STRONG APPROXIMATION THEOREMS FOR GEOMETRICALLY WEIGHTED RANDOM SERIES AND THEIR APPLICATIONS

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constants whose values are uninteresting and may vary from line to line. The expression \(a_n \sim b_n\) means \(a_n/b_n \to 1\) \((n \to \infty)\); \(a_n \approx b_n\) means that there exist \(C_1, C_2 > 0\) such that \(C_1 \leq a_n/b_n \leq C_2\) for \(n\) large enough.

The following theorem gives a general result on strong approximations for random geometric series.

**Theorem 1.1.** Let \(H(x)\) \((x \geq 0)\) be a monotone nondecreasing positive function with \(H(x) \to \infty\) \((x \to \infty)\) and \(H(2n) \leq C H(n),\) \((n \geq 0)\) and let \(\{\xi_n; n \geq 0\}, \{\eta_n; n \geq 0\}\) be two sequences of random variables with \(E|\xi_n|^p \leq C n^q,\) \(E|\eta_n|^p \leq C n^q,\) \((n \geq 0)\) for some \(p, q > 0.\) If

\[
\sum_{k=0}^{n} \xi_k - \sum_{k=0}^{n} \eta_k = O(H(n)) \text{ (or } o(H(n))\text{)} \quad \text{a.s. } (n \to \infty),
\]

then

\[
\sum_{n=0}^{\infty} \beta^n \xi_n - \sum_{n=0}^{\infty} \beta^n \eta_n = O\left(H\left(\frac{1}{1-\beta^2}\right)\right) \left(\text{or } O\left(H\left(\frac{1}{1-\beta^2}\right)\right)\right) \quad \text{a.s. } (\beta \not\sim 1).
\]

The following corollary comes from Theorem 1.1 immediately.

**Corollary 1.1.** Suppose \(\{X_n; n \geq 0\}\) is a sequence of i.i.d. random variables, or more generally, a stationary ergodic martingale difference with \(EX_0 = 0, EX_0^2 = \sigma^2, 0 < \sigma < \infty.\) Then there exists a sequence of i.i.d. normal random variables \(\{Y_n; n \geq 0\}\) with \(Y_n \overset{d}{=} N(0, \sigma^2)\) such that

\[
\lim_{\beta \not\sim 1} \frac{\sqrt{1-\beta^2}}{2 \log(1/(1-\beta^2))} \left|\sum_{n=0}^{\infty} \beta^n X_n - \sum_{n=0}^{\infty} \beta^n Y_n\right| = 0 \quad \text{a.s.}
\]

By Theorem 1.1 and Theorems 1.1, 1.2 of Shao [11], we have the following corollary.

**Corollary 1.2.** Let \(\{X_n; n \geq 0\}\) be a stationary stochastic sequence with \(EX_0 = 0, EX_0^2 < \infty\) and \(B_n = E(\sum_{k=0}^{n} X_k)^2 \to \infty\) \((n \to \infty)\) satisfying one of the following conditions:

(i) \(\{X_n; n \geq 0\}\) is \(\rho\)-mixing. The mixing coefficients satisfy

\[
\rho(n) \leq \log^{-r} n \quad \text{for some } r > 1;
\]

(ii) \(\{X_n; n \geq 0\}\) is \(\phi\)-mixing. The mixing coefficients satisfy

\[
\sum_{n=0}^{\infty} \phi^{1/2}(2^n) < \infty.
\]

Then \(\lim_{n \to \infty}(B_n/n) = \sigma^2\) for some \(0 < \sigma < \infty,\) and the conclusion of Corollary 1.1. holds true.
When \( \{X_n; n \geq 0\} \) is a sequence of i.i.d. random variables with higher than second moments by Theorem 1.1, Theorem 1 of Zhang [14] and the results of Komlos, Major and Tusnady [6, 7], we have the following conclusion.

**Corollary 1.3.** Suppose that \( \{X_n; n \geq 0\} \) is a sequence of i.i.d. random variables with \( EX_0 = 0, EX_0^2 = 1 \). Let \( H(x) \) \( (x \geq 0) \) be a nondecreasing positive continuous function such that for some \( \gamma > 0, x_0 > 0, x^{-2-\gamma}H(x) \) \( (x \geq x_0) \) is nondecreasing and \( x^{-1} \log H(x) \) \( (x \geq x_0) \) is nonincreasing.

