Existence and Uniqueness of Positive Eigenvalues for Certain Eigenvalue Systems

Ruying Xue
Department of Mathematics, Zhejiang University,
Hangzhou 310027, P. R. China
e-mail: ryxue@zju.edu.cn

Abstract
In this paper we consider certain eigenvalue systems. Imposing some reasonable hypotheses, we prove that the eigenvalue systems have a unique eigenvalue with positive eigenfunctions, and that the eigenfunction is unique up to a scalar multiple.

Key words and phrases: eigenvalue problem; eigenfunction; fixed point index; maximum principle.

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1 Introduction
Let Ω be a bounded domain in \( R^n \) with smooth boundary \( \partial \Omega \), \( \Delta_p \) denotes the p-Laplacian defined by \( \Delta_p = \text{div}(\text{grad} \psi_p(u)) \), for \( p \in (1, \infty) \), and \( \psi_p(u) = |u|^{p-1}u \).

It is well-known that the eigenvalue problem
\[
-\Delta_p u + a \psi_p(u) = \lambda a(x) \psi_p(u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega
\]
has a unique positive eigenvalue \( \lambda_0 > 0 \) associated to which the eigenvalue problem (1.1) possesses positive eigenfunctions. In fact,
\[
\lambda_0 = \inf \{ \int_\Omega |\nabla u|^p + a|u|^p dx : u \in W^{1,p}_0(\Omega) \text{ with } \int_\Omega a(x)|u|^p dx = 1 \},
\]
and the eigenfunctions for \( \lambda_0 \) are the minimizers of the functional
\[
J(u) = \int_\Omega |\nabla u|^p + a|u|^p dx - \lambda_0 \int_\Omega a(x)|u|^p dx, \quad u \in W^{1,p}_0(\Omega).
\]
Moreover, such a minimizer \( \phi_0 \in W^{1,p}_0(\Omega) \) exists and is unique up to a scalar multiple, and \( \phi_0 \in L^\infty(\Omega) \); we normalize it by \( \int_\Omega a(x)|\phi_0|^p dx = 1 \) and \( \phi_0 \geq 0 \) in \( \Omega \). Then also \( \phi_0 > 0 \) in \( \Omega \) and \( \phi_0 \in C^{1,\beta}(\bar{\Omega}) \) as a consequence of Lemma 2.3 in Section 2.

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The aim of this paper is to generalize the preceding results to eigenvalue problems of the form

\[
\begin{aligned}
-\Delta_p u + a\psi_p(u) &= \lambda^{p-1}\sum_{i=1}^{k} a_i(x)u^{\alpha_i} + \delta - \alpha, \quad \text{in } \Omega, \\
-\Delta_p v + b\psi_p(v) &= \lambda^{q-1}\sum_{j=1}^{N} b_j(x)v^{\beta_j} - \delta - \beta, \quad \text{in } \Omega, \\
u = v = 0 &\quad \text{on } \partial\Omega,
\end{aligned}
\]  

(1.3)

where \(1 < p < +\infty, 1 < q < +\infty, a \) and \( b \) are some positive numbers, \( 0 \leq \alpha_i \leq p - 1(i = 1, \ldots, k), 0 \leq \beta_j \leq q - 1(j = 1, 2, \ldots, N) \), \( a_i(x)(i = 1, 2, \ldots, k) \) and \( b_j(x)(j = 1, \ldots, N) \) are nonnegative, never vanishing functions belonging to \( C^\alpha(\Omega) \) for some \( \alpha \in (0, 1) \). Imposing some reasonable hypotheses we prove that the eigenvalue problem (1.3) has a unique positive eigenvalue for which the eigenfunctions are positive, and that the eigenfunction is unique up to a scalar multiple.

Under a positive eigenfunction of eigenvalue problem (1.3) we mean a weak solution \((u, v) \in W_0^1(\Omega) \times W_0^1(\Omega) \) with \( u > 0 \) and \( v > 0 \) in \( \Omega \). A nonnegative solution of eigenvalue problem (1.3) is a pair \((u, v) \in W_0^1(\Omega) \times W_0^1(\Omega) \) which does not vanish identically in \( \Omega \) and satisfies (1.3), each component is nonnegative.

This article is organized as follows. In Section 2 we recall some well-known results for the single equation with \( \Delta_p \). In Section 3 we prove that (1.3) has a unique eigenvalue with positive eigenfunctions. The uniqueness of the eigenfunction of (1.3) is considered in Section 4.

In this paper, we shall write \((u_1, v_1) \geq (u_2, v_2)\) if \( u_1 \geq u_2 \) and \( v_1 \geq v_2 \), \((u_1, v_1) > (u_2, v_2)\) if \( u_1 > u_2 \) and \( v_1 > v_2 \). We also denote by \( |u|_{0, \Omega} = \sup_{\Omega} |u| \), and denote by \( \|u\|_{A(x), p} = \int_{\Omega} A(x)|u|^p dx \).

