Existence of suitable weak solutions of complex Ginzburg–Landau equations and properties of the set of singular points

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In this paper, we consider the supercritical complex Ginzburg–Landau equation. We discuss the existence of suitable weak solution in $\Omega \times [0,T)$, where $\Omega$ is a bounded domain in $\mathbb{R}^n$ or the whole space. We also discuss the properties of the set of the singular points of the suitable weak solution in $\mathbb{R}^n$, which means that the possible singular points are located in a bounded ball for any given time and there is no singular point on the whole space after limited time. © 2003 American Institute of Physics. [DOI: 10.1063/1.1618360]

I. INTRODUCTION AND RESULTS

In this paper we consider the following supercritical complex Ginzburg–Landau (CGL) equations:

$$u_t = (1 + i\mu)\Delta u - (1 + i\nu)|u|^{2\sigma}u + Ru \quad \text{in} \quad \Omega \times [0,T),$$

$$u(x,0) = u_0(x), \quad x \in \Omega,$$

$$u(x,t) = 0, \quad x \in \partial \Omega, \quad 0 < t < T,$$

where $u(x,t)$ is a complex-valued field, the initial data $u_0 \in L^2$, $R > 0$, $\mu, \nu \in \mathbb{R}$ and $2/n < \sigma < (n + 4)/2n$, $\Omega$ is a bounded domain in $\mathbb{R}^n$ or the whole space. When $\Omega = \mathbb{R}^n$, the third line of (1) should be replaced by $u(x,t) \to 0$ as $|x| \to \infty$ for $0 < t < T$. This equation, most often considered with a cubic nonlinearity and $n = 3$, has a long history in physics as a generic amplitude equation near the onset of instabilities in fluid mechanical systems, as well as in the theory of phase transitions and superconductivity. In this paper, we concentrate on mathematical questions related to the regularity of weak solutions. In the case $\sigma = 1$, solutions of (1) have similar scaling properties as solutions of the Navier–Stokes equations. It might be a good model problem in connection with regularity questions regarding the Navier–Stokes equations.

For Eq. (1) in the periodic case, the existence of weak solutions in all cases was obtained in Ref. 1. It is pointed out in Ref. 1 that in the subcritical case ($\sigma n < 2$), all weak solutions are regular. When $\sigma n = 2$, it is known that singularities cannot develop from sufficiently regular initial data (see Ref. 2 for case $\sigma = 2, n = 2$ and Ref. 1 for the general case). When $\sigma n > 2$, regularity results are available only for special values of $\mu$ and $\nu$ (see Ref. 1).

For Eq. (1), the existence of a weak solution can be obtained by the Galerkin approximation. The full regularity for weak solutions of (1) in the subcritical and critical cases can be obtained by

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the usual bootstrap argument and a refined version of the standard bootstrap argument, respectively. In the supercritical case \((\sigma n > 2)\) it is not known whether weak solutions are regular for general values of \(\mu\) and \(\nu\).

In Ref. 3, the following version of CGL was considered:

\[
\begin{align*}
   u_t &= (1 + i \mu) Du - i \nu |u|^2 \sigma u + Ru \quad \text{in} \quad \Omega \times [0, T), \\
   u(x, 0) &= u_0(x), \quad x \in \Omega, \\
   u(x, t) &= 0, \quad x \in \partial \Omega, \quad 0 < t < T.
\end{align*}
\]

The partial regularity of the suitable weak solution was discussed and the existence of the suitable weak solution was mentioned but without proof in Ref. 3. In this paper we prove the existence of the suitable weak solution of (1). Actually, it also works for (2). For more details on (1) and (2), see Refs. 3, 1, and 2.

