A COUNTEREXAMPLE TO INFINITE-DIMENSIONAL VERSION
OF THE MORSE-SARD THEOREM

Jiang Haiyi

Abstract. In this note a positive answer for a question posed by S. Bates in 1992 is given, i.e., there exists a
$C^\infty$ rank-1 map $f: \mathbb{R}^2 \to \mathbb{R}^l$ such that $\partial A$ has nonempty interior for some subset $A \subset \mathbb{R}^l$ of critical
points with finite Hausdorff dimension. It is to say the Morse criticality theorem can not be generalized to
infinite-dimensional spaces.

§ 1 Introduction

The Morse-Sard theorem is a fundamental theorem in analysis especially in the basis of transversality theory and
differential topology. The classical Morse-Sard theorem states that the image of the set of critical points of
a function $f: \mathbb{R}^m \to \mathbb{R}^l$ of class $C^{m-l+1}$ has zero Lebesgue measure in $\mathbb{R}^l$ for $m \geq l \in \mathbb{N}$. It was proved by Morse [10] in
the case $l = 1$ and by Sard [20] in the general case. So it is called that.

In case $\mathbb{R}^m$ is replaced by an open set in a Hilbert or Banach space $B$ it is easy to see that the theorem
is false if either (i) $f$ is not required to be $C^\infty$, (ii) $B$ is not separable. It is natural to ask whether the following
generalization is true. Let $f: U \to \mathbb{R}^l$ be $C^\infty$ where $U$ is an open set in a separable Hilbert space $H$ then the set of critical values for $f$ forms a set of measure zero. Kupka has proved that this generalization is false in [3]. But in his example the set of critical points for $f$ is infinite-dimensional.

Later in 1993 Bates constructed a surjective $C^\infty$ map $f: \mathbb{R}^2 \to \mathbb{R}^2$ satisfying rank $Jf_v \leq 1$ for all
$v \in E$ where $E$ is any separable infinite-dimensional Banach space. But in his example the set of critical
points for $f$ is $E$ so it is infinite-dimensional too. For details see [4].

In view of the preceding remarks it would be interesting to determine precisely how large a set $A
\subset E$ must be in order that its image under some smooth rank-1 map into the plane has non-empty interior.
Bates concluded this discussion with the following questions in [5]. Does there exist a $C^\infty$ rank-1 map $f: \mathbb{R}^2 \to \mathbb{R}^m$ such that $\partial A$ has non-empty interior for some subset $A \subset \mathbb{R}^l$ of finite Hausdorff dimension for $m \in \mathbb{N}$? On the other hand can the Morse criticality theorem [10] be generalized to infinite-di-

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mensional spaces]

In this note we construct a $C^\infty$ map $f: H \to \mathbb{R}^d$ such that the set of critical points is the Cantor set $\emptyset \subset H$ with $\dim_H \emptyset \leq 4d$ and $\emptyset \not\subset \mathbb{R}^{d-1}$. It shows that the Morse criticality theorem cannot be generalized to infinite-dimensional spaces. Recently a similar result was discovered independently by Bates and Moreira [6].

§ 2 Definitions

Let $A$ be a non-empty bounded subset of metric $X$ and $d \geq 0$. For each $\delta > 0$ let

$$\mathcal{H}^d A = \inf \sum_i \text{diam} U_i \|_{\mathbb{R}^d} A \text{ is covered by sets } U_i \text{ with } 0 < \text{diam} U_i \leq \delta,$$

where the infimum is over all coverings of $A$ by a finite or countable collection of sets with diameters at most $\delta$. We may define

$$\mathcal{H}^d A = \lim_{\delta \to 0} \mathcal{H}^d A.$$

We call $\mathcal{H}^d A$ the $d$-dimensional Hausdorff measure of $A$.

It is easy to see that there is a number $d$ at which $\mathcal{H}^d A$ jumps from $\infty$ to $0$; we call this number $d$ the Hausdorff or Hausdorff-Besicovich dimension of $A$ which we denote by $\dim_H A$. Thus

$$\dim_H A = \sup d \|_{\mathbb{R}^d} A = \infty = \inf d \|_{\mathbb{R}^d} A = 0.$$

Let $t > 0$ be a real number and $X$ a metric space; we recall that a subset $A \subset X$ is $t$-finite if $\mathcal{H}^t A < \infty$; that $A \subset X$ is $t$-null if $\mathcal{H}^t A = 0$; and that $A \subset X$ is $t$-sigma-finite if $A$ is the countable union of $t$-finite sets.