### 2 Some Known Results for A Single Equation with \( \Delta_p \)

In this section, we give some known results for the following Dirichlet problem

\[
\begin{aligned}
-\Delta_p u + a\psi_p(u) &= f(x) \quad \text{in } \Omega, \\
u &= g \quad \text{on } \partial\Omega,
\end{aligned}
\]  

(2.1)

where \( a \in (0, \infty) \) is a constant, \( 0 \leq f \in L^\infty(\Omega) \) with the norm \( \|f\|_{L^\infty(\Omega)} \), and \( 0 \leq g \in C^{1,\alpha}(\partial\Omega) \) with H"older norm \( |g|_{1,\alpha, \partial\Omega} \). By variational methods the Dirichlet problem (2.1) has a unique weak solution \( u \in W_0^{1,p}(\Omega) \).

The following weak comparison principle and strong comparison principle near the boundary \( \partial\Omega \) are proved in [1].

**Lemma 2.1** Assume that \( u, u' \in W_0^{1,p}(\Omega) \), respectively, are weak solutions of

\[
\begin{aligned}
-\Delta_p u + a\psi_p(u) &= f \quad \text{in } \Omega, u = g \quad \text{on } \partial\Omega, \\
-\Delta_p u' + a\psi_p(u') &= f' \quad \text{in } \Omega, u' = g' \quad \text{on } \partial\Omega,
\end{aligned}
\]

with \( f \leq f' \) and \( g \leq g' \). Then \( u \leq u' \) almost everywhere in \( \Omega \).

**Lemma 2.2** Assume that \( u, u' \in C^{1,\alpha}(\Omega) \), respectively, are weak solutions of

\[
\begin{aligned}
-\Delta_p u + a\psi_p(u) &= f \quad \text{in } \Omega, u = 0 \quad \text{on } \partial\Omega, \\
-\Delta_p u' + a\psi_p(u') &= f' \quad \text{in } \Omega, u' = 0 \quad \text{on } \partial\Omega.
\end{aligned}
\]
with \( f, f' \in C^0(\bar{\Omega}), 0 \leq f \leq f' \), and \( f \) does not vanish identically in \( \Omega \). Then there exists a positive constant \( \delta > 0 \) small enough such that if \( \Sigma \subset \Omega \delta \) is a connected component of \( \Omega \delta \), then either \( u \equiv u' \) in \( \Sigma \) or else
\[
  u < u' \quad \text{in} \quad \Sigma - \partial \Omega \quad \text{and} \quad \frac{\partial u}{\partial \nu} > \frac{\partial u'}{\partial \nu} \quad \text{on} \quad \Sigma \cap \partial \Omega.
\]

here \( \nu \) denotes the exterior unit normal to \( \partial \Omega, \Omega \delta = \{ x \in \Omega; \text{dist}(x, \partial \Omega) < \delta \} \).

The following regularity result is proved in [2].

**Lemma 2.3** Assume that \( u \in W^{1,p}(\Omega) \) is a bounded weak solution of (2.1). Let \( C' > 0 \) be a constant such that
\[
  \|u\|_{\infty, \Omega} \leq C', \quad \|f\|_{\infty, \Omega} \leq C' \quad \text{and} \quad |g|_{1+\alpha, \partial \Omega} \leq C'.
\]

Then there exists a constant \( \beta, 0 < \beta < 1 \), depending solely on \( \alpha, p, a \) and \( n \), and another constant \( C, 0 \leq C < \infty \), depending solely upon \( \alpha, p, a, n, \Omega \) and \( C' \) such that \( u \in C^{1,\beta}(\bar{\Omega}) \) with the Hölder norm
\[
  |u|_{1+\beta, \Omega} \leq C.
\]

Vazquez[3] has proved the following strong maximum and boundary point principle for the Dirichlet problem (2.1). **Lemma 2.4** Assume that \( u \) is a function satisfying the following conditions:

1. \( u \in C^1(\bar{\Omega}) \), and \( u \geq 0 \) in \( \Omega \),
2. \( \Delta_p u \in L^2_{\text{loc}}(\Omega) \),
3. \( -\Delta_p u \geq a\psi_p(u) \) a.e. in \( \Omega \).

Then, if \( u \) does not vanish identically in \( \Omega \), it is positive everywhere in \( \Omega \) and
\[
  \frac{\partial u}{\partial \nu}(x_0) < 0 \quad \text{whenever} \quad u(x_0) = 0 \quad \text{for} \quad x_0 \in \partial \Omega.
\]

Here \( \nu = \nu(x_0) \) denotes the exterior unit normal of \( \partial \Omega \) at \( x_0 \).

Denote by \( C_0(\Omega) = \{ u \in C(\Omega), u = 0 \text{ on } \partial \Omega \} \) and let \( V_+ \) be the positive cone in \( C_0(\Omega) \times C_0(\Omega), V_+ = \{ (u, v) \in C_0(\Omega) \times C_0(\Omega), u \geq 0, v \geq 0 \text{ in } \Omega \} \). Let \( f(x, u, v), g(x, u, v) \) be nonnegative functions defined in \( \Omega \times [0, +\infty) \times [0, +\infty) \) and satisfy \( f, g \in C^\alpha(\Omega \times [0, +\infty) \times [0, +\infty)) \) for some \( \alpha \in (0, 1) \). For any \( (u, v) \in V_+ \), by variational methods there exists a unique weak solution \( (\bar{u}, \bar{v}) \in W^{1,p}_0(\Omega) \times W^{1,q}_0(\Omega) \) satisfying
\[
  \begin{aligned}
    -\Delta_p \bar{u} + a\psi_p(\bar{u}) &= f(x, u, v) \quad \text{in} \quad \Omega, \\
    -\Delta_q \bar{v} + a\psi_q(\bar{v}) &= g(x, u, v) \quad \text{in} \quad \Omega, \\
    \bar{u} = \bar{v} = 0 \quad \text{on} \partial \Omega.
  \end{aligned}
\]

