Existence time for solutions of semilinear different speed
considered by Fang [6]. In this paper, we devote to study the problem in dimension \( d \geq 2 \). We consider the following semilinear Klein-Gordon system

\[
\begin{align*}
\Box u + \mu^2 u &= F_1(u, v, \partial_t u, \partial_t v, \partial_x u, \partial_x v), \\
\Box v + v &= F_2(u, v, \partial_t u, \partial_t v, \partial_x u, \partial_x v), \\
\partial_{t^2} u |_{t=0} &= \epsilon_0, \\
\partial_{t^2} v |_{t=0} &= \epsilon_1, \\
v |_{t=0} &= \epsilon v_1,
\end{align*}
\]

with different propagation speeds in multi-space dimension, where \((t, x)\) are coordinates on \( \mathbb{R}^{1+d} \) with \( d \geq 2 \), \( \Box = \partial^2_t - \sum_{j=1}^d \partial_j^2 \) with different propagation speeds in \( d \geq 2 \), \( \Box = \partial^2_t - \partial^2_j \), \( \Box = \partial^2_t - \partial^2_j - \mu^2 \sum_{j=1}^d \partial^2_j \), \( 0 < \mu \neq 1 \), \( U(t, x) = (u(t, x), v(t, x)) \), \( u(t, x) = (u_1, \cdots, u_m), v(t, x) = (v_1, \cdots, v_{m_2}) \), \( m = m_1 + m_2 \). We assume that functions \( F_i \) satisfy:

\[
F_1(U, \partial_t U, \partial_x U) = G_0^1(u, v) + \sum_{1 \leq i \leq m_1, 1 \leq j \leq m_2} G_{i,j}^1(u)v_i \partial_j v_j + \sum_{1 \leq i \leq m_1, 1 \leq j \leq m_2} G_{i,j}^2(u)u_i \partial_j v_j \]

and

\[
F_2(U, \partial_t U, \partial_x U) = G_0^2(u, v) + \sum_{1 \leq i \leq m_1, 1 \leq j \leq m_2} \tilde{G}_{i,j}^1(v)\partial_i u_i + \sum_{1 \leq i \leq m_1, 1 \leq j \leq m_2} \tilde{G}_{i,j}^2(v)\partial_i v_j + \sum_{1 \leq i \leq m_1, 1 \leq j \leq m_2} \tilde{G}_{i,j}^3(v)\partial_i u_i \partial_j u_j \]

where \( G_{i,j}^k, \tilde{G}_{i,j}^k \) and \( \tilde{G}_{i,j}^k \), with \( k = 1, 2, \ell = 1, 2, 3 \), are polynomials vanishing at least at order 2 at \( 0 \), \( q_k, k = 1, 2, \ell = 1, 2, 3 \), are the bilinear form

\[
q_k(T, X; T', X') = TT' - XX'.
\]

The main theorem is the following:

**Theorem 1.1** Let \( N \in \mathbb{N}, N \geq (d + 3)/2 \). There is a constant \( c > 0 \), such that for any pair \((U_0, U_1)\) in the unit ball of \( H^N(\mathbb{R}^d) \times H^{N-1}(\mathbb{R}^d) \), and any \( \epsilon \in [0, 1] \), problem (2) has a unique solution \( U \in C^0([-T_c, T_c], H^N) \cap C^1([-T_c, T_c], H^{N-1}) \) with the existence time \( T_c > c\epsilon^{-2}|\log \epsilon|^{-\alpha} \) with \( \alpha = 2 \) if \( d \geq 3 \) and \( \alpha = 3 \) if \( d = 2 \).

In the whole of the paper, we shall denote by \( \| \cdot \| \) \( L^2 \)-norms, and call “universal constant” a constant which does not depend on the different parameters.

## 2 Reduction to a Local Problem

We shall use rescaling technique to reduce the proof of the theorem to a local existence problem with Cauchy data of limited smoothness. Let us denote by \( \Delta_0 \) (resp. \( \Delta_1, j \in \mathbb{N}^* \)) the Fourier multiplier on \( L^2(\mathbb{R}^d) \) with symbol \( \phi_0 = \mathbb{1}_{\{|\xi| < 1\}} \) (resp. \( \phi_j = \mathbb{1}_{\{2^{-j-1} < |\xi| < 2^{-j}\}} \)).

**Definition 2.1** If \( s \in \mathbb{R}, N \in \mathbb{N} \), we shall denote by \( H^N_0(\mathbb{R}^d) \) (or \( H^N(\mathbb{R}^d, \mathbb{R}^m) \)) the space of families of functions \((u^h)_{h \in [0, 1/2]} \), which are in \( L^2(\mathbb{R}^d) \) for fixed \( h \), such that there is a sequence \((\epsilon_j(h))_j \) in the unit ball of \( \ell^2(\mathbb{N}) \) and \( C > 0 \) satisfying

\[
\| \Delta_j u^h \| \leq Cc_j(h)h^s|\log h|^{-\alpha}(1 + 2^j h)^{-N}.
\]
where we put $\nu = 1$ if $d \geq 3$ and $\nu = \frac{3}{2}$ if $d = 2$. The best constant $C$ in (5) defines the norm $\| (u^h)_h \|_{H^\delta_N}$.

It is not difficult to know that Theorem 1.1 will be a consequence of the following result:

**Theorem 2.2** Let $N$ be an integer, $N \geq (d + 3)/2$. There exists $\delta > 0$ such that for any families $(V^h)_h \in [0,1/2]$, (resp. $(W^h)_h \in [0,1/2]$) in $H^\frac{d+1}{N}$ (resp. in $H^\frac{d+1}{N-1}$), of norms in these spaces smaller than $\delta$, the system

\[
\begin{cases}
\Box \mu u^h + \mu^2 \frac{1}{1} u^h = \frac{1}{\sqrt{2}} F_1(u^h, h\partial_t U^h, h\partial_x U^h), \\
\Box u^h + \frac{1}{\sqrt{2}} u^h = \frac{1}{\sqrt{2}} F_2(u^h, h\partial_t U^h, h\partial_x U^h), \\
U^h|_{t=0} = V^h, \partial_t U^h|_{t=0} = W^h
\end{cases}
\]  

(6)

has a solution $(U^h)_h \in C^0[-1,1], H^\frac{d+1}{N} \cap C^1[-1,1], H^\frac{d+1}{N-1}$.

We shall define some spaces. If $j,k \in \mathbb{N}$, and $t_\mu \in \{\mu,1\}$, we define

\[
\Phi_{j,k}^{\pm,t_\mu}(\tau, \xi) = \mathbb{I}_{\{t_\mu \geq 0\}} \mathbb{I}_{\{2^{j-1} \leq |\xi| < 2^j\}} \mathbb{I}_{\{2^{k-1} \leq |\tau| \leq t_\mu \sqrt{1+|\xi|^2} \leq 2^k\}}
\]

if $j > 0, k > 0$,

\[
\Phi_{0t}^{\pm,t_\mu}(\tau, \xi) = \mathbb{I}_{\{t_\mu \geq 0\}} \mathbb{I}_{\{|\xi| < 1\}} \mathbb{I}_{\{2^{k-1} \leq |\tau| \leq t_\mu \sqrt{1+|\xi|^2} \leq 2^k\}}
\]

if $k > 0$,

\[
\Phi_{j0}^{\pm,t_\mu}(\tau, \xi) = \mathbb{I}_{\{t_\mu \geq 0\}} \mathbb{I}_{\{2^{j-1} \leq |\xi| < 2^j\}} \mathbb{I}_{\{|\tau| \leq t_\mu \sqrt{1+|\xi|^2} \leq 1\}}
\]

if $j > 0$,

\[
\Phi_{00}^{\pm,t_\mu}(\tau, \xi) = \mathbb{I}_{\{t_\mu \geq 0\}} \mathbb{I}_{\{|\xi| < 1\}} \mathbb{I}_{\{|\tau| \leq t_\mu \sqrt{1+|\xi|^2} \leq 1\}}
\]

Let us define the corresponding Fourier multipliers as

\[
\triangle_{j,k}^{\pm,t_\mu} u = \mathcal{F}^{-1}(\Phi_{j,k}^{\pm,t_\mu}(\tau, \xi) \hat{u}(\tau, \xi))
\]

for $u \in L^2(\mathbb{R}^{1+d})$, where we used Fourier transform and inverse Fourier transform in space-time variables.