If $0 < p < \infty$ then $l^p$ consists of all real sequences $x = \{ x_n \}$ such that

$$\| x \|_p = \left( \sum_{n=1}^{\infty} | x_n |^p \right)^{1/p} < \infty,$$

with the norm

$$\| x \|_p = \left( \sum_{n=1}^{\infty} | x_n |^p \right)^{1/p}.$$

Similarly $l^\infty$ is the vector space of all bounded real sequences with the sup norm. As we know for $1 \leq p < \infty$ $\| \cdot \|_p$ is a separable Banach space but $l^\infty$ is not separable. For $l^p$-spaces we have the following comparable theorem.

Theorem 2.1. If $1 \leq p < q \leq \infty$ then $l^p \subset l^q$ holds. Moreover, the conclusion is proper.

To know the proof of this theorem [see 7].

Especially for $p = 2$ we denote by $l^2$ the space of real sequences $x = \{ x_n \}$ equipped with the norm

$$\| x \| = \left( \sum_{n=1}^{\infty} | x_n |^2 \right)^{1/2} < \infty.$$

It is well known that $l^2 || \cdot ||$ is a separable Hilbert space. Let $H = l^2$.

Let $H^*$ be the dual of $H$. A base of $H^*$ is formed by the linear functions $e_1 e_2 \ldots e_n \ldots$ where
\[ e_n^k \ x = x_n \quad \text{for} \quad x = \{ x_1, x_2, \ldots, x_n \}. \]

For any \( k \in \mathbb{N} \), let \( L^k \) be the space of all \( k \)-linear symmetric continuous functions on \( H \). \( \mathcal{L}^k \) is also a Hilbert space with norm \( \| \cdot \|_k \). Clearly \( \mathcal{L}^k = H^* \).

To each \( e_n \) and each \( k = 1, 2, \ldots, n \) there is canonically attached an element \( e_n^k \) in \( L^k \) defined by

\[ e_n^k \ x^1 \ x^2 \ldots \ x^k = e_n \ x^1 e_n \ x^2 \ldots e_n \ x^k \]

for any points \( x^1, x^2, \ldots, x^k \) in \( H \). Note that for any \( k \) \( \| e_n \|_k = 1 \).

To each \( x \in H \) we associate a continuous linear operator \( \mathcal{L} \) \( x \colon L^k \mathcal{L} = H^* \) for all \( k \geq 2 \) by\( \mathcal{L} \)\( \| \mathcal{L} \| \leq 1 \) and \( \mathcal{L} \)\( \| x \| = 1 \).

The norm of \( \mathcal{L} \) \( x \) is \( \| x \| \) for \( \| x \| = 1 \) and \( \| x \| = 1 \) and \( \| x \| = 1 \).

Now let \( \mathcal{L} \) \( U \to \mathbb{R}^1 \) be a function defined on the open set \( U \subset H \). We say that \( \mathcal{L} \) is of class \( \mathcal{C}^\infty \) if there exists a sequence of continuous maps \( D^k \mathcal{L} \colon x \in U \to D^k \mathcal{L} \) \( x \in H^* \) such that \( \mathcal{L} \) for any \( \varepsilon > 0 \) and any integer \( N > 0 \) there exists a \( \delta > 0 \) such that \( \| x \| = 1 \) and \( \| x \| = 1 \) then

\[ \| x \| = 1 \]

and

\[ \| D^k_{x+y} - D^k_x \| \leq 1 \quad \| x \| = 1 \]

for \( k = 1, 2, 3, \ldots, n \). The sequence \( D^1, D^2, \ldots \) is then unique and \( D^k \) is called the \( k \)-th \( \mathcal{C}^\infty \) derivative of \( \mathcal{L} \) at \( x \) and is written \( D^k \mathcal{L} \) \( x \).