(a) If \( EH(|X_0|) < \infty \), then there exists a sequence of i.i.d. standard normal random variables \( \{Y_n; n \geq 0\} \) such that

\[
\sum_{n=0}^{\infty} \beta^n X_n - \sum_{n=0}^{\infty} \beta^n Y_n = O\left( \text{inv } H\left(\frac{1}{1-\beta^2}\right) \right) \quad \text{a.s. (} \beta \nearrow 1). \]

(b) If \( x^{-1} \log H(x) \to 0 \) \( (x \to \infty) \) and \( EH(t|X_0|) < \infty \) for any \( t > 0 \), then there exists a sequence of i.i.d. standard normal random variables \( \{Y_n; n \geq 0\} \) such that

\[
\sum_{n=0}^{\infty} \beta^n X_n - \sum_{n=0}^{\infty} \beta^n Y_n = o\left( \text{inv } H\left(\frac{1}{1-\beta^2}\right) \right) \quad \text{a.s. (} \beta \nearrow 1). \]

The following theorem deals with the sequence of independent but not necessarily identically distributed random variables.

**Theorem 1.2.** Let \( \{X_n; n \geq 0\} \) be a sequence of independent random variables with \( EX_n = 0 \) and \( EX_n^2 < \infty \) \( (n \geq 0) \). Set \( B_n = \sum_{k=0}^{n} EX_k^2 \) and \( \tau(\beta) = \sum_{n=0}^{\infty} \beta^{2n} EX_n^2 \). Suppose \( B_n \to \infty \) \( (n \to \infty) \), \( \limsup_{n \to \infty} B_{2n}/B_n < \infty \) and for some \( p \geq 2 \),

\[
(1.5) \quad \sum_{n=0}^{\infty} \frac{E|X_n|^p I\{|X_n| > \varepsilon (B_n/\log \log B_n)^{1/2}\}}{(B_n \log \log B_n)^{p/2}} < \infty \quad \text{for any } \varepsilon > 0. \]

Then there exists a sequence of independent normal random variables \( \{Y_n; n \geq 0\} \) with \( Y_n \Rightarrow N(0, EX_n^2) \) such that

\[
(1.6) \quad \lim_{\beta \nearrow 1} \frac{\sum_{n=0}^{\infty} \beta^n X_n - \sum_{n=0}^{\infty} \beta^n Y_n}{(2\tau(\beta) \log \log \tau(\beta))^{1/2}} = 0 \quad \text{a.s.} \]

If (1.5) is replaced by

\[
(1.5') \quad \sum_{n=0}^{\infty} \frac{E|X_n|^p I\{|X_n| > \varepsilon (B_n/\log \log B_n)^{1/2}\}}{(B_n \log \log B_n)^{p/2}} < \infty \quad \text{for some } \varepsilon > 0, \]

then there exists a sequence of independent normal random variables \( \{Y_n; n \geq 0\} \) with \( Y_n \Rightarrow N(0, EX_n^2) \) such that

\[
(1.6') \quad \limsup_{\beta \nearrow 1} \frac{\sum_{n=0}^{\infty} \beta^n X_n - \sum_{n=0}^{\infty} \beta^n Y_n}{(2\tau(\beta) \log \log \tau(\beta))^{1/2}} \leq \Gamma \varepsilon \quad \text{a.s.}, \]

where \( \Gamma \) is a numerical constant.
PROOF OF THEOREM 1.1. Without loss of generality, we can assume $0 < p \leq 1$. Note that for any $0 < \beta < 1$,

$$E\left(\sum_{n=0}^{\infty} \beta^n |\xi_n|\right)^p \leq \sum_{n=0}^{\infty} \beta^{np} E|\xi_n|^p \leq C \sum_{n=0}^{\infty} \beta^{np} n^q < \infty.$$ 