We recall that \((\bar{u}, \bar{v})\) is a bounded, nonnegative weak solution of (2.2) by Lemma refl2.1, and that \((\bar{u}, \bar{v}) \in C^{1,\beta}_0(\Omega) \times C^{1,\beta}_0(\Omega) \) for some \( \beta \in (0, 1) \) by Lemma 2.3. We define the mapping \( T = (T_p, T_q) : (u, v) \mapsto (\bar{u}, \bar{v}) = T(u, v) \). Observe that \( T \) is a self-mapping of \( V_+ \). Moreover, \( T : V_+ \mapsto V_+ \) is continuous and relatively compact.
3 Existence and Uniqueness of The Eigenvalue

**Theorem 3.1** Assume that $\alpha_i$ and $\beta_j$ satisfy one of the following conditions:

(1). $0 \leq \alpha_i < p - 1, 0 \leq \beta_j < q - 1$ for all $i \in \{1, 2, ..., k\}, j \in \{1, 2, ..., N\}$,

(2). $\alpha_1 = 0$ and $\beta_1 = 0$.

Then the eigenvalue problem (1.3) has a unique eigenvalue $\lambda_0$ associated to which (1.3) possesses positive eigenfunctions.

**Proof:** First, we prove the existence of the eigenvalue of (1.3). Let $T$ be the mapping from $V_+$ to $V_+$ defined by (2.2) for the special functions

$$ f(x, u, v) = \sum_{i=1}^{k} a_i(x) u^{\alpha_i} v^{p-1-\alpha_i}, $$

$$ g(x, u, v) = \sum_{j=1}^{N} b_j(x) u^{q-1-\beta_j} v^{\beta_j}. $$

It is obvious that the eigenvalue problem (1.3) has an eigenvalue $\lambda_0$ with nonnegative eigenfunctions if and only if $\lambda_0 T$ has a fixed point in $V_+$. We denote by $\lambda_1$ the first eigenvalue of

$$ -\Delta_p u + \alpha_p u = \lambda_1^{p-1} u_0(x) u^{p-1}, u|_{\partial \Omega} = 0, u \in W_0^{1,p}(\Omega), $$

and denote by $\phi_1$ the eigenfunction associated to $\lambda_1$. Let $\lambda_2$ be the first eigenvalue of

$$ -\Delta_q u + b\psi_q u = \lambda_2^{q-1} b_1(x) u^{q-1}, u|_{\partial \Omega} = 0, u \in W_0^{1,q}(\Omega), $$

and let $\phi_2$ be the eigenfunctions associated to $\lambda_2$. It is well known that $0 \leq \phi_1 \in L^\infty(\Omega), 0 \leq \phi_2 \in L^\infty(\Omega)$. Combining Lemma 2.1, Lemma 2.3 with Lemma ?? we arrive at $0 < \phi_1 \in C^{1,\beta}(\Omega), 0 < \phi_2 \in C^{1,\beta}(\Omega)$, and

$$ \phi_1 > 0, \phi_2 > 0 \text{ in } \Omega \text{ and } \frac{\partial \phi_1}{\partial \nu} < 0, \frac{\partial \phi_2}{\partial \nu} < 0 \text{ on } \partial \Omega. \quad (3.1) $$

Since $\partial \Omega$ is bounded and closed, (3.1) implies that

$$ C_0 \phi_2(x) \leq \phi_1 \leq \frac{1}{C_0} \phi_2(x) \quad (3.2) $$

for some positive constant $C_0$.

Denote by $T_\epsilon(u, v) = T(u, v) + \epsilon(\phi_1, \phi_2)$. Since the mapping $T$ is continuous and relatively compact in $V_+$, the mapping $T_\epsilon : V_+ \rightarrow V_+$ is continuous and relatively compact. Moreover, we have

$$ \inf \{|T_\epsilon(u, v)|_{0,\tilde{\Omega}} : |u|_{0,\tilde{\Omega}} + |v|_{0,\tilde{\Omega}} = 1, (u, v) \in V_+\} \geq \epsilon|\phi_0|_{0,\tilde{\Omega}} + \epsilon|\phi_2|_{0,\tilde{\Omega}} > 0. $$

Then, by Theorem 5.4.33 in [2], there exists $\lambda_\epsilon > 0$ and $(u_\epsilon, v_\epsilon) \in V_+$ with $|u_\epsilon|_{0,\tilde{\Omega}} + |v_\epsilon|_{0,\tilde{\Omega}} = 1$ satisfying

$$ \lambda_\epsilon (u_\epsilon, v_\epsilon) = T_\epsilon(u_\epsilon, v_\epsilon) = T(u_\epsilon, v_\epsilon) + \epsilon(\phi_1, \phi_2). \quad (3.3) $$

