The strong approximation for the Kesten–Spitzer random walk

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Abstract

In this note we study the strong approximation for a one-dimensional simple random walk in a general i.i.d. scenery when the scenery has only finite lower moments. Namely, an approximation is obtained when the scenery has only finite 

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Let \(\sigma = \{\sigma_x\}_{x \in \mathbb{Z}}\) (sometimes also written \(\{\sigma(x)\}_{x \in \mathbb{Z}}\)) denote a sequence of independent and identically distributed real-valued random variables (i.i.d. r.v.’s) such that

\[
E(\sigma_0) = 0 \quad \text{and} \quad E(\sigma_0^2) = 1.
\]

(1)

Any realization of the sequence \(\{\sigma_x\}_{x \in \mathbb{Z}}\) is called a “scenery”. Let \(S = \{S_k\}_{k \in \mathbb{N}_0}\) be a simple symmetric random walk on \(\mathbb{Z}\) starting at \(S_0 = 0\), independent of \(\sigma\). The process \(K = \{K(n)\}_{n \in \mathbb{N}_0}\), defined by

\[
K(n) = \sum_{k=0}^{n} \sigma(S_k), \quad n \in \mathbb{N}_0
\]

(2)

is usually referred to as the Kesten–Spitzer random walk in random scenery (cf. Kesten and Spitzer, 1979, R.0@+esz, 1990).

There is a continuous analogue for \(K\) introduced and analyzed by Kesten and Spitzer (1979). To describe this, let \(B = \{B(t); t \geq 0\}\) and \(W = \{W(x); x \in \mathbb{R}\}\) be independent real-valued standard Brownian motions with \(B(0) = W(0) = 0\). Let \(\{L(t, x); t \geq 0, x \in \mathbb{R}\}\) denote the jointly continuous version of the local time

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process of $B$. Now, define the process $G$, which is called Brownian motion in Brownian scenery, by

$$G(t) = \int_{\mathbb{R}} L(t, x) \, dW(x), \quad t > 0. \quad (3)$$

It is proved by Kesten and Spitzer (1979) that

$$\{n^{-3/4}K([nt]); 0 \leq t \leq 1\} \overset{\text{in law}}{\to} \{G(t); 0 \leq t \leq 1\}, \quad (4)$$

where $\overset{\text{in law}}{\to}$ stands for weak convergence in $D[0,1]$. Khoshnevisan and Lewis (1998) studied a strong vision of (4) in the special case where the random scenery $\sigma$ is Gaussian. More precisely, they proved that, if $\sigma_0$ is a Gaussian $N(0,1)$ variables, then (on a suitably enlarged probability space) one can construct a pair of $K$ and $G$ such that, for any $\varepsilon > 0$,

$$\max_{0 \leq m \leq n} |K(m) - G(m)| \overset{\text{a.s.}}{=} o(n^{1/2+\varepsilon}), \quad n \to \infty. \quad (5)$$

Csáki et al. (1999) extended (5) to a more general random scenery. They proved the following theorem.

**Theorem A.** Let $\{\sigma_x\}_{x \in \mathbb{Z}}$ be a random scenery satisfying (1) and $E|\sigma_0|^p < \infty$ for all $p > 0$. Then on a suitably enlarged probability space there exists a coupling of $K$ and $G$, such that, for any $\varepsilon > 0$,

$$\max_{0 \leq m \leq n} |K(m) - G(m)| \overset{\text{a.s.}}{=} o(n^{5/8+\varepsilon}), \quad n \to \infty. \quad (6)$$

An application of Theorem A yields the law of iterated logarithm:

$$\limsup_{n \to \infty} \frac{K(n)}{(n \log \log n)^{3/4}} = \frac{2^{5/4}}{3} \quad \text{a.s.} \quad (7)$$

if $\{\sigma_x\}_{x \in \mathbb{Z}}$ satisfies (1) and $E|\sigma_0|^p < \infty$, for all $p > 0$.