Now we give some notations. Let \(D = \Omega \times (0, T)\), \(B_r\) be the ball in \(\mathbb{R}^n\) with radius \(r\), \(Q_r = Q(x, t) = \{(y, s) : |y - x| \leq r, t - r^2 \leq s < t\}\). Let \(E_0(u) = \sup_{0 < s < T} \frac{1}{2} \int_\Omega |u|^2 dx\), \(E_1(u) = \int_0^T \int_\Omega |\nabla u|^2 dx \, dt\), and \(E = E_0 + E_1\). Let \(H^1(\Omega) = \{u \in L^2(\Omega); \nabla u \in L^2(\Omega)\}\) and \(H^1_0(\Omega)\) be the completion of \(C^\infty(\Omega)\) in \(H^1(\Omega)\) and \(H^{-1}(\Omega)\) be the dual of \(H^1_0(\Omega)\). We omit \(\Omega\) if there is no confusion.

The definition of suitable weak solution is the following.

**Definition 1.1**: We say \(u\) is a suitable weak solution of (1) on \(D\), if the following conditions are satisfied:

(i) \(u \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))\).

(ii) \(u\) satisfies (1) in the sense of distributions on \(D\).

(iii) (Generalized energy inequality) For each real-valued \(\phi \in C^\infty_0(D)\) with \(\phi \geq 0\), the following inequality holds:

\[
2 \int_\Omega \int |\nabla u|^2 \phi \, dx \, dt + \int_0^T \int |u|^{2\sigma + 2} \phi \, dx \, dt \leq \int \int |u|^2 (\phi' + 2\Delta \phi) \, dx \, dt + 2\Re \left( (1 + i \mu) \int \int u \nabla \phi \nabla \phi \, dx \, dt \right) + 2R \int \int |u|^2 \phi \, dx \, dt.
\]

We have the following existence result.

**Theorem 1.1**: Let \(u_0 \in L^2(\Omega)\). Then there exists a weak solution \(u\) of (1) on \(D\) satisfying

\[
\begin{align*}
   u &\in L^2(0, T; H^1_0) \cap L^\infty(0, T; L^2), \\
   u(t) &\rightarrow u_0 \text{ weakly in } L^2 \text{ as } t \rightarrow 0,
\end{align*}
\]

and if \(\phi \in C^\infty(\bar{D})\), \(\phi \geq 0\), and \(\phi = 0\) near \(\partial \Omega \times (0, T)\), then for \(0 < t < T\),

\[
\begin{align*}
   \int_{\Omega \times [0, t]} |u|^2 \phi &+ 2 \int_0^t \int_\Omega |\nabla u|^2 \phi + \int_0^t \int |u|^{2\sigma + 2} \phi \leq \int_{\Omega} |u_0|^2 \phi(x, 0) + \int_0^t \int_\Omega |u|^2 (\phi' + 2\Delta \phi) + 2\Re \left( (1 + i \mu) \int_\Omega \int u \nabla \phi \nabla \phi \right) + 2R \int_0^t \int_\Omega |u|^2 \phi.
\end{align*}
\]
By the motivation of Ref. 4, we consider the property of the set of singular points of the suitable weak solution with more assumptions on the initial data. We consider

\[ \int_{\mathbb{R}^n} |u_0|^2 |x| \, dx = G < \infty. \quad (7) \]

Then there exists a weak solution of the initial value problem of (2) which is regular in the region \( \{(x,t) : |x|^2 t^{n-2} \sigma > K_1\} \), where \( K_1 = K_1(E,G) \) is a constant depending on \( E \) and \( G \).

Remark 1.1: The above theorem means that the set of singular points is restricted in a bounded domain for every time. Actually, there is no singular point in \( \mathbb{R}^n \) after a limited time, see Lemma 3.1.

The following interpolation inequality is useful in our following proofs.