Since $T$ is a compact operator defined on $V_+$, and $|u_\epsilon|_{0,\tilde{\Omega}} + |v_\epsilon|_{0,\tilde{\Omega}} = 1$, we may assume that there exists a pair $(w_0, v_0)$ in $V_+$ and a nonnegative number $\lambda^1$ such that

$$ T(u_\epsilon, v_\epsilon) \rightarrow (u_1, v_1) \text{ and } \lambda_\epsilon \rightarrow \lambda^1, \text{ as } \epsilon \rightarrow 0. $$
Clearly, if \( \lambda^1 > 0 \), we have
\[
(u_\epsilon, v_\epsilon) = \frac{1}{\lambda^1} T_\epsilon (u, v) + \epsilon (\phi_1, \phi_2) \to \frac{1}{\lambda^1} (u_0, v_0) \text{ def.} = (u, v),
\]
as \( \epsilon \to 0 \), and \(|u_0|_{0, \Omega} + |v_0|_{0, \Omega} = 1 \) and \((u_0, v_0) = \frac{1}{\lambda^1} T(u_0, v_0) \), so that it remains only to prove that \( \lambda^1 \neq 0 \).

To this end, we first note that (3.3) implies
\[
(u, v) \geq C_0 \phi_1, v \phi_2.
\]
Thus, there is a largest number \( t_\epsilon > 0 \) such that
\[
(u, v) \geq (t_\epsilon \phi_1, t_\epsilon \phi_2).
\]
(3.5) implies
\[
\sum_{i=1}^k a_i(x) u_i^{p_i - 1 - \alpha_i} \geq a_1(x) t_\epsilon^{p_1 - 1} \phi_1 \phi_2^{p_2 - 1 - \alpha_1} \geq C_0^{p_1 - 1} t_\epsilon^{p_1 - 1} a_1(x) \phi_1^{p_1 - 1},
\]
\[
\sum_{j=1}^N b_j(x) u_j^{q_j - 1 - \beta_j} \psi_j \geq b_1(x) t_\epsilon^{q_1 - 1} \phi_1 \phi_2^{q_2 - 1} \geq C_0^{q_1 - 1} t_\epsilon^{q_1 - 1} b_1(x) \phi_2^{q_2 - 1},
\]
By the weak comparison principle (Lemma 2.1), it follows from (3.6) and (3.7) that
\[
\lambda \epsilon (u, v) \geq T(u, v) \geq (C_0^{1 - \frac{\alpha_1}{p_1}}, C_0^{1 - \frac{\beta_1}{q_1}}) t_\epsilon \phi_1, C_0^{1 - \frac{\alpha_1}{p_1}} t_\epsilon \phi_2.
\]
But
\[
t_\epsilon \text{ is a largest number satisfying (3.5), (3.6) implies}
\]
\[
t_\epsilon \geq \min \{ C_0^{1 - \frac{\alpha_1}{p_1}}, C_0^{1 - \frac{\beta_1}{q_1}} \} t_\epsilon \phi_1^{-1}, \]
so that \( \lambda \epsilon \geq \min \{ C_0^{1 - \frac{\alpha_1}{p_1}}, C_0^{1 - \frac{\beta_1}{q_1}} \} > 0 \). Consequently \( \lambda^1 > 0 \), as required.

Denote by \( \lambda_0 = \frac{1}{\lambda^1} \), so \((u_0, v_0)\) is a nonnegative eigenfunction for \( \lambda_0 \). Combining (1) (or (2)) with the fact that \(|u_0|_{0, \Omega} + |v_0|_{0, \Omega} = 1 \) implies \( u_0 \neq 0, v_0 \neq 0 \). By Lemma 2.3 and Lemma 2.2 we arrive at \( u_0 \in C_0^{1+\beta} (\Omega), v_0 \in C_0^{1+\beta} (\Omega) \) for some \( \beta \in (0, 1) \), and \( u_0 > 0, v_0 > 0 \) in \( \Omega \). Hence \((u_0, v_0)\) is a positive eigenfunction of (1.3) for \( \lambda_0 \).