**Definition 2.3** Let $s,s'$ be real numbers, $N \in \mathbb{N}$. We shall denote by $H^{s,s'}_{N,t_\mu}(\mathbb{R}^{1+d})$ the space of families $(u^h)_h \in [0,1/2]$ of $L^2$-functions on $\mathbb{R}^{1+d}$, with values in $\mathbb{R}$ (or in $\mathbb{R}^m$), such that there is a sequence $(c_{j,k}(h))_{(j,k) \in \mathbb{N}^2}$ satisfying

\[
\sum_{j} \sum_{k} \| c_{j,k}(h) \|^2 \leq 1
\]

and a constant $C > 0$ with

\[
\| \triangle_{j,k}^{s,s'} u^h \| \leq C c_{j,k}(h) \log h^{-\nu} 2^{-2s'}(1 + 2^j h + 2^k h)^{-N}
\]

(7)

for any $j,k \in \mathbb{N}$, $h \in [0,1/2]$, where $t_\mu \in \{\mu,1\}$, $\nu = \frac{3}{2}$ if $d = 2$ and $\nu = 1$ if $d \geq 3$. The best constant $C$ in (7) defines the norm of $(u^h)_h$ in $H^{s,s'}_{N,t_\mu}$.
In the rest of this paper, we shall use the following notation: if \( b(\tau, \xi; \tau', \xi') \) is a locally bounded function on \( \mathbb{R}^{1+d} \times \mathbb{R}^{1+d} \), and if \( f' \) and \( g \) are \( L^2 \) functions with compactly supported Fourier transforms, we shall define \( B(f, g) \) by
\[
B(f, g)(\tau, \xi) = \int f(\tau - \tau', \xi - \xi') \hat{g}(\tau', \xi') b(\tau, \xi; \tau', \xi') \, d\tau' \, d\xi';
\]
(8)
If \( j, j' \) and \( j'' \) are nonnegative integers, we shall set \( j \ll j' \) if \( j \leq j' - 2 \) and \( j \sim j' \) if \( |j - j'| \leq 3 \). Then, if \( u, v \in L^2(\mathbb{R}^{1+d}) \) and \( \Delta_j^{r, r'} u \Delta_j^{r, r'} v \neq 0 \) for \((r, r', r'') \in \{\mu, 1\}^3 \), one has
\[
(j \ll j' \text{ and } j \sim j' \text{ or } (j' \ll j \text{ and } j \sim j' \text{ or } (j'' \geq 5 \text{ and } j \sim j')).
\]
(9)
To study \( B(u, v) \) for \((u, v) \in H^{d+1/2}_{N, r} \times H^{d+1/2}_{N, r'}\), we shall decompose
\[
u = \sum_{e \in \{+,-\}} \sum_{j, k} \Delta_j^e u, \quad \nu' = \sum_{e \in \{+,-\}} \sum_{j, k} \Delta_j^{e'} u',
\]
(10)
and estimate \( \Delta_{j, k}^{e''}(B(u, v)) \). Decompose it from (10) as
\[
\Delta_{j, k}^{e''}(B(u, v)) = \sum_{e, e', j, j', k, k'} \Delta_{j, k}^{e''}(B(\Delta_{j, k}^{e''} u, \Delta_{j, k}^{e''} v))
\]
(11)
for fixed \( e'' \in \{+,-\}, j'', k'' \in \mathbb{N}, T = (r, r', r'') \in \{\mu, 1\}^3 \). Using the same proof as that in [2], we have

**Lemma 2.4** There is a positive constant \( C \) such that for any \( J = (j, j', j'') \in \mathbb{N}^{3}, K = (k, k', k'') \in \mathbb{N}^{3}, E = (e, e', e'') \in \{+,-\}^{3} \) and \( T = (r, r', r'') \in \{\mu, 1\}^{3} \), we have
\[
\| \Delta_{j, k}^{e''}(B(\Delta_{j, k}^{e''} u, \Delta_{j, k}^{e''} v)) \| \leq C 2^{\frac{3}{2} + \frac{\mu}{2}} \| b \|_{L^{\infty}(A(T, J, K, E)))} \| \Delta_{j, k}^{e''} u \| \| \Delta_{j, k}^{e''} v \|
\]
where \( \tilde{j} = \min(j, j', j'') \), \( \tilde{k} = \min(k, k', k'') \).

### 3 Microlocal estimates

**Theorem 3.1** Let \( M_0 \) be a fixed positive integer. There is a constant \( C > 0 \) such that for any \((j, j', j'') \in \mathbb{N}^{3}, k \leq k' \leq k'' \in \mathbb{N}^{3}, \) \( e \in \{+,-\} \) and \( T = (r', r'', r'') \in \{\mu, 1\}^{3} \),
\[
\| \Delta_{j, k}^{e''}(B(\Delta_{j, k}^{e''} u, \Delta_{j, k}^{e''} v)) \| \leq C 2^{\frac{3}{2} + \frac{\mu}{2}} \| b \|_{L^{\infty}(A(T, J, K, E)))} \| \Delta_{j, k}^{e''} u \| \| \Delta_{j, k}^{e''} v \|
\]
(12)
for \( r \neq r' \).
\[
\| \Delta_{j, k}^{e''}(B(\Delta_{j, k}^{e''} u, \Delta_{j, k}^{e''} v)) \| \leq C 2^{\frac{3}{2} + \frac{\mu}{2}} \| b \|_{L^{\infty}(A(T, J, K, E)))} \| \Delta_{j, k}^{e''} u \| \| \Delta_{j, k}^{e''} v \|
\]
(13)
for \( r = r' \).
\[
\| \Delta_{j, k}^{e''}(B(\Delta_{j, k}^{e''} u, \Delta_{j, k}^{e''} v)) \| \leq C 2^{\frac{3}{2} + \frac{\mu}{2}} \| b \|_{L^{\infty}(A(T, J, K, E)))} \| \Delta_{j, k}^{e''} u \| \| \Delta_{j, k}^{e''} v \|
\]
(14)
for all \( T \), where \( \tilde{j} = \min(j, j', j'') \), \( \tilde{j} = \max(j, j', j'') \), \( T = (r', r'', r'') \in \{\mu, 1\}^{3} \).
Remark that if \( j = 0 \), estimates (12), (13) and (14) are implied by inequality (12). As a consequence, we will be free in the sequel to assume \( j > 0 \), \( j' > 0 \), \( j'' > 0 \). We put \( f = \Delta_{jk}^{+}u \), 
\( g = \Delta_{jk}^{+}v \) and study the \( L^2 \) norm of

\[
I_{\pm}^\pm (\tau, \xi) = \Phi_{jk}^{\pm'}(\tau, \xi) \int f(\tau - \tau', \xi - \xi') \hat{g}(\tau', \xi') b(\tau, \xi; \tau', \xi') \, d\tau' d\xi'.
\]  

(15)

By the preceding remark, we will always have \( |\xi| > 1 \), \( |\xi'| > 1 \), \( |\xi - \xi'| > 1 \). Let us remark that it is equivalent to prove (2.1.1) or the corresponding inequality in which \( \Delta_{jk}^{+}u, \Delta_{jk}^{+}v \) is replaced by \( \Delta_{jk}^{-}u, \Delta_{jk}^{+}v \), for \( \Delta_{jk}^{-}u = \Delta_{jk}^{+}u, \Delta_{jk}^{+}v = \Delta_{jk}^{+}v \).

To prove theorem 2.1.1, we define two functions depending on \( h \) on \( \mathbb{R}^{1+d} \times \mathbb{R}^{1+d}, \) for \( k \in \mathbb{N}, \ell \in \mathbb{Z} \), as

\[
F_\pm^\mu(\xi, \xi') = \mu \sqrt{h^{-2} + (\xi - \xi')^2} \pm \sqrt{h^{-2} + \xi'^2},
\]  

(16)

\[
\lambda_\pm^\mu(\xi, \xi') = 1_{\{\xi^2 < F_\pm^\mu(\xi, \xi') < (\ell + 1)^2\}}.
\]  

(17)

Put

\[
\theta = 2^{-j'} \xi, \theta' = 2^{-j'} \xi'.
\]

We have \( 1/2 \leq |\theta| \leq 1, 1/2 \leq |\theta'| \leq 1 \) since in the integrand of (15) we have \( \xi \in \text{Supp} \phi_{jk}^\nu \) and \( \xi' \in \text{Supp} \phi_{jk}^\nu \). Let us define also

\[
\rho = 2^{j'-j}, \sigma = (1 + 2^j h)^{-1}, m = 1 - \sigma = 2^j h (1 + 2^j h)^{-1},
\]  

(18)

and for a given positive number \( M \),

\[
K_M = \{(\theta, \theta', \sigma, \rho) \in \mathbb{R}^{1+d} \times \mathbb{R}^{1+d} \times [0, 1] \times [0, M]; 1/2 \leq |\theta| \leq 1, 1/2 \leq |\theta'| \leq 1 \}.
\]  

(19)