**Lemma 1.1:** If \( u \in H^1(\mathbb{R}^n) \), then

\[ \int_{B_r} |u|^q \, dx \leq C \left( \int_{B_r} |\nabla u|^2 \, dx \right)^a \left( \int_{B_r} |u|^2 \, dx \right)^{(q/2) - a} + \frac{C}{r^{2a}} \left( \int_{B_r} |u|^2 \, dx \right)^{q/2}, \quad (8) \]

where \( a = (n/4)(q-2) \), \( 2 \leq q \leq 2n/(n-2) \), \( C \) independent of \( r \). Moreover, if \( u \) has mean zero on \( B_r \) or if \( B_r \) is replaced by all of \( \mathbb{R}^n \), then the second term on the right-hand side in (8) may be omitted.

In the remainder of this paper we prove these two theorems, respectively.

### II. PROOF OF THE THEOREM 1.1

We begin with some lemmas concerning the relevant linear systems.

**Lemma 2.1:** Suppose \( f \in L^2(0,T;\mathbb{H}^{-1}) \), \( u \in L^2(0,T;\mathbb{H}^1) \), and

\[ u_t = (1 + i\mu) \Delta u + f \quad (9) \]

in the sense of distributions on \( D \). Then \( u \in L^2(0,T;\mathbb{H}^1) \),

\[ \frac{d}{dt} \int_{\Omega} |u|^2 \, dx = 2\Re \int_{\Omega} u_i \overline{u} \, dx \quad (10) \]

in the sense of distributions on \((0,T)\), and \( u \in C([0,T];L^2) \) after modification on a set of measure zero. Solution of (9) is unique in \( L^2(0,T;\mathbb{H}^1) \) for given initial data \( u_0 \in L^2 \).

**Proof:** The proof is standard, see Refs. 4 and 5.

**Lemma 2.2:** Let \( u_0 \in L^2 \) and \( w \in C^\infty(\overline{D};\mathbb{R}) \) with \( w \equiv 0 \). Then there exists unique function \( u \in C([0,T];L^2) \cap L^2(0,T;\mathbb{H}^1) \) such that

\[ u_t = (1 + i\mu) \Delta u - (1 + i\nu) w u + Ru \quad (11) \]

in the sense of distributions on \( D \) and \( u(0) = u_0 \).

**Proof:** The proof is similar to Ref. 4, and for the proof of uniqueness of solution we need the Gronwall inequality.

**Lemma 2.3:** Let \( u_0 \in L^2 \), \( w \in C^\infty(\overline{D};\mathbb{R}) \) with \( w \equiv 0 \) and \( u \) be a solution of (11). Then for any \( \phi \in C^\infty_c(\overline{D}) \) with \( \phi = 0 \) near \( \partial \Omega \times (0,T) \), for all \( t \in (0,T) \),
\[ \int_{\Omega \times \{t\}} |u|^2 \phi + 2 \int_0^t \int_{\Omega} |\nabla u|^2 \phi + \int_0^t \int_{\Omega} w|u|^2 \phi \]
\[ = \int_{\Omega} |u_0|^2 \phi(x,0) + \int_0^t \int_{\Omega} |u|^2 (\phi' + 2 \Delta \phi) + 2\Re \left( (1 + i \mu) \int_0^t \int_{\Omega} u \nabla \tilde{u} \nabla \phi \right) \]
\[ + 2 R \int_0^t \int_{\Omega} |u|^2 \phi. \quad (12) \]

**Proof:** Suppose for the moment that \( \phi \) also vanishes near \( t = 0 \); choose \( \Omega_1 \) so that \( \Omega_1 \subset \Omega \) and \( \text{supp} \phi \subset \Omega_1 \times (0,T) \). Writing \( F = -(1 + i \nu) w + Ru \in L^2(D) \), we have
\[ u_i = (1 + i \mu) \Delta u + F. \quad (13) \]