Now, we prove that \( \lambda_0 \) is unique. Suppose that \((u_1, v_1), (u_2, v_2)\) are two linearly independent positive eigenfunctions associated with the eigenvalues \( \lambda_1, \lambda_2 \), respectively, for the eigenvalue problem (1.3). From Lemma 2.1 and Lemma 2.2, we have \( \lambda_1, \lambda_2 > 0 \) and
\[
\frac{\partial u_1}{\partial \nu} < 0, \frac{\partial u_2}{\partial \nu} < 0, \frac{\partial v_1}{\partial \nu} < 0, \frac{\partial v_2}{\partial \nu} < 0, \text{ on } \partial \Omega.
\]
We may assume that \((0, 0) < (u_1, v_1) \leq (u_2, v_2)\) and \( \lambda_1 \leq \lambda_2 \). We claim \( \lambda_1 = \lambda_2 \). On the contrary, suppose \( \lambda_1 < \lambda_2 \). Let us consider the following elliptic system:
\[
\begin{cases}
-\Delta u + \psi p(u) = \Lambda_1^{p_1 - 1} \sum_{i=1}^k a_i(x) u_i^{p_i - 1 - \alpha_i}, \\
(\Lambda_2^{p_2 - 1} - \Lambda_1^{p_2 - 1}) \sum_{i=1}^k a_i(x) u_i^{p_2 - 1 - \alpha_i}, \text{ in } \Omega, \\
-\Delta v + \psi q(v) = \Lambda_2^{q_1 - 1} \sum_{j=1}^N b_j(x) v_j^{q_2 - 1 - \beta_j}, \\
(\Lambda_2^{q_2 - 1} - \Lambda_1^{q_2 - 1}) \sum_{j=1}^N b_j(x) v_j^{q_2 - 1 - \beta_j}, \text{ in } \Omega, \\
u = v = 0 \text{ on } \partial \Omega.
\end{cases}
\]
Observe that \((u_2, v_2)\) is a solution of (3.10). By Lemma 2.1, we have \((\xi u_1, \xi v_1) \leq \tilde{T}(\xi u_1, \xi v_1)\) for all \(\xi \in (0, +\infty)\), and \(\tilde{T}(\xi u_2, \xi v_2) \leq (\xi u_2, \xi v_2)\) for all \(\xi \in (1, +\infty)\), where \(\tilde{T}\) be the mapping from \(V_+\) to \(V_+\) defined by (2.2) for the special functions

\[
\tilde{f}(x, u, v) = \Lambda_1^{p-1} \sum_{i=1}^{k} a_i(x) u^{\alpha_i} v^{p-1-\alpha_i},
\]

\[
+ \left( \Lambda_2^{p-1} - \Lambda_1^{p-1} \right) \sum_{i=1}^{k} a_i(x) u_2^{\alpha_i} v_2^{p-1-\alpha_i}, \quad \text{in } \Omega,
\]

\[
\tilde{g}(x, u, v) = \Lambda_1^{q-1} \sum_{j=1}^{N} b_j(x) u^{q-1-\beta_j} v^{\beta_j},
\]

\[
+ \left( \Lambda_2^{q-1} - \Lambda_1^{q-1} \right) \sum_{j=1}^{N} b_j(x) u_2^{q-1-\beta_j} v_2^{\beta_j}, \quad \text{in } \Omega.
\]

Making use of (3.9) and the fact that \((0, 0) < (u_1, v_1) \leq (u_2, v_2)\), we can pick \(\xi \geq 1\) so large that \((u_2, v_2) \leq \xi (u_1, v_1)\). Then, by Lemma 2.1 we arrive at

\[
(u_2, v_2) \leq \xi (u_1, v_1) \leq \tilde{T}(\xi u_1, \xi v_1) \leq \ldots \leq \tilde{T}^k(\xi u_1, \xi v_1) \leq \ldots \leq \tilde{T}^k(\xi u_2, \xi v_2) \leq \xi (u_2, v_2) \leq \xi (u_3, v_3).
\]

The compactness of \(\tilde{T}^k\) implies \(\tilde{T}^k(\xi u_2, \xi v_2) \to (u_3, v_3)\) in \(V_+\) as \(k \to +\infty\), hence

\[
(u_2, v_2) \leq \tilde{T}(u_3, v_3) = (u_3, v_3) \leq \xi (u_2, v_2),
\]

and \((u_3, v_3)\) is a positive solution of (3.10). We claim that \((u_3, v_3) = (u_2, v_2)\). If this is the case, (3.12) implies that

\[
(u_2, v_2) = \xi (u_1, v_1),
\]

a contradiction to our assumption that \((u_1, v_1)\) and \((u_2, v_2)\) are not colinear. Then we have \(\Lambda_1 = \Lambda_2\) as desired.

On the contrary, suppose \((u_3, v_3) \neq (u_2, v_2)\). Without loss of generality, we assume that \((u_3, v_3) \leq (u_2, v_2)\) is false. Consequently, by (3.9) we can pick \(t \in (1, +\infty)\) which is the smallest number satisfying

\[
t^{-1}(u_3, v_3) \leq (u_2, v_2)
\]

We have

\[
t^{-p}\tilde{f}(u_3, v_3) < \tilde{f}(t^{-1}u_3, t^{-1}v_3) \leq \tilde{f}(u_2, v_2),
\]

and

\[
-\Delta_p(t^{-1}u_3) + a_{\psi_p}(t^{-1}u_3) = t^{-p} \tilde{f}(u_3, v_3) \quad \text{in } \Omega,
\]

\[
-\Delta_p(u_2) + a_{\psi_p}(u_2) = \tilde{f}(u_2, v_2) \quad \text{in } \Omega,
\]

\[
u_3 = u_2 = 0 \quad \text{on } \partial\Omega.
\]

By (3.14) and (3.15), the maximum principle and comparison principle near the boundary \(\partial\Omega\) force

\[
0 < t^{-1}u_3 < u_2 \text{ in } \Omega_3 - \partial\Omega \quad \text{and} \quad \frac{\partial u_2}{\partial \nu} < \frac{\partial t^{-1}u_3}{\partial \nu} < 0 \text{ on } \partial\Omega.
\]
Making use of (3.16), we find $\bar{t} \in (1, t)$ such that
\[
0 < \bar{t} - 1 u_3 \leq u_2 \text{ in } \bar{\Omega} - \partial \Omega \text{ and } \partial u_2 / \partial \nu \leq \partial \bar{t} - 1 u_3 / \partial \nu < 0 \text{ on } \partial \Omega,
\] (3.17)

and
\[
\bar{t} - 1 \bar{f}(u_3, v_3) \leq \bar{f}(u_2, v_2), \text{ for } x \in \Omega - \bar{\Omega}
\] (3.18)

by compactness of the set $\Omega - \bar{\Omega}$.