We rescale \( F_\pm^\mu \) defining the following functions on \( K_M \):

\[
G_\pm^\mu(\theta, \theta', \sigma, \rho) = h(1 + 2^j h)^{-1} F_\pm^\mu(2^{j'} \theta, 2^{j'} \theta')
\]

\[
= \mu (\sigma^2 + m^2 (\rho \theta - \theta')^2)^{1/2} \pm (\sigma^2 + m^2 \theta'^2)^{1/2}.
\]  

(20)

Lemma 3.2 i). The function \( \theta' \rightarrow G_\pm^\mu(\theta, \theta', \sigma, \rho) \) has no critical point if \( \sigma \leq \sqrt{1 - \mu^2} \).

ii). The critical point, \( \theta_\pm \), of the function \( \theta' \rightarrow G_\pm^\mu(\theta, \theta', \sigma, \rho) \), if it exists, is unique on \( K_M \) and is collinear to the parameter \( \theta \). Let \( \theta_\pm = \frac{\rho \theta}{1 + \lambda_\pm} \), we have \( \frac{1}{\mu} \leq |\lambda_\pm| \leq C(M) \) with some positive number \( C(M) \) depending only on \( M \).

proof: From

\[
\nabla_{\theta'} G_\pm^\mu(\theta, \theta', \sigma, \rho) = \frac{-\mu m^2 (\rho \theta - \theta')}{\sigma^2 + m^2 (\rho \theta - \theta')^2} \pm \frac{m^2 \theta'}{\sqrt{\sigma^2 + m^2 \theta'^2}} = 0
\]  

(21)

we know that \( \rho \theta - \theta' \) and \( \theta' \) are collinear, and hence there is some constant \( \lambda_\pm \) so that \( \rho \theta - \theta' = \lambda_\pm \theta' \). Obviously \( \lambda_\pm \neq 0 \) and \( 1 + \lambda_\pm \neq 0 \). It follows from (21) that

\[
\pm \lambda_\pm > 0 \quad \text{and} \quad \frac{\sigma^2 + m^2 \theta'^2}{(\lambda_\pm^2 - 1) \sigma^2} = \frac{1}{\mu^2 - 1},
\]  

(22)
which means $|\lambda_\pm| > 1$ because of $0 < \mu < 1$. Denote by

$$K(\lambda) = \frac{\sigma^2 + m^2\theta^2}{(\lambda^2 - \lambda_\pm^2)\sigma^2} = \frac{1 + \frac{m^2\rho^2\theta^2}{(1 - \lambda_\pm^2)\sigma^2}}{1 - \frac{1}{\lambda^2}}.$$ 

It is obvious that $K(\lambda)$ is a strictly decreasing function when $\lambda > 1$ and is a strictly increasing function when $\lambda < -1$. Moreover we have

$$K(\pm\mu^{-1}) > \frac{1}{1 - \mu^2} \quad \text{and} \quad \lim_{\lambda \to \pm\infty} K(\lambda) = \frac{1}{1 - \mu^2}.$$ 

Then the system

$$\pm \lambda > 0, K(\lambda) = \frac{1}{1 - \mu^2}$$

has a unique solution $\lambda_\pm$ with $|\lambda_\pm| > \frac{1}{\mu}$, and hence $G^\mu_\pm(\theta', \sigma, \rho)$ has a unique critical point.

That $\sigma \geq \sqrt{1 - \frac{\mu^2}{3}}$ follows from (2.1.13), the assumption $m + \sigma = 1$ and $\frac{1}{2} \leq |\theta'| \leq 1$. What remains is to prove that $|\lambda_\pm| < C(M)$ for some positive constant. By $K(\lambda_\pm) = \frac{1}{1 - \mu^2}$, $\sigma \geq \sqrt{1 - \frac{\mu^2}{3}}$ and $|\lambda_\pm| \geq \frac{1}{\mu}$ we know

$$\frac{1}{1 - \mu^2} \leq \frac{1 + M^2(1 + \lambda_\pm)^{-2}\sigma^{-2}}{1 - \lambda_\pm^{-2}} \leq \frac{1 + 9M^2(1 + \lambda_\pm)^{-2}(1 - \mu^2)^{-1}}{1 - \lambda_\pm^{-2}}.$$ (23)

Notice that

$$0 < \mu < 1, \quad \frac{1 + 9M^2(1 + \lambda_\pm)^{-2}(1 - \mu^2)^{-1}}{1 - \lambda_\pm^{-2}} \to 1 \quad \text{as} \quad \lambda_\pm \to \pm\infty.$$ 

We deduce from (23) that there exists a positive constant $C(M)$, depending only on $M$, such that $|\lambda_\pm| \leq C(M)$. □

In the sequel, we will write $\theta' = (\theta'_1, \theta''_1) \in \mathbb{R} \times \mathbb{R}^{d-1}$ and $\theta = |\theta|(1, 0, \ldots, 0)$. It follows from Lemma 3.2 that the critical point, $\theta_\pm$, of the function $\theta' \to G^\mu_\pm(\theta', \sigma, \rho)$, if it exists, is unique on $K_M$. Let us define

$$D_\pm = \left\{ (\theta, \theta', \sigma, \rho) \in K_M : \theta' = \theta_\pm, \sigma \geq \sqrt{1 - \frac{\mu^2}{3}} \right\}. \quad (24)$$

**Lemma 3.3** For any $M > 0$ and any neighborhood $V_\pm$ of $D_\pm$ in $K_M$, there is a positive constant $C$ and a real valued $C^1$ function $H_\pm$, defined on a neighborhood of the set $K_M - V_\pm$, such that

$$H_\pm \quad \text{is bounded on} \quad K_M - V_\pm, \quad C^{-1} \leq |\nabla_{\theta'} H_\pm(\theta, \theta', \sigma, \rho)| \leq C \quad \text{on} \quad K_M - V_\pm,$$

$$G_\pm(\theta, \theta', \sigma, \rho) = (\mu \pm 1)\sigma + m^2 H_\pm(\theta, \theta', \sigma, \rho) \quad \text{on} \quad K_M - V_\pm. \quad (25)$$

**proof:** Denote by

$$H_\pm = \frac{G_\pm - (\mu \pm 1)\sigma}{m^2}.$$
We first prove the boundedness of $H_{\pm}$. When $m \geq 1/2$, it is obvious that $|H_{\pm}| \leq C$ for some positive number $C$. When $0 \leq m \leq 1/2$, we have $\sigma = 1 - m \geq 1/2$ and

$$H_{\pm} = \frac{\mu (\rho - \theta')^2}{(\sigma^2 + m^2(\rho - \theta')^2)^{1/2}} + \frac{\theta^2}{(\sigma^2 + m^2\zeta \theta')^2)^{1/2}}$$

for some $\zeta \in (0, 1)$, and hence $|H_{\pm}| \leq C$ since $(\sigma^2 + m^2(\rho - \theta')^2) \geq 1/4$, $|\rho - \theta'| \leq M + 1$ and $1/2 \leq |\theta'| \leq 1$. Notice that

$$\nabla_{\theta'} H_{\pm} = \frac{-\mu (\rho - \theta')}{(\sigma^2 + m^2(\rho - \theta')^2)^{1/2}} \pm \frac{\theta'}{(\sigma^2 + m^2\zeta \theta')^2)^{1/2}}.$$ 

Since $|\nabla_{\theta'} H_{\pm}|_{(\sigma = 0)} = |\mu + 1| > 0$, we can choose $\sigma_0 > 0$ small so that

$$|\nabla_{\theta'} H_{\pm}| \leq 2|\mu + 1| \text{ on the set } K_M \cap \{\sigma = \sigma_0\}.$$ 

On the compact set $K_M \cap \{\sigma \geq \sigma_0\}$, the norm $|\nabla_{\theta'} H_{\pm}|$ is a continuous function and vanishes only if $\theta' = \theta_{\pm}$. The boundedness of $|\nabla_{\theta'} H_{\pm}|$ now follows from the definition of $V_{\pm}$.

Let us introduce, for $\delta$ small positive number, the following open subsets of $K_M$:

$$V_{\pm}^\delta = \left\{ (\theta, \theta', \sigma, \rho) \in K_M; |\theta' - \theta_{\pm}| < \delta, \sigma > \frac{\sqrt{1 - \mu^2}}{6} \right\} \quad (26)$$

**Lemma 3.4** Let $M$ be a fixed positive number. There is $C > 0$, $\delta > 0$, and two real valued $C^1$ functions $H_{\pm}^n$ and $H_{\pm}^2$, defined on an open neighborhood of $V_{\pm}^\delta \cap \{\rho > 0\}$, bounded with bounded first $\theta'$-derivative on $V_{\pm}^\delta \cap \{\rho > 0\}$, such that

$$C^{-1} \leq |H_{\pm}^n(\theta, \theta', \sigma, \rho)| \leq C, \quad n = 1, 2, \quad (27)$$

$$G_{\pm}^\mu(\theta, \theta', \sigma, \rho) = G_{\pm}^\mu(\theta, \theta_{\pm}, \sigma, \rho) + m^2 \sigma^2 (\theta' - \theta_{\pm})^2 H_{\pm}^1(\theta, \theta', \sigma, \rho) + m^2 \sigma^2 H_{\pm}^2(\theta, \theta', \sigma, \rho)$$

on $V_{\pm}^\delta \cap \{\rho > 0\}$.