The aim of this paper is to show (6) and (7) under the conditions of finite lower absolute moments. Our result reads as follows:

**Theorem 1.** Let $\{\sigma_x\}_{x \in \mathbb{Z}}$ be a random scenery satisfying (1) and $E|\sigma_0|^p < \infty$, where $2 \leq p \leq 4$. Then on a suitably enlarged probability space there exists a coupling of $K$ and $G$, such that, for any $\varepsilon > 0$,

$$\max_{0 \leq m \leq n} |K(m) - G(m)| \overset{\text{a.s.}}{=} o(n^{1/2+1/(2p)+\varepsilon}), \quad n \to \infty. \quad (8)$$

An application of Theorem 1 yields that (7) holds if $E|\sigma_0|^{2+\delta} < \infty$ for some $\delta > 0$.

Denote the number of walker’s visits to $x$ until time $n$ by

$$\zeta(n, x) = \sum_{k=0}^{n} I\{S_k = x\}, \quad n \in \mathbb{N}_0, \quad x \in \mathbb{Z}, \quad (9)$$

which is often referred to as the local time of the random walk $S$. Then (2) can be rewritten as

$$K(n) = \sum_{x \in \mathbb{Z}} \sigma_x \zeta(n, x), \quad n \in \mathbb{N}_0. \quad (10)$$

We shall prove Theorem 1 in two steps.
Step 1. Let \( \sigma = \{ \sigma_x \}_{x \in \mathbb{Z}} \) satisfy (1). Then there is a coupling of \( \sigma, S \) and \( B \) such that \( \sigma \) is independent of \((S, B)\) and for any \( \varepsilon > 0 \),
\[
\sum_{x \in \mathbb{Z}} \sigma_x (\xi(n, x) - L(n, x)) \overset{a.s.}{=} o(n^{1/2+\varepsilon}), \quad n \to \infty.
\] (11)

Proof. According to a theorem by Revuz (1981) and a result by Khoshnevisan and Lewis (1998), one can construct a random walk \( S \) from the Brownian motion \( B \) such that, for every \( \varepsilon > 0 \),
\[
\sup_{x \in \mathbb{Z}} |\xi(n, x) - L(n, x)| \overset{a.s.}{=} O(n^{1/4+\varepsilon}), \quad n \to \infty;
\] (12)
and for any \( q \geq 1 \),
\[
\sup_{x \in \mathbb{Z}} \mathbb{E}(|\xi(n, x) - L(n, x)|^q) \leq c_q n^{q/4}, \quad n \in \mathbb{N},
\] (13)
where \( c_q \) is a constant depending only on \( q \).

Definition
\[
I(N, n) = \sum_{x=-N}^{N} \sigma_x (\xi(n, x) - L(n, x)), \quad N, n \in \mathbb{N}.
\] (14)

Following the argument of Csaki et al. (1999), in order to prove (11) we only need to show that, for every \( \varepsilon > 0 \),
\[
I([n^{1/2+\varepsilon}], n) \overset{a.s.}{=} O(n^{1/2+\varepsilon}).
\] (15)

For each \( x \in \mathbb{Z} \), let \( X_x = \sigma_x I\{|\sigma_x| > |x|^{1/2}\}, Y_x = \sigma_x I\{|\sigma_x| \leq |x|^{1/2}\} - \mathbb{E}\sigma_x I\{|\sigma_x| \leq |x|^{1/2}\} \) and \( P_x = \sigma_x I\{|\sigma_x| > |x|^{1/2}\} - \mathbb{E}\sigma_x I\{|\sigma_x| > |x|^{1/2}\}. \)

Definition
\[
I_1(N, n) = \sum_{x=-N}^{N} X_x (\xi(n, x) - L(n, x)),
\]
\[
I_2(N, n) = \sum_{x=-N}^{N} Y_x (\xi(n, x) - L(n, x)),
\]
\[
I_3(N, n) = \sum_{x=-N}^{N} (\xi(n, x) - L(n, x)) \mathbb{E}\sigma_x I\{|\sigma_x| \leq |x|^{1/2}\},
\] (16)
for all \( N, n \in \mathbb{N} \). Note that
\[
\sum_{x=-\infty}^{\infty} \mathbb{P}(|\sigma_x| > |x|^{1/2}) \leq c \mathbb{E}\sigma_0^2 < \infty.
\]