We now consider the following elliptic equations:
\[
-\Delta \bar{p}(\bar{t} - 1 u_3) + a\psi \bar{p}(\bar{t} - 1 u_3) = \bar{t} - 1 \bar{f}(u_3, v_3) \text{ in } \Omega - \bar{\Omega},
\]
\[
-\Delta \bar{p}(u_2) + a\psi \bar{p}(u_2) = \bar{f}(u_2, v_2) \text{ in } \Omega - \bar{\Omega},
\]
\[
\bar{t} - 1 u_3 \leq u_2 \text{ on } \partial(\Omega - \bar{\Omega}).
\] (3.19)

By (3.17), (3.18) and (3.19), the weak comparison principle implies
\[
\bar{t} - 1 u_3 \leq u_2 \text{ in } \Omega - \bar{\Omega},
\] (3.20)

and
\[
\bar{t} - 1 u_3 \leq u_2 \text{ in } \Omega.
\] (3.21)

Similarly, we have
\[
\bar{t} - 1 v_3 \leq v_2 \text{ in } \Omega.
\] (3.21)

(3.19) and (3.20) force, for some $\bar{t} \in (1, t),$
\[
\bar{t} - 1 (u_1, v_1) \leq (u_2, v_2),
\]
a contradiction to our choice of $t$. \hfill \Box

**Remark** It is obvious that if $\alpha_1 = p - 1$, (1.3) has a unique eigenvalue for which eigenfunctions are the form $(u, 0)$ with $u > 0$, and if $\beta_1 = q - 1$, (1.3) has a unique eigenvalue for which the eigenfunctions have the form $(0, v)$ with $v > 0$. If (1.3) has an eigenvalue with positive eigenfunctions, from the proof of Theorem 4.1 we know that the eigenvalue is unique. Hence, (1.3) has at most three eigenvalues with nonnegative eigenfunctions. It is possible that (1.3) may have no eigenvalue with positive eigenfunctions.

Assume that $p = q > 1$, $a = b > 0$, and that there exist positive constants $C_1, C_2, C_1 \leq C_2$, such that
\[
C_1^{p-1} \sum_{j=1}^{N} b_j(x) t^{p-1-\beta_j} \leq \sum_{i=1}^{k} a_i(x) \alpha_i \leq C_2^{p-1} \sum_{j=1}^{N} b_j(x) t^{p-1-\beta_j}
\]
for all $(t, x) \in [C_1, C_2] \times \bar{\Omega}$. Then the eigenvalue problem (1.3) has a unique eigenvalue $\lambda_0$ for which the eigenfunctions are positive in $\Omega$.

**Proof**: We denote by
\[
f(x, u, v) = \max \left\{ \sum_{i=1}^{k} a_i(x) u^{\alpha_i} v^{p-1-\alpha_i}, C_1^{p-1} \sum_{j=1}^{N} b_j(x) w^{p-1-\beta_j} \right\}.
\]

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\[ g(x, u, v) = \max \{ C_2^{-p} \sum_{i=1}^{k} a_i(x) u^{\alpha_i} v^{p-1-\alpha_i}, \sum_{j=1}^{N} b_j(x) u^{p-1-\beta_j} v^{\beta_j} \}, \]

for all \((x, u, v) \in \bar{\Omega} \times [0, +\infty) \times [0, +\infty)\). Consider the following eigenvalue problem

\[ -\Delta u + a\psi_{\alpha}(u) = \lambda^{p-1} f(x, u, v) \text{ in } \Omega, \]
\[ -\Delta u + a\psi_{\beta}(u) = \lambda^{p-1} g(x, u, v) \text{ in } \Omega, \]
\[ u = v = 0 \text{ on } \partial\Omega. \]

(3.24)

Using the analogous proof to that of Theorem 3.1, we infer that (3.24) has a positive eigenvalue \(\lambda_0\) for which (3.24) possesses a nonnegative eigenfunction \((u_0, v_0)\).

Making use of the weak comparison principle, we have

\[ C_1 v_0 \leq u_0 \leq C_2 v_0 \text{ for all } x \in \bar{\Omega}. \]

(3.25)

A combination of (??) with the strong maximum principle implies \((u_0, v_0)\) is a positive eigenfunction for \(\lambda_0\). Thus, \(\lambda_0\) is a positive eigenvalue of (1.3) by the definition of the functions \(f(x, u, v)\) and \(g(x, u, v)\).