**proof:** Denote by

$$\theta = |\theta|(1, 0, \cdots, 0), \theta' - \theta_{\pm} = \eta = (\eta_1, \eta''), \quad A = [\sigma^2 + m^2(\rho - \theta_{\pm})^2]^{1/2}, \quad B = [\sigma^2 + m^2\theta_{\pm}^2]^{1/2}.$$ 

Then $\eta_1 = \theta' - \theta_{\pm}$, $\eta'' = \theta''$, $\theta_{\pm} = \frac{1}{1 + \lambda_{\pm}} |\theta|(1, 0, \cdots, 0)$ and

$$G_{\pm}^\mu(\theta, \theta', \sigma, \rho) = \mu A \left[ 1 + \frac{-2m^2(\rho - \theta_{\pm})\eta_1 + m^2\eta_2^2}{A^2} \right]^{1/2} \pm B \left[ 1 + \frac{2m^2\theta_{\pm} \eta_1 + m^2\eta_2^2}{B^2} \right]^{1/2}$$

$$= G_{\pm}^\mu(\theta, \theta_{\pm}, \sigma, \rho) + \mu \frac{(\rho - \theta_{\pm})^2}{4A^2} \eta_1 + \frac{\mu}{2B^2} \eta_1 \eta_2 + \frac{\mu}{12A^2} m^2 \sigma^2 \eta_1^2$$

$$+ \frac{\mu}{12A^2} M^2 \eta_2^2 + g_6 m^4 \eta_1^3 + g_6 m^4 \eta_1^3,$$

where $g_i (i = 1, 2, \cdots, 6)$ are bounded continuous functions on $K_M$. Notice that $\pm \lambda_{\pm} > 1/\mu$, $\sqrt{1 - \mu^2}/\sigma \leq A \leq C$, $\sqrt{1 - \mu^2}/\sigma \leq B \leq C$. Using $\nabla_{\theta'} G_{\pm}^\mu(\theta, \theta_{\pm}, \sigma, \rho) = 0$ and Lemma (3.2) we have

$$-\mu A (\rho - \theta_{\pm}) + \theta_{\pm} B = 0.$$
\[ \frac{\mu}{2} (2^{\mu} - 1) \leq \frac{\mu}{2A} \pm \frac{1}{2B} = \frac{\mu|\theta|}{2A} \leq \frac{\mu(1 + \lambda_{\pm})}{2A} \leq \frac{\mu(1 - \mu)(1 + C(M))}{2\mu}, \]

and

\[ C \leq \frac{\mu}{2A^3} \pm \frac{1}{2B^3} = \frac{\mu}{2A} \left(1 + \frac{\lambda_{\pm}}{\sigma^2} + \frac{m^2 \theta^2}{A^2 B^2} \right) \geq \frac{1}{C} > 0. \]

Let

\[ H^1_+ = \frac{\mu}{2A^3} \pm \frac{1}{2B^3} + \frac{g_0 m^2 \eta_1}{\sigma^2}, \quad H^2_+ = \frac{\mu}{2A} \pm \frac{1}{2B} + g_0 m^2 \eta''. \] (28)

The conclusion now follows from (28) and the fact \( \sigma \geq \frac{\sqrt{1 - \mu^2}}{6} \) and \( |\eta| < \delta \) small. \( \square \)

4 Proof of Theorem 3.1

We first prove the case \( r \neq r' \). Without loss of generality, let us consider the case \( r = \mu, r' = 1 \), that is, we want to estimate the integral (15) with \( f = \Delta^{\pm}_{k=1} u, g = \Delta^{\pm}_{k=1} v \). Remember that we assumed \( |j - j'| \leq M_{\delta} \). Together with condition (9), this implies that \( \rho = 2^{j'' - j'} \) stays smaller than a fixed positive constant \( M \). For \( \delta > 0 \) and \( \xi_{\pm} = 2^j \theta_{\pm} \) we denote by

\[ W^\delta_{\pm} = \left\{ (\xi, \xi') : \xi \in \text{Supp} \phi_{j''}, \xi' \in \text{Supp} \phi_{j'}, |\xi - \xi_{\pm}| < \delta 2^{j'}, \sigma \geq \frac{\sqrt{1 - \mu^2}}{6} \right\}, \]

and split \( \mathcal{I}_{\pm} = \mathcal{I}_{\pm} + \mathcal{I}_{\pm} \) where

\[ \mathcal{I}_{\pm} = \Phi_{j''} \int \hat{f} (\tau', \xi - \xi') \hat{g} (\tau', \xi') b (\tau, \xi; \xi', \xi') \mathbb{I}_{W^\delta_{\pm}} d\tau' d\xi'. \] (29)

To estimate \( \mathcal{I}_{\pm} \), we introduce a result considered in [3] (Lemma 2.1.3 and Lemma 2.1.4). For any triple \((j, j', j'')\) and any real number \( R \in [0, 2] \), let us fix a finite partition of \( \mathbb{R}^d - \{0\} \) in conical subdomain, \((\gamma_a)_{a \in S}, \) where \( S \) is a finite subset of \( S^{d-1} \), such that there is a universal constant \( c > 0 \) with

\[ \left\{ \xi', \xi'' \in \mathbb{R}^d - \{0\}, \left| \frac{\xi'}{|\xi'|} - a \right| < c 2^{j'' - j} R \right\} \subset \gamma_a \subset \left\{ \xi', \xi'' \in \mathbb{R}^d - \{0\}, \left| \frac{\xi'}{|\xi'|} - a \right| < 2^{j'' - j} R \right\}. \]

Set \( \mathcal{P} = \mathbb{N} \times S \) and for \( p = (q, a) \in \mathcal{P} \) define

\[ Q_p = \{ \xi' : 2^{j'' - 1} \leq |\xi'| \leq 2^{j'}, q 2^{j''} \leq |\xi'| \leq (q + 1) 2^{j''}, \xi' \in \gamma_a \}, \]

\[ \hat{Q}_p = \{ \eta : 2^{j'' - 1} \leq |\eta| \leq 2^{j'}, \text{ and } \exists (\xi, \xi') \text{ with } 2^{j'' - 1} \leq |\xi| \leq 2^{j'}, \xi' \in Q_p, \}

\[ \eta = \xi - \xi', |\xi' - (\xi' \frac{\xi}{|\xi'|} \frac{\xi}{|\xi'|})| \leq R 2^{j'} \}. \]

Then there is a universal constant \( C > 0 \) such that

\[ 1 = \sum_{p \in \mathcal{P}} \| Q_p \| \leq C. \]