We know that $\sum_{n=0}^{\infty} \beta^n \xi_n$ is a.s. absolutely convergent for any $0 < \beta < 1$. Similarly, so is $\sum_{n=0}^{\infty} \beta^n n^{-1}$. Let $N(\beta) = [1/(1 - \beta^2)]$. (We keep this in mind in the remainder of this paper.) Note that $H(2n) \leq CH(n)$ ($n \geq 0$) implies $H(kn) \leq C kQ$ ($n > 0$, $k > 1$) for some $C$, $Q > 0$; we have proved
Then, note that for $\beta$ near enough to $1$,

$$(1 - \beta^r) A^{-1}_{N(\beta)} \sum_{n=0}^{\infty} \beta^{nr} |S_n|$$

$$= (1 - \beta^r) A^{-1}_{N(\beta)} \sum_{n=0}^{N(\beta)} \beta^{nr} |S_n| + (1 - \beta^r) A^{-1}_{N(\beta)} \sum_{n=N(\beta)+1}^{\infty} \beta^{nr} |S_n|$$

$$\leq (1 - \beta^r)(N(\beta) + 1)$$

$$+ (1 - \beta^r)N(\beta) \frac{1}{N(\beta)} \sum_{n=N(\beta)+1}^{\infty} \exp\left(-\frac{nr}{2N(\beta)}\right) \frac{A_n}{A_{N(\beta)}} |S_n|$$

$$\leq (1 - \beta^r)(N(\beta) + 1)$$

$$+ (1 - \beta^r)N(\beta) \frac{1}{N(\beta)} \sum_{n=N(\beta)+1}^{\infty} \exp\left(-\frac{nr}{2N(\beta)}\right) C_0 \left(\frac{n}{N(\beta)}\right)^Q$$

$$\rightarrow \frac{r}{2} + \frac{rC_0}{2} \int_1^{\infty} \exp\left(-\frac{rx}{2}\right) x^Q dx, \quad \text{as} \quad \beta \not\to 1.$$ 

We have proved (1.7). The proof of (1.8) is similar.

**Proof of Theorem 1.2.** If $\limsup_{n \to \infty} B_{2n}/B_n < \infty$ then $B_{2n}/B_n \leq C$ for some $C > 0$. Then there exist $C_0, Q > 0$ such that $B_{kn}/B_n \leq C_0 k^Q$, $B_n \leq C_0 n^Q$ ($n \geq 0$, $k \geq 1$). Hence $EX_n^2 \leq C_0 n^Q$ ($n \geq 0$). By Theorem 1.1, we need only to show that

(1.10) $\tau(\beta) \approx B_{N(\beta)}$, $\beta \not\to 1$,

that there exists a sequence of independent normal random variables $\{Y_n; n \geq 0\}$ with $Y_n \equiv N(0, EX_n^2)$ such that

(1.11) $\left| \sum_{k=0}^{n} X_k - \sum_{k=0}^{n} Y_k \right| = o\left((B_n \log \log B_n)^{1/2}\right)$ a.s. ($n \to \infty$)

whenever (1.5) holds, and that there exists a sequence of independent normal random variables $\{Y_n; n \geq 0\}$ with $Y_n \equiv N(0, EX_n^2)$ such that for some numerical constant $\Gamma$,

(1.11') $\limsup_{n \to \infty} \left| \sum_{k=0}^{n} X_k - \sum_{k=0}^{n} Y_k \right| \leq \Gamma e^{1/2}$ a.s.

whenever (1.5) is replaced by (1.5').

We prove (1.10) first. First, $\tau(\beta) \geq \sum_{n=0}^{N(\beta)} \beta^{2n} EX_n^2 \geq \beta^{2N(\beta)} B_{N(\beta)}$ implies

(1.12) $\liminf_{\beta \not\to 1} \tau(\beta)/B_{N(\beta)} \geq e^{-1}$. 
On the other hand, it follows from Lemma 1.1 that
\begin{equation}
\limsup_{\beta \to 1} \frac{\tau(\beta)}{B_{N(\beta)}} \leq 1 + C_0 \int_1^\infty \exp(-x)x^Q \, dx < \infty.
\end{equation}

Hence we have proved (1.10).