It follows that
\[
\mathbb{P}(X_x \neq 0, i.o.) = 0.
\]

On the other hand, by (12), for each fixed \( x \in \mathbb{Z} \),
\[
X_x (\xi(n, x) - L(n, x)) \overset{a.s.}{=} o(n^{1/4+\varepsilon}), \quad n \to \infty,
\]
and therefore
\[ I_2([n^{1/2+\varepsilon}], n) \overset{a.s.}{=} o(n^{1/4+\varepsilon}), \quad n \to \infty. \quad (17) \]

For \( I_3 \), by (12) we have
\[
|I_3(N,n)| = \left| \sum_{x=-N}^{N} (\zeta(n,x) - L(n,x))E_\sigma_I \{ |\sigma_0| \geq |x|^{1/2} \} \right|
\leq \sum_{x=-N}^{N} |\zeta(n,x) - L(n,x)| E_\sigma_I \{ |\sigma_0| \geq |x|^{1/2} \}
\leq \sup_x |\zeta(n,x) - L(n,x)| \left( E_\sigma_I + \sum_{x=-N, x \neq 0}^{N} |x|^{-1/2} E_\sigma_0^2 \right)
\overset{a.s.}{=} O(N^{1/2})o(n^{1/4+\varepsilon/2}) \quad \text{as} \ n \to \infty, \ N \to \infty.
\]

Hence
\[
I_2([n^{1/2+\varepsilon}], n) \overset{a.s.}{=} O(n^{1/4+\varepsilon/2})o(n^{1/4+\varepsilon/2}) = o(n^{1/2+\varepsilon}), \quad n \to \infty. \quad (18)
\]

At last, by (16)–(18), it suffices to show the summability of \( P(I_2([n^{1/2+\varepsilon}], n) > n^{1/2+\varepsilon}) \). For Oxed \((S, B)\), by the Rosenthal inequality (cf., Petrov, 1995) we have for any \( q \geq 2 \),
\[
E[|I_2(N,n)|^q |(S,B)] \leq c_q \left\{ \left( \sum_{x=-N}^{N} (\zeta(n,x) - L(n,x))^2 E_\sigma_x^2 \right)^{q/2} + \sum_{x=-N}^{N} |\zeta(n,x) - L(n,x)|^q E_\sigma_x^q \right\}
\leq c_q \left\{ \left( \sum_{x=-N}^{N} (\zeta(n,x) - L(n,x))^2 E_\sigma_x^2 \right)^{q/2} + \sum_{x=-N}^{N} |\zeta(n,x) - L(n,x)|^q N^{(q-2)/2} E_\sigma_x^2 \right\}
\leq c_q N^{q/2-1} \sum_{x=-N}^{N} |\zeta(n,x) - L(n,x)|^q.
\]

By (13) and (19), we have
\[
E[|I_2(N,n)|^q] \leq c_q N^{q/2-1} \sum_{x=-N}^{N} E|\zeta(n,x) - L(n,x)|^q \leq c_q N^{q/2-n^{q/4}}.
\]

Hence
\[
P(I_2([n^{1/2+\varepsilon}], n) > n^{1/2+\varepsilon}) \leq c_q n^{-(1/2+\varepsilon)q/2} (1/2+\varepsilon)q/2 n^{q/4} = c_q n^{-(q/2)\varepsilon}, \]
which is summable for any \( q > 2/\varepsilon \). This concludes the proof of Step 1.