Estimate of \( \mathcal{I}^n_{\pm} \)
We have $T''_\pm = \sum_{p \in \mathcal{P}} T''_{\pm p}$ with
\[
T''_{\pm p} = \Phi^{\ell'', r''}_{j'' k''} \int \tilde{f}(\tau - \tau', \xi - \xi') \tilde{g}_p(\tau', \xi') b(\tau, \xi; \tau', \xi') \mathbb{I}_{-W^2_\pm} \, d\tau' \, d\xi',
\]
where $\tilde{f}_p = \tilde{f} \mathbb{1}_{Q_p}$, $\tilde{g}_p = \tilde{g} \mathbb{1}_{Q_p}$. We write also $T''_{\pm p} = \sum \ell \ell' T''_{\pm p, \ell}$ with
\[
T''_{\pm p, \ell} = \Phi^{\ell'', r''}_{j'' k''} \int \tilde{f}(\tau - \tau', \xi - \xi') \tilde{g}_p(\tau', \xi') b(\tau, \xi; \tau', \xi') \mathbb{I}_{-W^2_\pm} \chi^{\ell''}_{\pm}(\xi, \xi') \, d\tau' \, d\xi'.
\]
The absolute value of $T''_{\pm p, \ell}$ is smaller than
\[
2^{k'} \phi_j(\xi) \left( \int |\tilde{f}(\tau - \tau', \xi - \xi')|^2 |\tilde{g}_p(\tau', \xi')|^2 |b|^{2} \mathbb{I}_{-W^2_\pm} \chi^{\ell''}_{\pm}(\xi, \xi') \, d\tau' \, d\xi' \right)^{1/2} \times \left( \int \chi^{\ell, \mu}_{\pm}(\xi, \xi') \mathbb{I}_{Q_p}(\xi) \mathbb{I}_{W^2_\pm} \, d\xi' \right)^{1/2}.
\]
The last integral is bounded from above by
\[
2^{j''} \int \mathbb{1}_{\{\ell L^{-1/2} \leq H_\pm \leq (\ell+1) L^{-1/2}\}} \mathbb{I}_{-W^2_\pm} \chi^{\ell''}_{\pm}(\xi, \xi') \, d\tau' \, d\xi' \, Q(\xi) \, d\theta',
\]
where $\tilde{Q}$, obtained from $Q_p$ by rotation, is contained in a cube of diameter $C2^{j''}$. Using Lemma 3.3, (31) can be controlled by
\[
2^{j''} \int \mathbb{1}_{\{\ell L^{-1/2} \leq H_\pm \leq (\ell+1) L^{-1/2}\}} \mathbb{I}_{-W^2_\pm} \chi^{\ell''}_{\pm}(\xi, \xi') \, d\tau' \, d\xi' \, Q(\xi) \, d\theta'
\]
with $L = 2^{k-j''}$. Since $\frac{1}{2} \leq |\nabla \phi_j| \leq C$, we can cover the domain of integration by a universally bounded finite number local charts, over which $H_\pm$ can be taken as a local coordinate satisfying universal bounds. This allows us to bound (32) by
\[
C 2^{j''} \frac{2^{k-j''}}{m} 2^{(d-j'')(d-1)}.
\]
Plugging into (30), using almost orthogonality and summing in $\ell$ and $p$, we get
\[
\|T''_\pm\| \leq C 2^{j''} \frac{2^{k-j''}}{m} 2^{(d-j'')(d-1)} \|f\| \|g\|
\]
(33) since $|j-j''| \leq M_0$ implies that $j''$ can be controlled by $j$.

**Estimate of $T'_\pm$**

We set $\xi' = \xi'_\pm + 2^{j'} \eta$, and let us decompose $T'_\pm = \sum \ell T'_\pm \ell$ with
\[
T'_\pm \ell(\tau, \xi) = \Phi^{\ell', r'}_{j' k'} \int \tilde{f}(\tau - \tau', \xi - \xi') \tilde{g}(\tau', \xi') b(\tau, \xi; \tau', \xi') \mathbb{I}_{W^2_\pm} \, d\tau' \, d\xi'.
\]
Using almost orthogonality (see [3]), we have $\|T'_\pm\| \leq C(\sum_{\ell} \|T'_\pm \ell\|^2)^{1/2}$ and
\[
|T'_\pm \ell| \leq 2^{j'/2} \left( \int |\tilde{f}(\tau - \tau', \xi - \xi')|^2 |\tilde{g}(\tau', \xi')|^2 b^2 \chi^{\ell', \mu}_{\pm}(\xi, \xi') \, d\tau' \, d\xi' \right)^{1/2} \times 2^{j'/2} \left( \int \mathbb{1}_{W^2_\pm(\xi, \xi' + 2^{j'} \eta)} \chi^{\ell', \mu}_{\pm}(\xi, \xi' + 2^{j'} \eta) \, d\eta \right)^{1/2}.
\]
(34)
\[ \int \mathbb{1}_{\{|\eta'|<\delta, |\eta_1'|<\delta\}^2} (e^{L-T-c\frac{\eta_1^2}{H_\pm^1} + \eta'\sigma^2 H_\pm^2 < (\ell+1)L-T}) \, d\eta; \]  

where \( L = 2^{k-j}/m, T = G^m_\pm(\theta, \theta', \sigma, \rho)/m^2 \). Performing the change of variables inverse to \( w_1 = \eta_1(H_\pm^1)^{\frac{j}{2}} \) and \( w_2 = \eta''(H_\pm^2)^{\frac{j}{2}} \), we get the upper bound of (35)

\[ C \int \mathbb{1}_{\{|w_1|+w_2|\leq C\delta\}^2} (e^{L-T-c\frac{\eta_1^2}{w_1^2} + \eta'\sigma^2 w_2^2 < (\ell+1)L-T}) \, dw_2 \leq C2^{k-j'}(1+2^{j}h)^2(2^{j'}h)^{-1}. \]

Plugging into (34), using the fact that \( j' \) can be controlled by \( j \) and summing in \( \ell \) we obtain

\[ \|I_\pm\| \leq C2^{j} \mathbb{I}_{\pm \gamma} + \frac{j}{2} (1+2^{j}h)^{\frac{j}{2}} (2^{j'}h)^{-\frac{j}{2}} \|b\| \|f\| \|g\|. \]

For the case of \( r = r' \) in Theorem 3.1, we can get the conclusion from Theorem 2.1.1 in [3].

We wish to bound \( \|\Delta_{j'k'}^m \theta'\theta''(B(\Delta_{j,k}^+ u, \Delta_{j,k}^+ r'v))\| \) when \( k' \leq k \leq k'' \), \( e'' = \pm \) and \( j \) is much smaller than \( j' \sim j'' \) or \( j' \) much smaller than \( j \sim j'' \). We will study only the second case, since the first one is similar. The result is the following one:

**Proposition 4.1** There is a positive integer \( N_0 \) such that if \( j' \leq j'' - N_0, k' \leq k \leq k'' \) and \((r,r',r'')\in\{\mu,1\}\), one has a universal constant \( C \) with

\[ \left\| \Delta_{j'k'}^m \theta'\theta''(B(\Delta_{j,k}^+ u, \Delta_{j,k}^+ r'v)) \right\| \leq C2^{j} \mathbb{I}_{\pm \gamma} + \frac{j}{2} (1+2^{j}h)^{\frac{j}{2}} (2^{j'}h)^{-\frac{j}{2}} \|b\| \|f\| \|g\|. \]  

To prove this proposition, we shall use the following notations

\[ \xi = 2^{j'} \theta, \xi' = 2^{j'} \theta', \sigma = (1+2^{j'}h)^{-1}, m = 2^{j''}h(1+2^{j'}h)^{-1}, \rho = 2^{j'-j'}. \]

We will have \( 1/2 \leq |\theta| \leq 1, 1/2 \leq |\theta'| \leq 1 \) and \( \rho \) small. As before, we set \( \theta = |\theta|(1,0, \ldots, 0), \theta' = (\theta', \theta'') \). Let us also set

\[ \begin{align*}
G^m_\pm(\theta, \theta', \sigma, \rho) & = h(1+2^{j'}h)^{-1}F^m_\pm(2^{j'} \theta, 2^{j'} \theta') \\
& = \mu(\sigma^2 + m^2(\theta - \rho \theta'))^{1/2} \pm (\sigma^2 + m^2 \rho^2 \theta^2)^{1/2}
\end{align*} \]

if \( r = 1, r' = \mu \). We shall study the local behavior of \( G^m_\pm \). For \( \delta > 0 \) small, let us define

\[ \begin{align*}
V_\pm^\delta & = \{ (|\theta|, \theta', \sigma, \rho); 1/2 \leq |\theta| \leq 1, 1/2 \leq |\theta'| \leq 1, \rho \in [0,1], \sigma \in [0,1], \\
& \quad \sigma/\rho < \delta, \pm \theta' > 0, |\theta''| < \delta \}, \\
K_\pm^\delta & = \{ (|\theta|, \theta', \sigma, \rho); 1/2 \leq |\theta| \leq 1, 1/2 \leq |\theta'| \leq 1, \rho \in [0,1], \sigma \in [0,1] \} - V_\pm^\delta
\end{align*} \]

**Lemma 4.2** i) There is \( \delta > 0, C > 0 \), and a real valued \( C^1 \) function, \( H_\pm \), defined on an open neighborhood of \( V_\pm^\delta \), such that on \( V_\pm^\delta \cap \{ \rho < \delta \} \)

\[ \begin{align*}
G^m_\pm(\theta, \theta', \sigma, \rho) & = \mu |\theta| + \rho m H_\pm(\theta, \theta', \sigma, \rho), \\
|H_\pm(|\theta|, \theta', \sigma, \rho)| & \leq C, C^{-1} \leq |\partial H_\pm(\theta, \theta', \sigma, \rho)| \leq C.
\end{align*} \]
ii) For any fixed $\delta > 0$ small, there is $\rho_0 > 0$, $C > 0$ and a real valued $C^1$ function, $H_{\pm}$, defined on an open neighborhood of $K_{\pm}^\delta \cap \{ \rho < \rho_0 \}$, such that on $K_{\pm}^\delta \cap \{ \rho < \rho_0 \}$

$$G^\mu_\pm (|\theta|, \theta', \sigma, \rho) = \pm \sigma + \mu (\sigma^2 + m^2 \theta^2)^{1/2} + \rho m^2 H_{\pm} (|\theta|, \theta', \sigma, \rho),$$

$$|H_{\pm} (|\theta|, \theta', \sigma, \rho)| \leq C, \quad C^{-1} \leq |\nabla \theta | H_{\pm} (|\theta|, \theta', \sigma, \rho)| \leq C. \quad (40)$$