For a \( \mathcal{C}^\infty \) differentiable map \( \mathcal{L} \colon H \to \mathbb{R}^m \) where \( m \in \mathbb{N} \) and \( H \) is a separable Hilbert space. The rank \( r \) of a point \( x \in H \) is defined as the rank of the tangent map \( D^1 \mathcal{L} \) \( x \in H \). The dimension of the image \( D^1 \mathcal{L} \) \( x \in H \) is \( r \). Evidently \( D^1 \mathcal{L} \) \( x \in H \) is \( D^1 \mathcal{L} \) \( x \in H \) for all \( x \in H \). \( A \subset H \) is a set of rank \( r \) for \( \mathcal{L} \) \( H \to \mathbb{R}^m \) if the rank of \( D^1 \mathcal{L} \) \( x \) is at most \( r \) for every \( x \in A \).

### §3 Morse criticality theorem

The classical Morse criticality theorem is as follows.

**Theorem 3.1.** Let \( A \) be a subset of \( \mathbb{R}^m \) and \( k \) a non-negative integer. There exists a sequence of sets \( A, i \in [0, 1] \) and maps \( \varphi_i, i \in \mathbb{N} \) such that \( A_0 = \varnothing \subset A \subset \mathbb{R} \) \( \cap A \subset \mathbb{N} \) and for \( i \geq 1 \) \( \varphi_i \) is a \( C^1 \) homeomorphism of the ball \( B^m_{\varepsilon_i} \) into \( \mathbb{R}^m \) with \( A_i \subset B^m_{\varepsilon_i} \) and \( B^m_{\varepsilon_i} \) is the set of \( x \in \mathbb{R}^m \) with \( \| x \| < \varepsilon_i \).

\[ \| \varphi_i \| < \varepsilon_i \| x \| \]

for any \( f \in C^k \) vanishing on \( A \) there exist monotone functions \( \delta \varepsilon_i \) \( x \in \mathbb{R}^m \) with \( \lim \delta \varepsilon_i \varepsilon_i = 0 \) such that

\[ \| \varphi_i \| < \delta \varepsilon_i \| x \| \]

for all \( x, y \in B^m_{\varepsilon_i} \) with \( \varphi_i \) \( y \in A \).
This theorem has been proved in [1] in this theorem the m-dimensional Euclidean space is finite-dimensional. The problem of Bates is to ask if \( \mathbb{R}^n \) can be replaced by an infinite-dimensional space in particular the separable Hilbert space \( L^2 \).

In the next section we will prove there exists a \( C^\infty \) rank-1 map \( f : \mathbb{F}^2 \rightarrow \mathbb{R}^1 \) such that \( \mathbb{F} A \) has nonempty interior for some subset \( A \subseteq \mathbb{F}^2 \) of critical points with finite Hausdorff dimension. If the Morse criticality theorem in the infinite-dimensional space is true then we can easily prove that \( \mathbb{F} A \) is a 0-dimensional subset in \( \mathbb{R}^1 \). It is a contradiction.

§ 4 Construction of the example

Now we define the function \( F : H \rightarrow \mathbb{R} \) as follows. For \( x \in H \setminus \{ x_1 \dot{} x_2 \dot{} \ldots \} \in H \) set
\[
E x \equiv \sum_{n=1}^{\infty} 2^{-n} 3 \cdot 2^{-\frac{3}{4}} x_n^2 - 2x_n^3.
\]
This series is clearly convergent for all \( x \in H \). We now show that \( F \) is of class \( C^\infty \).

The sequence of maps \( D^k \) \( x \in H \rightarrow D^k x \) \( \in L^1 \) \( H^+ \) is given by
\[
D^1_x = \sum_{n=1}^{\infty} 2^{-n} 3 \cdot 2^{-\frac{3}{4}} x_n^2 - 6x_n e_n e_n^2
\]
\[
D^2_x = \sum_{n=1}^{\infty} 2^{-n} 3 \cdot 2^{-\frac{3}{4}} - 12x_n e_n e_n^2
\]
\[
D^3_x = 12 \sum_{n=1}^{\infty} 2^{-n} e_n e_n^2
\]
\[
D^k_x \equiv 0 \quad \text{for all } k \geq 4.
\]

It is easy to know that \( D^k \) converges and defines a member of \( L^1 \) \( H^+ \) for each \( x \in H \) and \( k \in \mathbb{N} \). And it is easy to verify that the map \( x \rightarrow D^k x \) is continuous for every \( k \in \mathbb{N} \) for any \( r \in H \) there is a constant \( M_k > 0 \) only depends on \( k \) such that \( \| D^k x \| \leq M \| r \|^2 \). It then follows that \( F \) is of class \( C^\infty \).