Step 2. Let \( \{\sigma_i\}_{i \in \mathbb{Z}} \) satisfy (1) and \( E|\sigma_0|^p < \infty \), where \( 2 \leq p \leq 4 \). Then there is a coupling of \( \sigma, S \) and \( W \) such that \( \sigma, W \) is independent of \((S,B)\) and for any \( \varepsilon > 0 \),
\[
\sum_{x \in \mathbb{Z}} \sigma_x L(n,x) - \int_{\mathbb{R}} L(n,x) dW(x) \overset{a.s.}{=} o(n^{1/2+1/(2p)+\varepsilon}), \quad n \to \infty. \quad (20)
\]
Proof. Following the line of the proof of Proposition 2.2 of Csáki et al. (1999), it suffices to show that, for any 0 < \( b > \frac{1}{2} \) and any \( \epsilon > 0 \),

\[
J([n^b], n) \overset{\text{a.s.}}{=} o(n^{1/4+b(1/2+1/p)+\epsilon}), \quad n \to \infty,
\]  

(21) 

where

\[
J(N, n) = \mathbb{W}_n \left( \int_0^{\rho(N)} A_n^2(x) \, dx \right), \quad N, n \in \mathbb{N},
\]

(22) 

\[
\{\mathbb{W}_n(t); \ t \geq 1\} \quad \text{is a Brownian motion for each } n \in \mathbb{N},
\]

\[
A_n(x) = L(n, x) - L(n, j) \quad \text{if } x \in (\rho_{j-1}, \rho_j),
\]

\[
\rho(n) = \rho_n = \sum_{i=1}^{n} T_i, \quad n \in \mathbb{N}
\]

and \( \{T_i\}_{i=1}^{\infty} \) is a sequence of i.i.d. non-negative random variables with \( ET_1 = Var(\sigma_0) = 1 \) and \( ET_1^{p/2} \leq c_p E|\sigma_0|^p < \infty \).

By (2.21) and (2.22) of Csáki et al. (1999) we have for any \( a \geq 0 \) that

\[
\max_{1 \leq j \leq n} \max_{0 \leq s \leq n} |\mathbb{W}_s| \overset{\text{a.s.}}{=} O(n^{a/2}(\log n)^{1/2}), \quad n \to \infty,
\]

(23) 

\[
\sup_{|x-y| \leq n^a} |L(t, x) - L(t, y)| \overset{\text{a.s.}}{=} o(1^{1/4+a/2+\epsilon}), \quad t \to \infty, \quad \forall \epsilon > 0.
\]

(24) 

On the other hand, since the sequence \( \{T_i\}_{i \in \mathbb{N}} \) is i.i.d. with \( ET_i = 1 \), \( E|T_i|^{p/2} < \infty \) and \( 1 \leq p/2 \leq 2 \), by the classical LLN or LIL, we obtain

\[
\rho(n) \overset{\text{a.s.}}{=} \begin{cases} 
  n + o(n^{2/p}) & \text{if } 2 \leq p < 4, \\
  n + O((\log n)^{1/2}) & \text{if } p = 4,
\end{cases} \quad n \to \infty.
\]

(25) 

Now \( 0 < b > \frac{1}{2} \) and \( \epsilon > 0 \). For any \( 0 < x < \rho([n^b]) \), there exists \( 1 \leq j \leq [n^b] \) such that \( x \in (\rho_{j-1}, \rho_j) \). Since by (25), \( |x-j| \leq n^{2b/p+\epsilon} \) and \( x \leq 2n^b \), we can apply (24) to conclude that

\[
\sup_{0 \leq x \leq \rho([n^b])} |A_n(x)| \overset{\text{a.s.}}{=} o(n^{1/4+b/p+\epsilon}), \quad n \to \infty.
\]

Therefore, by (25),

\[
\int_0^{\rho([n^b])} A_n^2(x) \, dx \leq \rho([n^b]) \sup_{0 \leq x \leq \rho([n^b])} A_n^2(x) \overset{\text{a.s.}}{=} o(n^{1/4+b+2b/p+2\epsilon}).
\]

(26) 

By (23) and (26), we have

\[
J([n^b], n) \overset{\text{a.s.}}{=} O(n^{1/4+b(1/2+1/p)+\epsilon}(\log n)^{1/2}) = o(n^{1/4+b(1/2+1/p)+2\epsilon}).
\]

The proof of Step 2 is now completed.

Putting together (10), (11) and (20), we get (8).

References


