T_n + 1) + 12 \). So, in any case we
have $M_n(R) \leq 4 \log (\lambda(R)/T_n + 1) + 12$. Hence, we have

$$\sup_{R \subseteq D_{i+1}} |W(R)| \leq Y_{n+1} + \left\{ 4 \log \left( \frac{T_{n+1}}{T_n} + 1 \right) + 12 \right\} \cdot \sup_{(x,y) \in D''_{i+1}} |W(x,y)|,$$

$$Y_{n+1} \leq \sup_{R \subseteq D_{i+1}} |W(R)| + \left\{ 4 \log \left( \frac{T_{n+1}}{T_n} + 1 \right) + 12 \right\} \cdot \sup_{(x,y) \in D''_{i+1}} |W(x,y)|.$$

(3.11)

Now, by (2.1) or Theorem 1.12.6 of Csőrgő and Révész (1981), we have that for $n$ large enough,

$$J''_{n+1} = P \left\{ 4 \log \left( \frac{T_{n+1}}{T_n} + 1 \right) + 12 \right\} \cdot \sup_{(x,y) \in D''_{i+1}} \beta_{T_{n+1}} |W(x,y)| > \epsilon \right\}
\leq P \left\{ \sup_{(x,y) \in D''_{i+1}} |W(x,y)| > \epsilon' n^{-p} \beta_{T_{n+1}}^{-1} \right\}
\leq P \left\{ \sup_{(x,y) \in D''_{i+1}} |W(x,y)| > \epsilon' n^{-p} T_{n+1}^{1/2} \left( \log \log T_{n+1} \right)^{-1/2} \right\}
\leq C(1 + \log b_{T_{n+1}} (2T_n)^{-1/2}) \exp \left\{ -\epsilon'' \frac{T_{n+1}}{n^2 p (\log \log T_{n+1}) T_n} \right\}
\leq C \log \log T_{n+1} \exp \left\{ -\epsilon'' \frac{e^{(n+1)p}}{e^{n^2 p T_n}} \right\}
\leq C(\log n) e^{-2 \epsilon} \leq C e^{-n}. \quad (3.12)$$

where $\epsilon'$ is a positive constant dependent only on $\epsilon$ and $p$, whose value can differ from line to line. By the Borel–Cantelli lemma we have

$$\limsup_{n \to \infty} \left\{ 4 \log \left( \frac{T_{n+1}}{T_n} + 1 \right) + 12 \right\} \cdot \sup_{(x,y) \in D''_{i+1}} \beta_{T_{n+1}} |W(x,y)| \leq \epsilon \quad \text{a.s.} \quad (3.13)$$

Let

$$u = \frac{1}{C_2 (1 + \epsilon)^2} \log b_{T_{n+1}} T_{n+1}^{-1/2} \log \log T_{n+1}$$

by Theorem 2.2 we have that for $n$ large enough,

$$J'_{n+1} = P \left\{ \sup_{R \subseteq D_{i+1}} \beta_{T_{n+1}} |W(R)| \leq C_2^{1/2} (1 + \epsilon) \right\}
\geq \exp \left\{ -\frac{C_2 (1 + \epsilon) \log b_{T_{n+1}} T_{n+1}^{-1/2}}{u} \right\}
= \exp \left\{ -\frac{1}{1 + \epsilon} \log \log T_{n+1} \right\} = (n+1)^{-p(1+\epsilon)}. \quad (3.14)$$
Choose \( p \) such that \( 1 < p < 1 + \varepsilon/2 \). Then

\[
\sum_{n=1}^{\infty} J_{n+1} = \infty.
\]  

(3.15)

By (3.11), (3.12) and (3.14), we have that for \( n \) large enough,

\[
J'_{n+1} = P(Y_{n+1} \leq C_2^{1/2}(1 + \varepsilon) + \varepsilon) \\
\geq J_{n+1} + J''_{n+1} \geq (n + 1)^{-p(1+\varepsilon)} - ce^{-n}.
\]  

(3.16)