From the formula for \( D^k \) \( x \equiv D^1_x \) it follows that \( x \) is a critical point if and only if \( x_n = 0 \) \( x_n \dot{} x_n \in \mathbb{Z} \). Let \( \gamma = \{ x \} x_1 \dot{} x_2 \dot{} x_n \in \mathbb{Z} \). Then we have \( \mathbb{F} \gamma \equiv \mathbb{F} x \equiv \mathbb{F} \equiv \mathbb{F} x \equiv \mathbb{F} 0 \equiv 0 \equiv ^0 \equiv ^0 \).

It is easy to see that \( F \) is a rank-1 map by the definition of rank-1. And \( \gamma \) is a set of rank 0 for \( F \). Then this \( F \) is a \( C^\infty \) map such that the set of critical points is the Cantor set \( \gamma \subset H \) with \( \text{dim} \gamma \equiv 4 \equiv 0 \equiv ^0 \equiv 0 \equiv ^0 \).

It remains to verify that \( \text{dim} \gamma \equiv 4 \equiv 0 \equiv ^0 \equiv 0 \equiv ^0 \equiv 0 \equiv ^0 \). Let \( U^x = \{ y \in H \} \| x - y \| < \varepsilon \) be an open set for \( x \in H \varepsilon > 0 \) in \( H \). And fix \( k \) let \( \varepsilon_k = \frac{2^{-k/4}}{1 - 2^{-1/4}} \) then \( U^k \) can cover at most one of \( \{ x_n \} x_n \in \mathbb{Z} \) \( 2^{-i/4} \) for \( i \equiv k \) and \( x_i = 0 \) for \( i > k \). Thus at least \( 2^k + 1 \) open sets of radius \( \varepsilon_k \) are required to...
cover $\mathcal{C}$ so we have
\[
\dim_{\mathbb{F}} \mathcal{C} \leq \dim_{\mathbb{F}} \mathcal{C} \leq \lim_{k \to \infty} \frac{\log 2^k + 1}{2^{-k/4}} = 4.
\]

For $m \geq 2$ the function $f : \mathbb{R}^{m-1} \otimes \mathbb{R} H \to \mathbb{R}^m$ can be defined as follows for $x = x_1, x_2, \ldots, \tilde{x} \in H$ and $y \in \mathbb{R}^{m-1}$ by $y \otimes_{\otimes_{\mathbb{R}}} x_1, x_2, \ldots, \tilde{x} \in \mathbb{R}^m$ and $y \in \mathbb{R}^{m-1}$. Then $f : \mathbb{R}^{m-1} \otimes \mathbb{R} \mathcal{C} = \mathbb{R}^{m-1}$ but $f$ is a rank-$m$ map.

On the other hand for $m \geq 2$ the function $f : \mathbb{H} \to \mathbb{R}^m$ can be defined as follows for $x = x_1, x_2, \ldots, \tilde{x} \in H$ and $y \in \mathbb{R}^m$ where the number of $x$'s is $m$. Then the set $\mathcal{C} \subset H$ is a subset of critical set for $f$ and $\dim_{\mathbb{F}} \mathcal{C} \leq 4$ but $\mathcal{C} = \{ y = y \in \mathbb{R}^m \mid y_1 \in \mathbb{R} \bigcup \bigcup y_1 = y_2 = \ldots = y_m \}$ is a one-dimensional subset in $\mathbb{R}^m$. And $f$ is a rank-$m$ map.

For $m \geq 2$ the function $f : \mathbb{H} \to \mathbb{R}^m$ can be defined as follows for $x = x_1, x_2, \ldots, \tilde{x} \in H$ and $y \in \mathbb{R}^m$ where the number of $0$ is $m - 1$. Then the set $\mathcal{C} \subset H$ is a subset of critical set for $f$ and $\dim_{\mathbb{F}} \mathcal{C} \leq 4$ but $\mathcal{C} = \{ y \in \mathbb{R}^{m-1} \}$ is a one-dimensional subset in $\mathbb{R}^m$. Meanwhile $f$ is a rank-$1$ map.

References