**Proof**: i) Set $\nu = \frac{\mu \sigma}{m \rho}$ and write

$$G^\mu_\pm (|\theta|, \theta', \sigma, \rho) = \mu m |\theta| - m \rho (\sigma_1 + |\theta'|) + \frac{m \rho (\theta'' - \nu)}{2} \left( \frac{\mu \rho}{|\theta| - \rho \theta_1} \pm 1 - (\theta'' - \nu)^2 g_1 \pm g_2 \right)$$

where $|\theta| - \rho \theta_1 \geq \frac{1}{4}, |\theta'| \geq \frac{1}{4}, g_1$ and $g_2$ are two bounded $C^1$ functions on $V_{\pm}^\delta$ for $\delta$ small. Denote by

$$H_{\pm} = (1 \pm \mu) \theta_1 + \frac{\theta'' - \nu}{2} \left( \frac{\mu \rho}{|\theta| - \rho \theta_1} \pm 1 - (\theta'' - \nu)^2 (g_1 \pm g_2) \right).$$

Obviously we have on $V_{\pm}^\delta$, for $\delta$ small,

$$G^\mu_\pm (|\theta|, \theta', \sigma, \rho) = \mu m |\theta| + \rho m H_{\pm} (|\theta|, \theta', \sigma, \rho),$$

$$|H_{\pm} (|\theta|, \theta', \sigma, \rho)| \leq C, \quad 1 - \frac{\mu}{2} \leq \left| \frac{\partial H_{\pm}}{\partial \theta_1} (|\theta|, \theta', \sigma, \rho) \right| \leq 2(1 - \mu).$$

ii) Let

$$H_{\pm} = \frac{1}{m^2 \rho} \left[ G^\mu_\pm (|\theta|, \theta', \sigma, \rho) - \mu (\sigma^2 + m^2 \theta^2)^{1/2} \pm \sigma \right]. \quad (41)$$

This definition implies at once that $H_{\pm}$ is bounded. For $\rho < \rho_0$ small, we have

$$\nabla \theta_1 H_{\pm} = -\frac{\mu |\theta|}{(\sigma^2 + m^2 \theta^2)^{1/2}} \pm \frac{\rho \theta_1}{(\sigma^2 + m^2 \rho \theta^2)^{1/2}} + \rho g_1$$

where $g_1$ is a bounded function. Obviously $|\nabla \theta_1 H_{\pm}| \leq C$ on $K_{\pm}^\delta$ and

$$|\nabla \theta_1 H_{\pm}|_{|\sigma| = 0} \geq |\mu \pm 1| \geq 1 - \mu > 0.$$  

Hence there exists a small positive number $\sigma_0$ such that $|\nabla \theta_1 H_{\pm}| \geq \frac{1 - \mu}{2}$ on $K_{\pm}^\delta \cap \{ 0 \leq \sigma \leq \sigma_0 \}$. For $\sigma_0 \leq \sigma \leq 1$, we can choose $\rho_0 > 0$ small so that

$$|\nabla \theta_1 H_{\pm}| \geq \frac{\mu}{4} \geq |\nabla \theta_1 | \rho g_1 \geq \frac{\mu}{16} > 0.$$  

Then, we have $\frac{1}{8} \leq |\nabla \theta_1 H_{\pm}| \leq C$ on $K_{\pm}^\delta \cap \{ \rho < \rho_0 \}$ for $\rho_0$ small. \(\square\)

**Lemma 4.3** There is a positive integer $N_0$ such that if $j' \leq j'' - N_0$, $k' \leq k \leq k''$, one has a universal constant $C$ with

$$\left\| \Delta^\mu_{j''} \nabla^{j'} (B(\Delta^+_{jk} \hat{u}, \Delta^\pm_{j'k'} \hat{v})) \right\| \leq 2^{j - j' + \frac{1}{2} + \frac{k'}{2}} (1 + 2^2 h)(2^2 h)^{-\frac{1}{2}} \times |b|_{L^\infty(A^\gamma_{j,k,E})} \left\| \Delta^+_{j'k'} \hat{u} \right\| \| \Delta^\pm_{j'k'} \hat{v} \| \left\| \nabla^{j'} (B(\Delta^+_{j'k'} \hat{u}, \Delta^\pm_{j''k''} \hat{v})) \right\|. \quad (42)$$
proof: If \( r = r' \), the result comes from Proposition 2.3.1 in [3]. If \( r \neq r' \), without loss of generality, we consider the case \( r = \mu, r' = 1 \). To do this, we denote \( f = \Delta_{j_k}^{+} u, g = \Delta_{j_k}^{-1} v \) and

\[
\mathcal{I}_{\pm} = \Phi_{j_k}^{e'\mu'} \int \hat{f}(\tau - \tau', \xi - \xi') \hat{g}(\tau', \xi') b(\tau, \xi; \tau', \xi') d\tau' d\xi'.
\]

Let us define

\[
W_{\delta} = \{(\xi, \xi'); 2^{j''-1} \leq |\xi| \leq 2^{j''}, 2^{j'-1} \leq |\xi'| \leq 2^{j'}, 2^{j''} - j'(1 + 2^{j''} h)^{-1} < \delta, \pm \xi' > 0, |\xi'| < 2^{j'} \delta\}
\]

and decompose \( \mathcal{I}_{\pm} = \mathcal{I}_{\pm}' + \mathcal{I}_{\pm}'' \) with

\[
\mathcal{I}_{\pm}' = \Phi_{j_k}^{e'\mu'} \int \hat{f}(\tau - \tau', \xi - \xi') \hat{g}(\tau', \xi') b_{\pm}(\xi, \xi') W_{\pm}^{1} d\tau' d\xi',
\]

where \( \delta \) is fixed so that Lemma 4.2 is true. Taking the integer \( N_0 \) of the statement of proposition 4.1 large enough, we may assume that \( \rho \) is smaller than \( \rho_0 \) of Lemma 4.2 ii).

**Estimate of \( \mathcal{I}_{\pm}' \)**

We write \( \mathcal{I}_{\pm}' = \sum_{\ell=0}^{+\infty} \mathcal{I}_{\pm}'^{\ell} \) with

\[
\mathcal{I}_{\pm}'^{\ell} = \Phi_{j_k}^{e'\mu'} \int \hat{f}(\tau - \tau', \xi - \xi') \hat{g}(\tau', \xi') b_{\pm}(\xi, \xi') W_{\pm}^{1} d\tau' d\xi'.
\]

and get

\[
|\mathcal{I}_{\pm}'^{\ell}| \leq \phi_{j''}(\xi) \left( \int |\hat{f}(\tau - \tau', \xi - \xi')|^2 |\hat{g}(\tau', \xi')|^2 |b|^2 \chi_{\pm}(\xi, \xi') d\tau' d\xi' \right)^{1/2} 
\times 2^{k'/2} \left( \int W_{\pm}^{1} \chi_{\pm}(\xi, \xi') d\xi' \right)^{1/2}.
\]

The last integral can be written

\[
2^{j''} \int W_{\pm}^{1}(b, \theta', \sigma, \rho) W_{\pm}^{1}(b, \theta' \rho) W_{\pm}^{1} \{ \{2^{k''} m < G_{\pm}^{\mu} \leq (\ell + 1)2^{k''} m \} d\theta',
\]

By Lemma 4.2, (44) can be controlled from above by

\[
C 2^{j''} \frac{2^{k''-j''}}{\rho} = C 2^{j''-d+1+k}.
\]

Plugging into (43), we can get

\[
\|\mathcal{I}_{\pm}'\| \leq C 2^{j'/2} \left( \frac{k'}{\rho} + \frac{k'}{\rho} + \frac{k'}{\rho} \right) \|f\| \|g\| \|b\|_{L^\infty(A^{r}_{J,K,E})}.
\]

**Estimate of \( \mathcal{I}_{\pm}'' \)**

We decompose \( \mathcal{I}_{\pm}'' = \sum_{\ell} \mathcal{I}_{\pm}''^{\ell} \) and have

\[
|\mathcal{I}_{\pm}''^{\ell}| \leq \phi_{j''}(\xi) \left( \int |\hat{f}(\tau - \tau', \xi - \xi')|^2 |\hat{g}(\tau', \xi')|^2 |b|^2 \chi_{\pm}(\xi, \xi') d\tau' d\xi' \right)^{1/2}.
\]
\[ \times 2^{k/2} \left( \int \phi_{
u}^{\prime}(\xi') \mathbb{1}_{W^d_{\pm} \times \mathbb{R}^d_+}(\xi, \xi') \, d\xi' \right)^{1/2}. \]  