The proof of the uniqueness of \(\lambda_0\) is the same as that in Theorem 3.1, we omit it. \(\square\)

**Corollary 3.2** Assume that \(p = q > 1, a = b > 0\), and one of the following conditions is satisfied. Then (1.3) has a unique eigenvalue with positive eigenfunctions.

there exist \(\alpha_1\) and \(\beta_1\) such that

\[ a_1(x) > 0, b_1(x) > 0 \text{ in } \bar{\Omega}, \]
\[ p - 1 \leq \alpha_1 + \beta_j < 2(p - 1) \text{ for all } j \in \{1, 2, ..., N\}, \]
\[ p - 1 \leq \alpha_i + \beta_1 < 2(p - 1) \text{ for all } i \in \{1, 2, ..., k\}, \]

\(\alpha_1 < p - 1\) (or \(\beta_1 < p - 1\)), \(a_i(x) \equiv a_i > 0, b_j(x) \equiv b_j > 0\) for \(i = 1, 2, ..., k\), \(j = 1, 2, ..., N\), here \(a_i\) and \(b_j\) are positive numbers, and \(\sum_{i=1}^{k} a_i(x) = \sum_{j=1}^{N} b_j(x) \text{ for all } x \in \bar{\Omega}. \)

If \(\alpha_i\) and \(\beta_1\) satisfy the condition (1), we can prove that the conditions in Theorem 3 hold. More precisely, we choose positive constant \(C_1\) small enough such that

\[ C_1^{p-1} \sum_{j=1}^{N} b_j(x) u^{p-1-\beta_j} v^{\beta_j} \leq a_1(x) u^{\alpha_1} \]

for \((x, u, v) \in \bar{\Omega} \times (0, +\infty) \times (0, +\infty)\) with \(C_1 v \leq u\), and a positive number \(C_2\) large enough such that

\[ \sum_{i=1}^{k} a_i(x) u^{\alpha_i} v^{p-1-\alpha_i} \leq C_2^{p-1} b_1(x) u^{p-1-\beta_1} v^{\beta_1} \]

for \((x, u, v) \in \bar{\Omega} \times (0, +\infty) \times (0, +\infty)\) with \(u \leq C_2 v\). Therefore, Theorem 3.2 implies that (1.3) has a unique eigenvalue with positive eigenfunctions. \(\square\)
4 Uniqueness of Eigenfunctions

In this section, we prove that, under some reasonable hypotheses, the eigenfunction associated to \( \lambda_0 \) is unique up to a scalar multiple. Consider the following eigenvalue problem

\[
-\Delta_p u + a\psi_p(u) = \lambda_0^p - 1 A(x) \sum_{i=1}^{k} a_i u^{p-1-\alpha_i} \quad \text{in } \Omega,
\]

\[
-\Delta_p v + a\psi_p(v) = \lambda_0^p - 1 A(x) \sum_{j=1}^{N} b_j v^{q-1-\beta_j} \quad \text{in } \Omega,
\]

\[u = v = 0 \quad \text{on } \partial \Omega \quad (4.1)\]

where \( a_i \) and \( b_j \) are positive numbers, \( A(x) \) is a continuous, nonnegative, never vanishing function in \( \Omega \).

**Theorem 4.1** Assume that \( \alpha_1 < p - 1 \) (or \( \beta_1 < p - 1 \)). Then, the eigenvalue problem (4.1) has a unique eigenvalue \( \lambda_0 \) for which the eigenfunctions are positive, the eigenfunction associated to \( \lambda_0 \) is unique up to a scalar multiple.

**Proof:** It is obvious that we can choose a positive constant \( C \) such that

\[
\sum_{i=1}^{k} C^{p-1-\alpha_i} a_i \equiv \sum_{j=1}^{N} b_j C^{-p+1}.
\]

We denote by \( \lambda_1, \phi_1(x) \) the first eigenvalue and the eigenfunction, respectively, of the following eigenvalue problem:

\[
-\Delta_p \phi_1 + a\psi_p(\phi_1) = \lambda_1^p - 1 A(x) \sum_{i=1}^{k} a_i C^{p-1-\alpha_i} \phi_1^{p-1} \quad \text{in } \Omega,
\]

\[\phi_1 = 0 \quad \text{on } \partial \Omega \quad (4.2)\]

It is well-known that

\[
\lambda_1^{p-1} = \inf \{ \int_{\Omega} |\nabla u|^p + a|u|^p dx : \int_{\Omega} A(x) \sum_{i=1}^{k} a_i |u|^{p-1} dx = 1, u \in W_0^{1,p} \}
\]

and the eigenfunction for \( \lambda_1 \) is a minimizer, such a minimizer exists and is unique up to a scalar multiple. Obviously, \( \lambda_1 \) is an eigenvalue of (4.1), and \( (C^{-1} \phi_1, \phi_1) \) is a positive eigenfunction of (4.1) associated to \( \lambda_1 \).

In analogy with our proof of Theorem 3.1, we infer that (4.1) has a unique eigenvalue with positive eigenfunctions, and hence \( \lambda_0 = \lambda_1 \).