Now we will prove (1.11). If (1.5) holds, then there exists a sequence of nonincreasing positive numbers \( \{\varepsilon_n; n \geq 0\} \) satisfying \( 1 > \varepsilon_n \to 0, \)
\( \varepsilon_n^{n^{-2}} \log \log B_n \not\to \infty, \varepsilon_n (B_n / \log \log B_n)^{1/2} \not\to \infty (n \to \infty) \) such that
\begin{equation}
\sum_{n=0}^\infty \frac{\mathbb{E}[X_n|^I |X_n| > \varepsilon_n (B_n / \log \log B_n)^{1/2}]}{(B_n \log \log B_n)^{p/2}} < \infty.
\end{equation}

Let
\begin{equation}
\xi_n = X_n \left[ |X_n| \leq \varepsilon_n \left( \frac{B_n}{\log \log B_n} \right)^{1/2} \right] \\
- \mathbb{E}X_n \left[ |X_n| \leq \varepsilon_n \left( \frac{B_n}{\log \log B_n} \right)^{1/2} \right],
\end{equation}
\begin{equation}
\tilde{\xi}_n = X_n - \xi_n.
\end{equation}
Then \( \xi_n I(|\xi_n| \leq (B_n / \log \log B_n)^{1/2}) = \xi_n \) and
\begin{equation}
\sum_{n=0}^\infty \mathbb{P}\{ |\xi_n| > \varepsilon (B_n / \log \log B_n)^{1/2} \} < \infty
\end{equation}
for any \( \varepsilon > 0 \). By Theorem 1.1 of Shao [12] (see also [10]), there exists a sequence of i.i.d. normal random variables \( \{\eta_n; n \geq 0\} \) with \( \eta_n \sim N(0, 1) \) such that
\begin{equation}
\sum_{i=0}^n \xi_i - \sum_{i=0}^n \eta_i (\text{Var} \xi_i)^{1/2}
= o \left( \left( \frac{B_n}{\log \log B_n} \right)^{1/2} \log \sum_{i=0}^n \frac{(\log \log B_i)E\xi_i^2}{B_i} \right) \text{ a.s. } (n \to \infty),
\end{equation}
which together with
\begin{equation}
\sum_{i=0}^n \frac{(\log \log B_i)E\xi_i^2}{B_i} \leq C \log B_n \log \log B_n
\end{equation}
implies that
\begin{equation}
\sum_{i=0}^n \tilde{\xi}_i - \sum_{i=0}^n \eta_i (\text{Var} \tilde{\xi}_i)^{1/2} = o((B_n \log \log B_n)^{1/2}) \text{ a.s. } (n \to \infty).
\end{equation}
Set \( Y_n = (\text{Var} \tilde{X}_n)^{1/2} \eta_n \) \( (n \geq 0) \). According to (1.15) and (1.16), in order to prove (1.11) we need only to show that
\begin{equation}
\sum_{i=0}^n \tilde{\xi}_i = o((B_n \log \log B_n)^{1/2}) \text{ a.s. } (n \to \infty)
\end{equation}
and

\[(1.18)\quad \sum_{i=0}^{n} (\text{Var} X_i)^{1/2} - (\text{Var} \xi_i)^{1/2}) \eta_i = o((B_n \log \log B_n)^{1/2}) \quad \text{a.s.} \quad (n \to \infty).\]

First, we apply Proposition 2.2. of [2] to prove (1.17). Let \(S_n = \sum_{k=0}^{n} \xi_k\), \(a_n = (2B_n \log \log B_n)^{1/2}\). It is easy to see that hypothesis (2.9) of [2] is fulfilled, since

\[
P\left(\left|\frac{S_n}{a_n}\right| \geq \epsilon \right) \leq \frac{1}{\epsilon^2} \frac{ES_n^2}{a_n^2} \leq \frac{1}{\epsilon^2} \frac{1}{2 \log \log B_n}.
\]

From (1.14), it follows that hypothesis (2.3) of [2] is fulfilled. Now, let \(\{n_k\}\) satisfy (2.2) of [2]. That is,

\[
\lambda a_{n_k} \leq a_{n_{k+1}} \leq \lambda^3 a_{n_{k+1}}
\]

for some \(\lambda > 1\). Let \(I(k) = \{n_k + 1, \ldots, n_{k+1}\}\) and

\[
N_1 = \left\{k \in N; \sum_{j \in I(k)} \frac{EX_j^2 I\{|X_j| > \epsilon_j(B_j/\log \log B_j)^{1/2}\}}{a_j^p} \leq (2 \log \log B_{n_{k+1}})^{-p}\right\}.
\]