Mollifying (in \( \mathbb{R}^{n+1} \)) each term of (13), we obtain sequences of smooth functions \( \{u_m\} \) and \( \{F_m\} \) such that
\[ \frac{d}{dt} u_m = (1 + i \mu) \Delta u_m + F_m, \quad (14) \]
with \( u_m \to u \) in \( L^2(D) \), \( \nabla u_m \to \nabla u \) in \( L^2(D) \), and \( F_m \to F \) in \( L^2(D) \). Multiplying (14) by \( \tilde{u}_m \phi \), integrating by part on \( D \), and taking the real part, by Lemma 2.1, we have
\[ - \int \int |u_m|^2 \phi' = 2\Re \left( (1 + i \mu) \int \int |\nabla u_m|^2 \phi + \int \int u_m \nabla \tilde{u}_m \nabla \phi + \int \int |u_m|^2 \Delta \phi \right) \]
\[ + 2 \Re \int \int F_m \tilde{u}_m \phi. \]

We pass to the limit as \( m \to \infty \) and obtain
\[ - \int \int |u|^2 \phi' = -2 \int \int |\nabla u|^2 \phi + 2\Re \left( (1 + i \mu) \int \int u \nabla \tilde{u} \nabla \phi \right) + 2 \Re \int \int |u|^2 \Delta \phi \]
\[ + 2 \Re \int \int F \tilde{u} \phi. \]

On the other hand,
\[ 2\Re \int \int F \tilde{u} \phi = 2\Re \left( \int \int -(1 + i \nu) w|u|^2 \phi + R|u|^2 \phi \right) = -2 \int \int w|u|^2 \phi + 2 \Re \int \int |u|^2 \phi. \]

So we have
\[ 2 \int \int |\nabla u|^2 \phi + \int \int w|u|^2 \phi = \int \int |u|^2 (\phi' + 2 \Delta \phi) + 2 \Re \int \int |u|^2 \phi \]
\[ + 2 \Re \left( (1 + i \mu) \int \int u \nabla \tilde{u} \nabla \phi \right). \quad (15) \]

For the general \( \phi \in C^\infty(\bar{D}) \), \( \phi \geq 0 \), by the argument in Ref. 4, we obtain (12).

Utilize the “retard mollifier” \( \Psi_\delta(u) \) as in Ref. 4, we have the following lemma.

**Lemma 2.4:** For any \( u \in L^\infty(0,T;L^2) \cap L^2(0,T;H_0^1) \),
\[ \sup_{0 < r < T} \int_{\Omega} |\Psi_\delta(u)|^2(x,t) \, dx \leq C E_\delta(u), \quad (16) \]
\[ \int_D |\nabla \Psi_d(u)|^2 \, dx \, dt \leq CE_1(u), \]  
where \( C \) denotes an universal constant.

Now we are ready to prove Theorem 1.1. For any large integer \( N \), let \( \delta = T/N \), and solve
\[ \frac{d}{dt} u_N = (1 + i \mu) \nabla u_N - \left( 1 + i \nu \right) \Psi_d(\Psi_d(u_N)^{2\sigma}) u_N + R u_N, \]  
\[ u_N \in L^2(0,T;H^1_0) \cap C([0,T];L^2), \]  
\[ u_N(0) = u_0. \]

By induction, due to Lemma 2.2, such \( u_N \) exist on each \((m\delta,(m+1)\delta), 0 \leq m \leq N - 1\). By (10), we have
\[ \int_{\Omega \times [t]} |u_N|^2 + 2 \int_0^t \int_{\Omega} |\nabla u_N|^2 + 2 \int_0^t \int_{\Omega} \Psi_d(\Psi_d(u_N)^{2\sigma}) |u_N|^2 = \int_{\Omega} |u_0|^2 + 2R \int_0^t \int_{\Omega} |u_N|^2, \]
then
\[ \int_{\Omega \times [t]} |u_N|^2 \leq \int_{\Omega} |u_0|^2 + 2R \int_0^t \int_{\Omega} |u_N|^2. \]