Hence \( \sum_{n=1}^{\infty} J'_{n+1} = \infty \). Then, by the Borel–Cantelli lemma and the independence of \( \{Y_n\}_{n=1}^{\infty} \), we have

\[
\liminf_{n \to \infty} Y_{n+1} \leq C_2^{1/2}(1 + \varepsilon) + \varepsilon \quad \text{a.s.}
\]  

(3.17)

By (3.11), (3.13) and (3.17) we have

\[
\liminf_{T \to \infty} \sup_{R \subset C_T} \beta_T |W(R)| \\
\leq \liminf_{T \to \infty} \sup_{R \subset C_T} \beta_{T+1} |W(R)| \leq C_2^{1/2}(1 + \varepsilon) + 2\varepsilon \quad \text{a.s.}
\]  

(3.18)

which implies (3.10).

Next, we prove

\[
\liminf_{T \to \infty} \sup_{(x, y) \in D_T} \beta_T |W(x, y)| \geq \frac{1}{(2C_2)^{1/2}} \quad \text{a.s.}
\]  

(3.19)

Let

\[
A_k = \{T : e^k \leq T \leq e^{k+1}\}, \quad k \geq 0, \\
A_{k, j} = \{T : e^{j'} \leq T \leq e^{(j+1)'}, T \in A_k\}, \quad k \geq 0, j \geq 0, \\
b(T_{k, j}) = \inf \{b_T : T \in A_{k, j}\}, \\
D_{k, j} = \{(x, y) : xy \leq e^{j'}, 0 \leq x, y \leq b(T_{k, j})\},
\]  

(3.20)

where \( 0 < p < 1 \) will be defined later. Note that \( A_T \to 0 \) and

\[
\frac{k}{p \log (j + 1)} \leq A_T \quad \text{for} \ T \in A_{k, j}.
\]

Also, for any \( \varepsilon > 0 \), there exists \( j_0 \) such that

\[
A_{k, j} = \emptyset \quad \text{for} \ k \geq \varepsilon \log j, \ j \geq j_0.
\]

Hence,

\[
\liminf_{T \to \infty} \beta_T \sup_{(x, y) \in D_T} |W(x, y)| \\
\geq \liminf_{j \to \infty} \inf_{k \leq \varepsilon \log j} \liminf_{T \to \infty} \sup_{(x, y) \in D_{k, j}} \beta_T |W(x, y)|
\]
By Theorem 2.2 we have

$$J_j = P \left\{ \inf_{k \leq \varepsilon \log j} e^{-j^{p/2}} \left( \frac{\log j^p}{k+1} \right)^{1/2} \left( \frac{\log j^p}{k+1} \right)^{-3/2} \sup_{(x, y) \in D_{k,j}} |W(x, y)| \leq \frac{1}{(2C_2)^{1/2}(1+\varepsilon)} \right\}$$

$$\leq \sum_{k \leq \varepsilon \log j} \exp \left( -2(1+\varepsilon) \frac{\log j^p}{k+1} \log b(T_{k,j}) e^{-j^{p/2}} \right)$$

$$\leq \sum_{k \leq \varepsilon \log j} \exp \left( -\frac{k}{k+1} 2(1+\varepsilon) \log j^p \right) \leq \varepsilon (\log j)^{j^{-p(1+\varepsilon)}}. \quad (3.22)$$

Choose $0 < p < 1$ such that $p(1+\varepsilon) > 1$. Then

$$\sum_{j=j_0}^{\infty} J_j < \infty, \quad (3.23)$$

which implies

$$\lim_{j \to \infty} \inf_{k \leq \varepsilon \log j} e^{-j^{p/2}} \left( \frac{\log j^p}{k+1} \right)^{1/2} \left( \frac{\log j^p}{k+1} \right)^{-3/2} \sup_{(x, y) \in D_{k,j}} |W(x, y)| \geq \frac{1}{(2C_2)^{1/2}(1+\varepsilon)} \quad \text{a.s.} \quad (3.24)$$

Hence, we have proved (3.19). By (3.10), (3.19) and (2.7) we have proved (1.9) and (1.10).

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References