For each \(k \in N_1\), we have

\[
\frac{1}{B_{n_{k+1}}} \sum_{j \in I(k)} \frac{EX_j^2 I\{|X_j| > \epsilon_j(B_j/\log \log B_j)^{1/2}\}}{a_j^p} \leq \sum_{j \in I(k)} \frac{2^{p/2} B_j}{B_{n_{k+1}}} \frac{(\log \log B_j)^p}{\epsilon_j^{-p/2} \log \log B_j} \frac{EX_j^2 I\{|X_j| > \epsilon_j(B_j/\log \log B_j)^{1/2}\}}{a_j^p} \leq \frac{2^{p/2}(\log \log B_{n_{k+1}})^p}{\epsilon_{n_k}^{-p/2} \log \log B_{n_k}} \sum_{j \in I(k)} \frac{EX_j^2 I\{|X_j| > \epsilon_j(B_j/\log \log B_j)^{1/2}\}}{a_j^p} \leq \frac{2^{-p/2}}{\epsilon_{n_k}^{-p/2} \log \log B_{n_k}} \to 0,
\]

which together with (1.14) implies (see [2])

\[
\sum_k \exp \left\{-\frac{\delta a_{n_{k+1}}^2}{\sum_{j \in I(k)} EX_j^2 I\{|X_j| > \epsilon_j(B_j/\log \log B_j)^{1/2}\}}\right\} \leq \sum_k \exp \left\{-\frac{\delta a_{n_{k+1}}^2}{\sum_{j \in I(k)} EX_j^2 I\{|X_j| > \epsilon_j(B_j/\log \log B_j)^{1/2}\}}\right\} < \infty
\]

for every \(\delta > 0\).
It follows that hypothesis (2.8) of [2] is fulfilled. Thus, by Proposition 2.2 of [2] we have proved (1.17).

Now, note that
\[
(c_2 \cdot \lambda_{1/2} \cdot \lambda_{1/2})^2
\]
\[
\limsup_{n \to \infty} B_{2n}/B_n < \infty. \text{ Set }
\]
\[
\tau(\beta) = \sum_{n=0}^{\infty} \beta^{2n} EY_n^2, \quad 0 < \beta < 1,
\]
\[
\tilde{\xi}(\beta) = \frac{\sum_{n=0}^{\infty} \beta^n Y_n}{(2\tau(\beta) \log \log \tau(\beta))^{1/2}}, \quad 0 < \beta < 1.
\]

Then:

(i) \[\mathcal{C}(\{\tilde{\xi}(\beta)\}) = [-1, 1] \quad \text{a.s.};\]

(ii) \[\lim_{\beta \to 1} d(\tilde{\xi}(\beta), [-1, 1]) = 0 \quad \text{a.s.},\]

where \(\mathcal{C}(\{\tilde{\xi}(\beta)\})\) denotes the cluster set (set of all limit points) of \(\tilde{\xi}(\beta)\) as \(\beta\) tends to one and \(d(x, A) = \inf_{y \in A} |x - y|\).

From Corollary 1.1, Corollary 1.2 and Proposition 2.1 the following theorem follows immediately.

**Theorem 2.1.** Let \(\{X_n; n \geq 0\}\) satisfy the conditions in Corollary 1.1 or Corollary 1.2. Set

\[
\tilde{\xi}(\beta) = \frac{\sqrt{1 - \beta^2}}{\sqrt{2 \log \log (1/(1 - \beta^2))}} \sum_{n=0}^{\infty} \beta^n X_n, \quad 0 < \beta < 1.
\]

Then

\[\mathcal{C}(\{\tilde{\xi}(\beta)\}) = [-\sigma, \sigma] \quad \text{a.s.},\]

\[\lim_{\beta \to 1} d(\tilde{\xi}(\beta), [-\sigma, \sigma]) = 0 \quad \text{a.s.}\]

By Theorem 1.2 and Proposition 2.1, we have the following theorem.

**Theorem 2.2.** Let \(\{X_n; n \geq 0\}\) be a sequence of independent random variables with \(EX_n = 0\) and \(EX_n^2 < \infty (n \geq 0)\). Set \(B_n = \sum_{k=0}^{n} EX_k^2\) and \(\tau(\beta) = \sum_{n=0}^{\infty} \beta^{2n} EX_n^2\). Suppose \(B_n \to \infty (n \to \infty)\), \(\limsup_{n \to \infty} B_{2n}/B_n < \infty\) and for each \(\varepsilon > 0\) there exists \(p \geq 2\) such that

\[\sum_{n=0}^{\infty} E[|X_n|^p I\{|X_n| > \varepsilon(B_n/\log \log B_n)^{1/2}\}] (B_n \log \log B_n)^{p/2} < \infty.\]

Let

\[
\tilde{\xi}(\beta) = \frac{\sum_{n=0}^{\infty} \beta^n X_n}{(2\tau(\beta) \log \log \tau(\beta))^{1/2}}, \quad 0 < \beta < 1.
\]

Then (i) and (ii) in Proposition 2.1 hold true.
In particular, we have the following Kolmogorov type law of the iterated logarithm.