By the Grownwall inequality, for \( t \in (0,T) \),
\[ \int_{\Omega \times [t]} |u_N|^2 \, dx \leq \exp(2RT) \int_{\Omega} |u_0|^2, \]
then for \( t \in (0,T) \),
\[ \int_{\Omega \times [t]} |u_N|^2 + 2 \int_0^t \int_{\Omega} |\nabla u_N|^2 + 2 \int_0^t \int_{\Omega} \Psi_d(\Psi_d(u_N)^{2\sigma}) |u_N|^2 \leq (1 + 2RT \exp(2RT)) \int_{\Omega} |u_0|^2, \]
so we have
\[ \{u_N\} \text{ is bounded in } L^2(0,T;L^2) \cap L^2(0,T;H^1_0). \]  
(21)

We claim that
\[ \frac{d}{dt} u_N \in L^p(0,T;H^{-s}), \]  
(22)
where \( s > n/2 \) for some \( 1 < p < \infty \).

By (18) and (21), it is sufficient to check that \( \Psi_d(\Psi_d(u_N)^{2\sigma}) u_N \in L^p(0,T;H^{-s}) \). By Lemma 1.1, we have
\[ \int_{\Omega} |u_N|^{2\sigma+1} \leq C \left( \int_{\Omega} |\nabla u_N|^2 \right)^{a} \left( \int_{\Omega} |u_N|^2 \right)^{(2\sigma+1)/2 - a} + \frac{C}{r^{2a}} \left( \int_{\Omega} |u_N|^2 \right)^{(2\sigma+1)/2}, \]
where
\[ a = \frac{n(2\sigma - 1)}{4}. \]
For $\sigma < (4 + n)/2n$, let $p = 4[n(2\sigma - 1)]/1 > 1$, we have
\[
\int_0^T \left( \int_\Omega |u_N|^{2\sigma + 1} \, dx \right)^p \, dt \leq C \left( E_0^{(2\sigma + 1)/2 - a} \right)^p E_1 + \frac{CT}{T} E_0^{(2\sigma + 1)/2} < \infty,
\]
so $\Psi_\delta(|\Psi_\delta(u_N)|^{2\sigma})u_N \in L^p(0,T;H^{-s})$.

When $\Omega$ is bounded in $\mathbb{R}^n$, by (21), (22) and Theorem 2.1 in Chap. III in Ref. 5, $\{u_N\}$ stays in a compact set of $L^2(0,T;L^2)$, then there exists a strongly convergent subsequence of $\{u_N\}$, still denoted by $\{u_N\}$, and $u^*$ with
\[
u_N \rightarrow u^* \text{ in } L^2(0,T;L^2).
\]
By Lemma 1.1, let $r_0 = \text{diam}(\Omega)$,
\[
\int_\Omega |u_N|^{2(n+2)/n} \, dx \leq C \left( \int_\Omega |\nabla u_N|^2 \, dx \right)^{2n} + \frac{C}{r_0^{2n}} \left( \int_\Omega |u_N|^2 \, dx \right)^{(n+2)/n},
\]
then integrate from 0 to $T$,
\[
\int_0^T \int_\Omega |u_N|^{2(n+2)/n} \, dx \, dt \leq CE_1E_0^{2n} + \frac{CE_0^{(n+2)/n}}{r_0^{(n+2)/n}} < \infty,
\]
so $\{u_N\}$ is bounded in $L^{2(n+2)/n}(D)$. By (23), for $2 \leq q < [2(n+2)/n]$,
\[
u_N \rightarrow u^* \text{ in } L^q(0,T;L^q).
\]
When $\Omega = \mathbb{R}^n$, we can only obtain that there exists a subsequence, still denoted by $\{u_N\}$, and $u^*$ satisfying $u_N \rightarrow u^*$ strongly in $L^2(0,T;L^2_{loc})$, $\{u_N\}$ is bounded in $L^{2(n+2)/n}(D)$ and then $u_N \rightarrow u^*$ strongly in $L^q(0,T;L^q_{loc})$ for $2 \leq q < [2(n+2)/n]$, which is enough to get our results.