(45)

If \( \rho < \rho_0 \), by Lemma 4.2 we write the last integral
\[ 2^{j/d} \int \mathbb{1}_{\mathcal{K}^j_{\pm}}(\theta, \theta', \sigma, \rho) \mathbb{1}_{(tL-T < H_{\pm} < (t+1)L-T)} \, d\theta' \]
where \( L = 2^{k-j''}/\rho m \), \( T = (\pm \sigma + \mu(\alpha^2 + m^2\theta^2))^{\frac{1}{2}} \rho^{-1} m^{-2} \). Since \( \frac{1}{L} \leq |\nabla_{\theta'} H_{\pm}| \leq C \), we can take \( H_{\pm} \) as a local coordinate satisfying universal bounds, and hence we can bound (46) by
\[ 2^{j/d} 2^{k-j''}/\rho m \leq 2^{j(d-1)+k(1+2^{j''}h)(2^{j''}h)^{-1}}. \]
Plugging into (45), summing in \( \ell \), we get
\[ \| T'' \| \leq C 2^{j(d-1)+k/2 + \frac{k}{2}} (1 + h^{2^j} (2^j h)^{-\frac{1}{2}} \| f \| \| g \| \| b \|. \]

\[ \square \]

To end this section, let us state a theorem which gathers the results of theorem 3.1 and proposition 4.1.

**Theorem 4.4** i) Assume \( j_3 < j_1 \sim j_2 r_1 = r_2 \), and either \((k_3 = k', e_1 \neq e_2)\) or \((k_3 \neq k', e_1 = e_2)\). One has, for a universal constant \( C \),
\[ \| \Delta_{jk'}^{e''}r'' (B(\Delta_{jk'}^{e}u, \Delta_{jk'}^{e'}v)) \| \leq C 2^{j(d-1) - \frac{k}{2} + \frac{d}{2}} (1 + 2^j h)(2^j h)^{-\frac{1}{2}} \times \| \Delta_{jk'}^{e}u \| \| \Delta_{jk'}^{e'}v \| \| b \|. \]

(47)

ii) In all the other cases,
\[ \| \Delta_{jk'}^{e''}r'' (B(\Delta_{jk'}^{e}u, \Delta_{jk'}^{e'}v)) \| \leq C 2^{j(d-1) - \frac{k}{2} + \frac{d}{2}} (1 + 2^j h)(2^j h)^{-\frac{1}{2}} \times \| \Delta_{jk'}^{e}u \| \| \Delta_{jk'}^{e'}v \| \| b \|. \]

(48)

where \( \tilde{j} = \min(j, j', j'') \), \( j = \max(j, j', j'') \).

## 5 The proof of the main theorem

In this section, we first show that the nonlinearities involved in the right hand side of (1.2.2) are bounded on \( H_{\nu,r}^d \) for convenient values of \( s, s', N \) and \( r \in \{\mu, 1\} \), then we prove the main result.

Let \( B(u,v) \) be one of the following expressions when \( u \) and \( v \) are functions on \( \mathbb{R}^{d+1} \).
\[ h^{-2} uv, \text{ or } h^{-1} u_{\partial_1} v, \text{ or } h^{-1} u_{\partial_2} v, \text{ or } \partial_1 u_{\partial_2} v - \partial_2 u_{\partial_1} v. \]

We have (see [3])

**Proposition 5.1** Let \( N \geq (d+3)/2 \), the map \( B(u,v) \) is continuous on \( H^\frac{d+1}{2} \times H^{\frac{d+1}{2}} \)
with values in \( H^\frac{d+1}{2} \) for \( r \in \{\mu, 1\} \).

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Now we prove that the nonlinearity $u\partial_x v$ in the right hand side of (6) is bounded operator from $H_{N,\mu}^{d+1,\frac{1}{2}} \times H_{N,\mu}^{d+1,\frac{1}{2}}$ to $H_{N-1,\mu}^{d+1,\frac{1}{2}}$ (resp. from $H_{N+1,\mu}^{d+1,\frac{1}{2}} \times H_{N,\mu}^{d+1,\frac{1}{2}}$ to $H_{N-1,\mu}^{d+1,\frac{1}{2}}$). Denote by $b = 1/h^2, \xi'/h$, or $\tau'/h$. It is obvious that
\[
\|b\|_{L^\infty} \leq \frac{1}{h^2} (1 + 2^j h + 2^k h).
\] (49)

Let
\[
B(u, v) = \frac{1}{h^2} uv, \quad \text{or} \quad \frac{1}{h} u\partial_x v, \quad \text{or} \quad \frac{1}{h} u\partial_t v.
\]

**Proposition 5.2** Let $N \geq (d+3)/2$. The map $(u, v) \to B(u, v)$ is continuous on $H_{N,\mu}^{d+1,\frac{1}{2}} \times H_{N,\mu}^{d+1,\frac{1}{2}}$ with values in $H_{N-1,\mu}^{d+1,\frac{1}{2}}$.

**proof:** We shall prove this proposition by writing
\[
\Delta_{j'k'}^{\nu}(B(u, v)) = \sum_{j,j'} \sum_{k,k'} \Delta_{j'k'}^{\nu}(B(\Delta_{jk}^{\nu} u, \Delta_{jk}^{\nu} v)),
\] (50)

**a). Estimate of**
\[
\sum_{j,j'} \sum_{k,k'} \|\Delta_{j'k'}^{\nu}(B(\Delta_{jk}^{\nu} u, \Delta_{jk}^{\nu} v))\|.
\] (51)

The general term of (51) is bounded from above by
\[
C h^{-2} (1 + 2^j h + 2^k h)(2^j h)^{d+1} (1 + 2^j h + 2^k h)^{d+1/2} \|\Delta_{jk}^{\nu} u\| \|\Delta_{jk}^{\nu} v\|
\]
Using the assumption $(u, v) \in H_{N,\mu}^{d+1,\frac{1}{2}} \times H_{N,\mu}^{d+1,\frac{1}{2}}$, we can bound this expression by
\[
C h^{d+1} (1 + 2^j h + 2^k h)^{d+1/2} (1 + 2^j h + 2^k h)^{d+1/2} (1 + 2^j h + 2^k h)^{d+1/2}
\]
\[
\|\log h\|^{-2\nu}\|u\|_{H_{N,\mu}^{d+1/2,1/2}} \|v\|_{H_{N,\mu}^{d+1/2,1/2}} \sum_{k,k'} |c_{jk}| \Theta^T(J, K, E)
\] (52)
with $\sum_j \sum_{k} |c_{jk}|^2 \leq 1$ and $\sum_j \sum_{k'} |c_{jk'}|^2 \leq 1$, where $\Theta^T(J, K, E)$ is the characteristic function of the set $\mathcal{E}^T$ and
\[
\mathcal{E}^T = \{(J, K, E) \in \mathbb{N}^3 \times \mathbb{N}^3 \times \{+, -\}^3, \exists (\tau, \xi) \in \text{Supp} \Phi_{j'k'}^{\nu}, \exists (\tau', \xi') \in \text{Supp} \Phi_{j'k'}^{\nu}, \tau - \tau' \leq E \}
\] (53)

Let us set
\[
d_{j'k'}^{\nu} = \sum_{j,j'} \sum_{k,k'} 2^{-\nu} h^{d+1/2} 2^{d+1/2} (1 + 2^j h + 2^k h)^{d+1/2} (1 + 2^j h + 2^k h)^{d+1/2} (1 + 2^j h + 2^k h)^{d+1/2}
\]
\[
\times (1 + 2^j h + 2^k h)^{d+1/2} (1 + 2^j h + 2^k h)^{d+1/2} |\log h|^{-\nu} \|u\|_{H_{N,\mu}^{d+1/2,1/2}} \|v\|_{H_{N,\mu}^{d+1/2,1/2}} |c_{jk}| \Theta^T(J, K, E)
\] (54)
then we have

\[
\alpha_j \leq C \sum_{j,j'} 2^{j(\frac{d}{2})} h^{\frac{d}{2} - 1} \log h^{-\nu}(1 + h \inf(2^j, 2^{j'}))^N c_j c_{j'} \Theta^T(J)
\]

for \(\ell^2\) sequences \((c_j)_{j'}, (c'_{j'})_{j'}\), and the result follows from Lemma 3.1.3 in [3], since \(N \geq (d+3)/2\), \(\nu \geq 1\) if \(d \geq 3\) and \(\nu \geq 3/2\) if \(d = 2\).