**Corollary 2.1.** Let \( \{X_n; \ n \geq 0\} \) be a sequence of independent random variables with \( EX_n = 0 \) and \( EX_n^2 < \infty \) \((n \geq 0)\). Suppose \( B_n = \sum_{i=0}^n EX_i^2 \to \infty \) \((n \to \infty)\) and \( \limsup_{n \to \infty} B_{2n}/B_n < \infty \). Let \( \tau(\beta) \) and \( \xi(\beta) \) be defined as in Theorem 2.2. If there exists a sequence of positive numbers \( \{k_n; \ n \geq 0\} \) with \( k_n \to 0 \) \((n \to \infty)\) such that \( |X_n| \leq k_n (B_n/\log \log B_n)^{1/2} \), then (i) and (ii) in Proposition 2.1 hold true.

For the sequence of i.i.d. random variables with possible infinite variance, we have the following results on the law of the iterated logarithm corresponding to those of Feller [4] (see also [3]).

**Theorem 2.3.** Let \( \{X_n; \ n \geq 0\} \) be a sequence of i.i.d. symmetric random variables. Suppose the function \( H(\lambda) = E(X_0^2 I\{|X_0| < \lambda\}) \) \((\lambda \geq 0)\) satisfies

\[
\limsup_{\lambda \to \infty} \frac{H(2\lambda)}{H(\lambda)} < \infty.
\]

For any \( n \geq 1 \), let \( a_n \) be the largest solution of the equation

\[
\lambda^2 = nH(\lambda) \log \log \lambda
\]

satisfying \( a_n \uparrow \infty \). Set \( \tau(\beta) = \sum_{n=0}^\infty \beta^{2n} E(X_0^2 I\{|X_0| \leq a_n\}) \) \((0 < \beta < 1)\) and

\[
\bar{\xi}(\beta) = \frac{\sum_{n=0}^\infty \beta^n X_n}{(9\pi(\beta^n \log \log \tau(\beta)^{1/2})} , \quad 0 < \beta < 1.
\]
It can be also proved that $a_n \sim (B_n \log \log B_n)^{1/2}$. By (2.5), it follows that

\begin{equation}
(2.7) \quad \limsup_{n \to \infty} \frac{B_{2n}}{B_n} < \infty.
\end{equation}

Hence, by Theorem 1.1 we have

\begin{equation}
(2.8) \quad \lim_{\beta \to 1} \frac{|\sum_{n=0}^{\infty} \beta^n X_n - \sum_{n=0}^{\infty} \beta^n Y_n|}{(2\tau(\beta) \log \log \tau(\beta))^{1/2}} = 0 \quad \text{a.s.}
\end{equation}

And so, by Proposition 2.1 and (2.8) we have proved Theorem 2.3. \qed

APPENDIX

**Proof of Proposition 2.1.** To prove Proposition 2.1, we need a lemma as follows.

**Lemma A.1.** Let $\{u_n; \ n \geq 0\}$ be a nonincreasing sequence of positive numbers and $\{\xi_n; \ n \geq 0\}$ be a sequence of real numbers. Then for each $n \geq 0$,

$$\left| \sum_{i=0}^{n} u_i \xi_i \right| \leq u_0 \max_{i \leq n} \left| \sum_{j=0}^{i} \xi_j \right|.$$
To prove (A.1), we need only to show
\begin{equation}
\limsup_{k \to \infty} \frac{\sup_{0 \leq \beta \leq \beta_k} |\sum_{n=0}^{\infty} \beta^n Y_n|}{(2\tau(\beta_k) \log \log \tau(\beta_k))^{1/2}} \leq 1 \quad \text{a.s.}
\end{equation}