By (21),
\[
u_N \rightarrow u^* \text{ weakly in } L^2(0,T;H^1_0),
\]
and
\[
u_N \rightarrow u^* \text{ weak-star in } L^\infty(0,T;L^2).
\]
By the definition of $\Psi_\delta$, for some $r < (n+2)/n\sigma$,
\[
\Psi_\delta(|\Psi_\delta(u_N)|^{2\sigma}) \rightarrow |u^*|^{2\sigma} \text{ in } L^r(0,T;L^r),
\]
and
\[
\Psi_\delta(u_N) \rightarrow u^* \text{ strongly in } L^q(D).
\]
So $u^*$ is the solution of (18) in the sense of distributions. From now on, the proof is similar to that in Ref. 4.

### III. PROOF OF THE THEOREM 1.2

In this section, we consider the initial value problem of (2) for simplicity. Let $\Omega = \mathbb{R}^n$ and $u$ be a suitable weak solution of (2) on $\mathbb{R}^n \times (0,\infty)$ with initial data $u_0 \in L^2$, and for $\phi \in C^\infty_c(\mathbb{R}^n \times \mathbb{R})$ with $\phi \geq 0$,
\[
\int_{\Omega \times [t]} |\nabla u|^2 \phi + 2 \int_0^t \int_{\Omega} |\nabla u|^2 \phi \leq \int_{\Omega} |u_0|^2 \phi(x,0) + 2 R \int_0^t \int_{\Omega} |u|^2 \phi + \int_0^t \int_{\Omega} |u|^2 (\phi' + 2 \Delta \phi)
\]
\[
+ 2 R \left\{ (1 + i \mu) \int_0^t \int_{\Omega} u \nabla \bar{u} \nabla \phi \right\}.
\]

(29)

Here we need a corollary in Ref. 3.

**Proposition 3.1:** Let \( \alpha > 2 \), \( 2 \sigma + 1 < \alpha < [2(n+2)]/n \), there exists \( \epsilon_0 > 0 \), if \( u \) is a suitable weak solution of (2) on \( Q_r \), and moreover,

\[
\frac{1}{r^{n+2-\alpha/\sigma}} \int_{Q_r} |u|^\alpha \leq \epsilon_0,
\]

(30)

then \( u \) is \( C^{\alpha_0} \) in \( Q_{r/4} \) for some \( \alpha_0 \in (0,1/2) \).

By the same idea in Ref. 3, we let \( v = e^{-Rt} u \), if \( u \) is a suitable weak solution of (2), then \( v \) is a suitable weak solution of the following problem:

\[
v_i = (1 + i \mu) \Delta v - i v e^{2\sigma R} |v|^{2\sigma} \quad \text{in} \quad \Omega \times (0,T),
\]

\[
v(x,0) = u_0(x), \quad x \in \Omega,
\]

\[
v(x,t) = 0, \quad x \in \partial \Omega, \quad t \in (0,T),
\]

(31)

and the corresponding generalized energy inequality

\[
\int_{\Omega \times [t]} |v|^2 \phi + 2 \int_0^t \int_{\Omega} |\nabla v|^2 \phi \leq \int_{\Omega} |v_0|^2 \phi(x,0) + \int_0^t \int_{\Omega} |v|^2 (\phi' + 2 \Delta \phi)
\]

\[
+ 2 R \left\{ (1 + i \mu) \int_0^t \int_{\Omega} v \nabla \bar{v} \nabla \phi \right\}
\]

(32)

holds for any real-valued \( \phi \in C^\infty(D) \) with \( \phi \geq 0 \) and \( \phi(t) \in C_0^\infty(\mathbb{R}^n) \) for each \( t \in [0,T] \).

To prove Theorem 1.2, we need the following lemmas.