\(\beta\). Estimate for \(e'' \neq e\) of

\[
\sum_{j,j'} \sum_{k,k' \leq k} \|\Delta_{j,k}^{e''} (B(\Delta_{j,k}^e u, \Delta_{j,k'}^{-1}v))\|.
\]

We bound the general term of (55) by

\[
C h^{d-\frac{d+3}{2}} \frac{2^{j} h}{\nu} (1 + 2^j h)(2^j h)^{-\frac{d}{2} + \frac{3}{2}} (1 + 2^j h + 2^j h)^{-N+1} + 2 \nu \|u\|_{H_N^{\nu+1}}^2 \|v\|_{H_N^{\nu+1}}^2 \sum_{j,j'} c_j c_{j'} \Theta^T(J, K, E)
\]

with \(\sum_j (\sum_k |c_{j,k}|)^2 \leq 1\) and \(\sum_j (\sum_{k'} |c'_{j,k'}|)^2 \leq 1\). Set

\[
d_{j,k} = \sum_{j''} 2^{j(\frac{d}{2})} h^{\frac{d}{2} - 1} (1 + 2^j h)(2^j h)^{-\frac{d}{2} + \frac{3}{2}} (1 + 2^j h + 2^j h)^{-N+1} + 2 \nu \|u\|_{H_N^{\nu+1}}^2 \|v\|_{H_N^{\nu+1}}^2 \sum_{j''} c_{j''} \Theta^T(J, K, E)
\]

Then

\[
d_{j\nu} \leq C \sum_{j,j'} 2^{j(\frac{d}{2})} h^{\frac{d}{2} - 1} \log h^{-\nu}(1 + h \inf(2^j, 2^{j'}))^N c_j c_{j'}
\]

for \(\ell^2\) sequences \((c_j)_{j'}, (c'_{j'})_{j'}\). Now the result follows from Lemma 3.1.3 in [3], since \(N \geq (d+3)/2\), \(\nu \geq 1\) if \(d \geq 3\) and \(\nu \geq 3/2\) if \(d = 2\).

\(\gamma\) Estimate for \(e'' = e\) of

\[
\sum_{j,j'} \sum_{k,k' \leq k} \|\Delta_{j,k}^{e''} (B(\Delta_{j,k}^e u, \Delta_{j,k'}^{-1}v))\|.
\]

We bound the general term of (56) by, using the fact \(j' < j \sim j' \),

\[
C h^{d-\frac{d+3}{2}} \frac{2^{j} h}{\nu} (1 + 2^j h)(2^j h)^{-\frac{d}{2} + \frac{3}{2}} (1 + 2^j h + 2^j h)^{-N+1} \times (1 + 2^j h + 2^j h)^{-N} \|\log h^{-\nu} \|_{H_N^{\nu+1}}^2 \|v\|_{H_N^{\nu+1}}^2 \sum_{j''} c_{j''} \Theta^T(J, K, E)
\]

with \(\sum_j (\sum_k |c_{j,k}|)^2 \leq 1\) and \(\sum_j (\sum_{k'} |c'_{j,k'}|)^2 \leq 1\). Set

\[
d_{j''} \leq \sum_{j,j'} 2^{j(\frac{d}{2})} h^{\frac{d}{2} - 1} (1 + 2^j h)(2^j h)^{-\frac{d}{2} + \frac{3}{2}} (1 + 2^j h + 2^j h)^{-N+1} \times (1 + 2^j h + 2^j h)^{-N} (1 + 2^j h + 2^j h)^{-N-1} \|\log h^{-\nu} \|_{H_N^{\nu+1}}^2 \sum_{j''} c_{j''} \Theta^T(J, K, E)
\]

(57)

We get

\[
d_{j''} \leq C \sum_{j,j'} 2^{j(\frac{d}{2})} h^{\frac{d}{2} - 1} \log h^{-\nu}(1 + h \inf(2^j, 2^{j'}))^N c_j c_{j'}
\]

for \(\ell^2\) sequences \((c_j)_{j'}, (c'_{j'})_{j'}\). Now the result follows from Lemma 3.1.3 in [3] as well. \(\square\) We will use the following proposition which is similar to proposition 3.1.6 in [3]:
Proposition 5.3 Let $N \geq (d+3)/2$. The map $(u, v) \mapsto u \cdot v$ is continuous on $H^{d+1, \frac{1}{2}}_{\mu, r} \times H^{d+1, \frac{1}{2}}_{\mu, r}$ with values in $H^{\frac{d+1}{2}}_{N-1, r}$, $r \in \{\mu, 1\}$.

The following proposition may be found in [2] proposition 3.2.2:

Proposition 5.4 Let $s \in \mathbb{R}$, $N \in \mathbb{N}$, $N \geq (d+3)/2$, and $t_{\mu} \in \{\mu, 1\}$,

$$(u_{0})_{h} \in H^{s}_{N} (\mathbb{R}^{d}), (u^{h})_{h} \in H^{s-1}_{N-1} (\mathbb{R}^{d}), (f^{h})_{h} \in H^{s-1, 1/2}_{N-1, t_{\mu}} (\mathbb{R}^{d+1}).$$

Let $(u^{h})_{h}$ be the solution of

$$
\begin{cases}
\Box_{t} u^{h} + h^{-2} (t_{\mu})^{2} u^{h} = f^{h}, \\
|u^{h}|_{t=0} = u_{0}^{h}, \quad \partial_{t} u^{h}|_{t=0} = u_{1}^{h},
\end{cases}
$$

(58)

Then for any $\chi \in C_{0}^{\infty} (\mathbb{R})$, $(\chi (t) u^{h})_{h}$ is in $H^{s, 1/2}_{N, t_{\mu}}$ and the map

$$
((u_{0})_{h}, (u^{h})_{h}, (f^{h})_{h}) \mapsto (\chi (t) u^{h})_{h}
$$
is continuous on the preceding spaces.

Proof of theorem 2.2: Take $\chi \in C_{0}^{\infty} (\mathbb{R})$, $\chi \equiv 1$ on a neighborhood of $[-1, 1]$. Let $U^{0} = (U^{0}, V^{0})$ be the solution of $\Box_{t} U^{0} + h^{-2} \mu^{2} U^{0} = 0, \Box V^{0} + h^{-2} V^{0} = 0, U^{0}|_{t=0} = V^{0}, \partial_{t} U^{0}|_{t=0} = W^{0}$, and for $n \geq 0$ define $U^{n+1} = (U^{n+1}, V^{n+1})$ by

$$
\Box_{t} U^{n+1} + h^{-2} \mu^{2} U^{n+1} = h^{-2} F_{1} (\chi (t) U^{n}, h \partial_{t} (\chi (t) U^{n}), h \partial_{x} (\chi (t) U^{n})),
$$

$$
\Box V^{n+1} + h^{-2} V^{n+1} = h^{-2} F_{2} (\chi (t) U^{n}, h \partial_{t} (\chi (t) U^{n}), h \partial_{x} (\chi (t) U^{n})),
$$

$$
U^{n+1}|_{t=0} = V^{h}, \quad \partial_{t} U^{n+1}|_{t=0} = W^{h}.
$$

The assumption (3) on $F$, together with propositions 5.1, 5.2 and 5.3, implies that if $\chi (t) U^{n}$ is in $H^{(d+1)/2, 1/2}_{N, \mu}$, $\chi (t) V^{n}$ is in $H^{(d+1)/2, 1/2}_{N, 1}$, the right hand side of the first equation (50) in $H^{(d-1)/2, 1/2}_{N-1, \mu}$ and the second is in $H^{(d-1)/2, 1/2}_{N-1, 1}$. Proposition 5.4 shows that $\chi (t) U^{n+1}$ is then in $H^{(d+1)/2, 1/2}_{N, \mu}$ and $\chi (t) V^{n+1}$ is then in $H^{(d+1)/2, 1/2}_{N, 1}$. If $(V^{h})_{h}$ and $(W^{h})_{h}$ are small enough, one deduces from that the boundedness and convergence of sequence $(\chi (t) U^{n})_{h}$. The limit provides the solution we are seeking for on $[-1, 1]$. Since $\chi (t) U \in H^{(d+1)/2, 1/2}_{N, \mu}$ implies $U \in C^{0} [1, 1], H^{(d+1)/2}_{N, \mu}$ and $\partial_{t} U \in C^{0} [-1, 1], H^{(d-1)/2}_{N-1, \mu}$, and from $\chi (t) V \in H^{(d+1)/2, 1/2}_{N, 1}$ implies $V \in C^{0} [1, 1], H^{(d+1)/2}_{N, 1}$ and $\partial_{t} V \in C^{0} [-1, 1], H^{(d-1)/2}_{N-1, 1}$, we get the properties of the statement of the theorem. \qed

Bibliography