From Lemma A.1, it follows that for any $0 \leq \beta \leq \beta_k,$
\begin{equation}
\begin{aligned}
\left| \sum_{n=0}^{\infty} \beta^n Y_n \right| &= \left| \sum_{n=0}^{\infty} \left( \frac{\beta}{\beta_k} \right)^n \beta_k^n Y_n \right| \\
&\leq \left( \frac{\beta}{\beta_k} \right)^0 \sup_{0 \leq m \leq \infty} \left| \sum_{n=0}^{m} \beta_k^n Y_n \right| \leq \sup_{0 \leq m \leq \infty} \left| \sum_{n=0}^{m} \beta_k^n Y_n \right|.
\end{aligned}
\end{equation}

This implies
\begin{equation}
\sup_{0 \leq \beta \leq \beta_k} \left| \sum_{n=0}^{\infty} \beta^n Y_n \right| \leq \sup_{0 \leq m \leq \infty} \left| \sum_{n=0}^{m} \beta_k^n Y_n \right|.
\end{equation}

Then
\begin{equation}
\begin{aligned}
P\left( \frac{\sup_{0 \leq \beta \leq \beta_k} |\sum_{n=0}^{\infty} \beta^n Y_n|}{(2\tau(\beta_k) \log \log \tau(\beta_k))^{1/2}} \geq 1 + \varepsilon \right) \\
&\leq P\left( \sup_{0 \leq m \leq \infty} \left| \sum_{n=0}^{m} \beta_k^n Y_n \right| \geq (1 + \varepsilon)(2\tau(\beta_k) \log \log \tau(\beta_k))^{1/2} \right) \\
&\leq 2P\left( \left| \sum_{n=0}^{\infty} \beta_k^n Y_n \right| \geq (1 + \varepsilon)(2\tau(\beta_k) \log \log \tau(\beta_k))^{1/2} \right) \\
&= 2P(|N(0, 1)| \geq (1 + \varepsilon)(2 \log \log \tau(\beta_k))^{1/2}) \\
&\leq 2 \exp\left( -(1 + \varepsilon) \log \log \tau(\beta_k) \right) = 2 \left( \frac{k}{\log \log k} \right)^{(1+\varepsilon)},
\end{aligned}
\end{equation}

which together with the Borel–Cantelli lemma implies (A.5). We have proved (A.1). Now, we show (A.2). Set $S_n = \sum_{k=0}^{n} Y_k (n \geq 1),$ $S_{-1} = 0.$ We have
\begin{equation}
\begin{aligned}
\limsup_{n \to \infty} \frac{|S_n|}{\sqrt{2B_n \log \log B_n}} &\leq \limsup_{n \to \infty} \frac{|W(B_n)|}{\sqrt{2B_n \log \log B_n}} \\
&\leq \limsup_{t \to \infty} \frac{|W(t)|}{\sqrt{2t \log \log t}} \leq 1 \quad \text{a.s.}
\end{aligned}
\end{equation}

where $\{W(t); \ t \geq 0\}$ is a standard Wiener process.
From (A.8) and Lemma 1.1, it follows that for any $N_0 \geq 1$,

$$
\limsup_{\beta \nearrow 1} \frac{\left| \sum_{n=N_0 N(\beta)+1}^{\infty} \beta^n Y_n \right|}{(2\tau(\beta) \log \log \tau(\beta))^{1/2}} \\
\leq e \limsup_{\beta \nearrow 1} \frac{\left| \sum_{n=N_0 N(\beta)+1}^{\infty} \beta^n Y_n \right|}{(2B_N(\beta) \log \log B_N(\beta))^{1/2}} \\
\leq eC_0 \int_{N_0}^{\infty} \exp\left(-\frac{x}{2}\right) x^Q \, dx + eC_0 \exp\left(-\frac{N_0}{2}\right) N_0^Q \to 0 \quad (N_0 \to \infty).
$$

(A.9)

Similarly, we have

$$
\limsup_{\beta \nearrow 1} \frac{\sum_{n=N_0 N(\beta)+1}^{\infty} \beta^{2n} EY_n^2}{\tau(\beta)} \\
\leq e \limsup_{\beta \nearrow 1} \frac{\sum_{n=N_0 N(\beta)+1}^{\infty} \beta^{2n} EY_n^2}{B_N(\beta)} \\
\leq eC_0 \int_{N_0}^{\infty} \exp(-x) x^Q \, dx + eC_0 \exp(-N_0) N_0^Q \to 0 \quad (N_0 \to \infty).
$$