**Lemma 3.1:** There exists an absolute constant \( C \) such that if \( t \equiv CE^{2\sigma(n\sigma-2)} \), then \( u \) is regular at \((x,t)\).

**Proof:** By Lemma 1.1,

\[
\int_{\mathbb{R}^n} |u|^\alpha dx \leq C \left( \int_{\mathbb{R}^n} |u|^2 dx \right)^{(a/2)(1-n(1/2 - 1/\alpha))} \left( \int_{\mathbb{R}^n} |\nabla u|^2 dx \right)^{(n\alpha/2)(1/2 - 1/\alpha)},
\]

then

\[
\int_0^T \int_{\mathbb{R}^n} |u|^\alpha dx \leq C \left( \int_{\mathbb{R}^n} |u|^2 dx \right)^{(a/2)(1-n(1/2 - 1/\alpha))} \int_0^T \left( \int_{\mathbb{R}^n} |\nabla u|^2 dx \right)^{(n\alpha/2)(1/2 - 1/\alpha)} dt
\]

\[
\leq CE_0^{(a/2)(1-n(1/2 - 1/\alpha))} E_{1/2}^{(n\alpha/2)(1/2 - 1/\alpha)} T\Gamma(1-(n\alpha/2)(1/2 - 1/\alpha))
\]

\[
\leq CE^{a/2} T^{1-(n\alpha/2)(1/2 - 1/\alpha)}.
\]

When \( E \equiv (\epsilon_0/C)^{2\alpha T^{n/2 - 1/\alpha}} \), i.e., \( T \equiv C(\epsilon_0, \alpha) E^{2\sigma(n\sigma-2)} \),

\[
\int_0^T \int_{\mathbb{R}^n} |u|^\alpha dx \leq \epsilon_0 T^{(n\alpha/2)-a/\sigma}.
\]

By Proposition 3.1, if
\[ \int \int_{Q_{r,t}} |u|^n \, dx \, dt \leq \epsilon_0 r^{n+2-\alpha/\sigma}. \]
then \( u \) is regular in \( Q_{r,t} \), so Lemma holds.

**Lemma 3.2:** Let \( \tau = \epsilon + |x|, \epsilon \geq 0. \) If

(i) \( r \geq 2, \gamma + n/r > 0, \alpha + n/2 > 0, \beta + n/2 > 0, 1/2 \leq a \leq 1, \)

(ii) \( \gamma + n/r = a(\alpha + (n-2)/2) + (1-a)(\beta + n/2), \)

(iii) \( a(\alpha - 1) + (1-a)\beta \leq \gamma \leq a\alpha + (1-a)\beta, \)
then

\[ |\tau \gamma u|_{L^2} \leq C |\tau^a \nabla u|_{L^2}^{\alpha} |\tau^\beta u|_{L^2}^{1-a}. \] (33)

For \( n = 3 \), the proof of the lemma has been given in Ref. 4; for the general case, the proof is similar.

**Lemma 3.3:** Let \( u \) be a suitable weak solution of (2) with (29) holds and \( G < \infty \). Then for a.e. \( t > 0, \)

\[ \frac{1}{2} \int_{\mathbb{R}^n} |u|^2 |x| \, dx + \int_0^t \int_{\mathbb{R}^n} |\nabla u|^2 \, dx \, dt \leq A(t), \] (34)

where \( A(t) = G + C t^{1/2} E. \)

**Proof:** By Lemma 3.2,

\[ \int_{\mathbb{R}^n} \left| \frac{u}{|x|} \right|^2 \, dx \leq C \left( \int_{\mathbb{R}^n} |u|^2 \, dx \right)^{1/2} \left( \int_{\mathbb{R}^n} |\nabla u|^2 \, dx \right)^{1/2}, \]
then

\[ \int_0^t \int_{\mathbb{R}^n} \left| \frac{u}{|x|} \right|^2 \, dx \, dt \leq C E_0^{1/2} E_1^{1/2} t^{1/2} \leq C E t^{1/2}. \]

Let \( \chi(t) \) be \( C^0 \) on \( s \geq 0 \) with \( 0 \leq \chi \leq 1, \) \( \chi = 1 \) for \( s \leq 1 \) and \( \chi = 0 \) for \( s \geq 2 \). For constants \( 1 > \lambda, \gamma > 0, \) we use the test function \( \phi(x) = 1/(2(\lambda^2 - |x|^2)^{1/2} \chi(\epsilon/|x|)) \) in the energy inequality (32). It is easy to verify that \( |\nabla \phi| \leq C, |\Delta \phi| \leq C |x|^{-1}, \) with \( C \) independent of \( \lambda \) and \( \epsilon. \) Then

\[ \int_{\mathbb{R}^n \times \{t\}} |u|^2 \phi \, dx + 2 \int_0^t \int_{\mathbb{R}^n} |\nabla u|^2 \phi \, dx \
\leq \int_{\mathbb{R}^n} |u_0|^2 |x| \, dx + C \int_0^t \int_{\mathbb{R}^n} \left( \frac{|u|^2}{|x|} + |u||\nabla u| \right) \, dx \, dt. \]

Let \( \epsilon \rightarrow 0 \) and then \( \lambda \rightarrow 0, \)

\[ \int_{\mathbb{R}^n \times \{t\}} |u|^2 |x| \, dx + 2 \int_0^t \int_{\mathbb{R}^n} |\nabla u|^2 |x| \, dx \, dt \leq G + C E t^{1/2}. \]

Now we prove Theorem 1.2. Let \( S \) be the set of singular points of \( u. \) By Lemma 3.1, if \( (x,t) \in S, \) then

\[ t < C E^{2\sigma/(n \sigma - 2)}. \] (35)

Let \( r^2 = 5/4 \), \( Q = Q_r(x,r^2), \) by Proposition 3.1,

\[ \int \int_{Q_r} |u|^n \, dx \, dt > C \epsilon_0 r^{n+2-\alpha/\sigma}. \] (36)
Let $R = |x|$ and suppose for the moment that $R > 2r$. Take $\gamma = \alpha = 1/2$, $a = n/(n+2)$, and $p = [2(n+1)]/n$. By Lemma 3.2,\[ ||x^{1/2}u||_{L^p}^p \leq C||x^{1/2}||_{L^2}^2||x^{1/2}u||_{L^2}^{4/n}. \]

Integrate from 0 to $t$,
\[
\int_0^t \int_{\mathbb{R}^n} |x|^{(n+2)/n} |u|^{(2(n+2))/n} dx \, dt \leq CA(t)^{2n} \int_0^t \int_{\mathbb{R}^n} |u|^{2(n+2)/n} dx \, dt \leq CA(t)^{2 + n}/n.
\]

By (36) and the Hölder inequality,
\[
Ce^{\sigma t + 2/2 - a/2}\leq \int_0^t \int_{\mathbb{R}^n} |u|^{a/2} dx \, dt \leq C^{\alpha t^{n + 2 - an/2}} \left( \int_0^t \int_{\mathbb{R}^n} |u|^{2(n+2)/n} dx \, dt \right)^{an/(2(n+2))}
\]
\[
\leq Ce^{\sigma t + 2 - an/2}R^{-a/2}A(t)^{a/2} = K(E,G)R^{n + 2/2 - an/4}R^{-a/2},
\]
so
\[
R^{2t - 2/2}\leq K_0(E,G,\alpha).
\]

For $R \leq 2r$,
\[
R^{2t - 2/2}\leq 5t^{n + 1 - 2\sigma} \leq C(E^{2\alpha n/(n+1)}R^{n + 1 - 2\sigma}.
\]

Therefore, Theorem 1.2 holds with $K_1 = C \max(K_0, E^{2\alpha n/(n+1)}R^{n + 1 - 2\sigma})$.