(A.10)

To prove (A.2), we need only to show that for any $b \in (-1, 1)$ and any $\delta > 0$ small enough, there exists a subsequent $\beta_k \nearrow 1$ such that

$$
P(\tilde{\xi}(\beta_k) \in (b - 2\delta, b + 2\delta) \text{ i.o.}) = 1.
$$

(A.11)

Set $\tau_{N_0}(\beta) = \sum_{n=0}^{N_0 N(\beta)} \beta^{2n} EY_n^2$. Choose $\beta_k$ such that $1 - \beta_k^2 = \exp(-k \log \log k)$.

Then $N(\beta_{k-1})/N(\beta_k) \to 0 \quad (k \to \infty)$. Define $\tau^*_k(\beta_k) = \sum_{n=N_0 N(\beta_{k-1})+1}^{N_0 N(\beta_k)} \beta_k^{2n} EY_n^2$.

Noting (A.9) and (A.10), we need only to show that for $N_0$ large enough,

$$
P\left(\frac{\sum_{n=0}^{N_0 N(\beta_k)} \beta_k^n Y_n}{(2\tau_{N_0}(\beta_k) \log \log \tau_{N_0}(\beta_k))^{1/2}} \in (b - \delta, b + \delta) \text{ i.o.}\right) = 1.
$$

(A.12)

From $e^{-1} B_N(\beta) \leq \tau_{N_0}(\beta) \leq \tau(\beta) \leq C B_N(\beta)$, it follows that

$$
\frac{\sum_{n=0}^{N_0 N(\beta_{k-1})} \beta_k^{2n} EY_n^2}{\tau_{N_0}(\beta_k)} \leq e \frac{B_{N_0 N(\beta_{k-1})}}{B_N(\beta_k)} \leq C_0 e \left(\frac{N_0 N(\beta_{k-1})}{N(\beta_k)}\right)^Q \to 0, \quad k \to \infty.
$$

Then

$$
\limsup_{\beta \nearrow 1} \frac{\left| \sum_{n=0}^{N_0 N(\beta_{k-1})} \beta_k^n Y_n \right|}{(2\tau_{N_0}(\beta_k) \log \log \tau_{N_0}(\beta_k))^{1/2}} = 0 \quad \text{a.s.}
$$

Hence, we need only to show that for $N_0$ large enough,

$$
P\left(\frac{\sum_{n=N_0 N(\beta_k)+1}^{N_0 N(\beta_{k-1})+1} \beta_k^n Y_n}{(2\tau^*_k(\beta_k) \log \log B_{N(\beta_k)})^{1/2}} \in (b - \delta, b + \delta) \text{ i.o.}\right) = 1.
$$

(A.13)
Note the independence. By the Borel–Cantelli lemma, we need only to prove

\[
(A.14) \quad \sum_{k=1}^{\infty} P\left( \frac{\sum_{n=N_0 N(\beta_{k-1})+1}^{N_0 N(\beta_k)} \beta_k^n Y_n}{(2\tau_{N_0}(\beta_k) \log \log B_{N(\beta_k)})^{1/2}} \in (b - \delta, b + \delta) \right) = \infty.
\]

Now, it can be shown that for \( k \) large enough

\[
P\left( \frac{\sum_{n=N_0 N(\beta_{k-1})+1}^{N_0 N(\beta_k)} \beta_k^n Y_n}{(2\tau_{N_0}(\beta_k) \log \log B_{N(\beta_k)})^{1/2}} \in (b - \delta, b + \delta) \right)
= P\left( N(0, 1) \in ((b - \delta)(2 \log \log B_{N(\beta_k)})^{1/2}, (b + \delta)(2 \log \log B_{N(\beta_k)})^{1/2}) \right)
\geq \exp(-b^2 \log \log B_{N(\beta_k)}) \frac{1}{\sqrt{2\pi}} \int_{-\delta(2 \log \log B_{N(\beta_k)})^{1/2}}^{\delta(2 \log \log B_{N(\beta_k)})^{1/2}} e^{-x^2/2} dx
\geq \frac{1}{\exp(-b^2 \log \log C_{\alpha}(N(\beta_k))^Q)}
\]


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