Now we prove that the eigenfunction for \( \lambda_0 \) is unique up to a scalar multiple. Without loss of generality, we assume \( C = 1 \). Suppose \( (u_1, v_1) \) is another positive eigenfunction for \( \lambda_0 \), that is, \( (u_1, v_1) \) satisfies

\[
-\Delta_p u_1 + a\psi_p(u_1) = \lambda_0^p - 1 A(x) \sum_{i=1}^{k} a_i u_1^{p-1-\alpha_i} \quad \text{in } \Omega,
\]

\[
-\Delta_p v_1 + a\psi_p(v_1) = \lambda_0^p - 1 A(x) \sum_{j=1}^{N} b_j v_1^{q-1-\beta_j} \quad \text{in } \Omega,
\]

\[u_1 = v_1 = 0 \quad \text{on } \partial \Omega \quad (4.4)\]

(4.5)
Combining (4.11) with (4.12) we deduce that

\[ \int_\Omega \| \nabla u_1 \|^p + a |u_1|^p dx \leq \lambda_1^{p-1} \sum_{i=1}^k \int_\Omega A(x) a_i u_i^p dx \frac{a_i}{a_i + p} \int_\Omega A(x) a_i v_i^p dx \]

(4.7)

Combining (4.9) and (4.10) we deduce from (4.6) that

\[ \int_\Omega \| \nabla v_1 \|^p + a |v_1|^p dx \leq \lambda_1^{p-1} \sum_{j=1}^N \int_\Omega A(x) b_j u_j^p dx \frac{b_j}{b_j + p} \int_\Omega A(x) b_j v_j^p dx \]

(4.8)

Denote by

\[ A_i = \frac{\| u_1 \|^p_{A(x)a_i,p}}{\| u_1 \|^p_{A(x)x,a_i,p}}, \quad B_i = \frac{\| v_1 \|^p_{A(x)a_i,p}}{\| u_1 \|^p_{A(x)x,a_i,p}} \]

\[ C_j = \frac{\| u_1 \|^p_{A(x)b_j,p}}{\| v_1 \|^p_{A(x)b_j,p}}, \quad D_j = \frac{\| v_1 \|^p_{A(x)b_j,p}}{\| v_1 \|^p_{A(x)b_j,p}} \]

here \( \| u \|_{B(x),p} \) denotes \( \int_\Omega B(x) u^p dx \). Thus, we have

\[ \int_\Omega (\| \nabla u_1 \|^p + a |u_1|^p dx \leq \lambda_1^{p-1} K \sum_{i=1}^k A_i B_i^{1-p-1(\alpha_i+1)} \int_\Omega A(x) a_i |u_i|^p dx \]

(4.9)

and

\[ \int_\Omega (\| \nabla v_1 \|^p + a |v_1|^p dx \leq \lambda_1^{p-1} \sum_{j=1}^N D_j C_j^{1-p-1(\beta_j+1)} \int_\Omega A(x) b_j |v_j|^p dx \]

(4.10)

with

\[ \sum_{i=1}^k A_i = 1, \sum_{j=1}^N D_j = 1, \quad B_i = \frac{\| v_1 \|^p_{A(x)b_j,p}}{\| u_1 \|^p_{A(x)b_j,p}} \]

(4.11)

By the definition of \( \lambda_1 \) it follows from (4.9) and (4.10) that

\[ \sum_{i=1}^k A_i B_i^{1-p-1(\alpha_i+1)} \geq 1 \quad \text{and} \quad \sum_{j=1}^N D_j C_j^{1-p-1(\beta_j+1)} \geq 1. \]

(4.12)

Combining (4.11) with (4.12) we deduce that

\[ B_i = C_j = 1, \]

and

\[ \sum_{i=1}^k A_i B_i^{1-p-1(\alpha_i+1)} = 1, \quad \sum_{j=1}^N D_j C_j^{1-p-1(\beta_j+1)} = 1. \]

(4.13)

A combination of (4.9), (4.10) and (4.13) with the fact that the minimizer defined in (4.3) is unique up to a scalar multiple yields

\( (u_1, v_1) = d(\phi_1, \phi_1) \)
for some $d \in (0, +\infty)$.

**Case (ii)** \( \alpha_1 < p - 1, \beta_j = p - 1 \) for all \( j \in \{1, 2, ..., N\} \). In this case, from (4.7) we have

\[
\int_{\Omega} (|\nabla v_1|^p + a|v_1|^p) dx \leq \lambda_1^{p-1} \int_{\Omega} \sum_{j=1}^{N} A(x) b_j |v_j|^p dx,
\]

which implies

\[
v_1 = d \phi_1 \text{ for some } d \in (0, +\infty).
\]

It is obvious that \( d \phi_1 \) and \( u_1 \) are two positive solutions of

\[
-\Delta_p u + \mathcal{A}(u) = \lambda_1^{p-1} \sum_{i=1}^{k} a_i u^{\alpha_i} v_1^{p-1-\alpha_i} \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega.
\]

(4.14)

The fact that (4.14) possesses only a positive solution (cf. [1, Theorem 1]) implies \( u_1 = d \phi_1 \).

**Case (iii)** \( \beta_1 < p - 1, \alpha_i = p - 1 \) for all \( i \in \{1, 2, ..., k\} \). In this case, the proof is the same as that in case (ii), we omit it. \(
\]

**Remark** When \( \alpha_i = p - 1 (i = 1, 2, ..., k), \beta_j = p - 1 (j = 1, 2, ..., N) \), we easily deduce that (4.1) has a unique eigenvalue with positive eigenfunctions if and only if \( a_i \) and \( b_j \) satisfy the equality

\[
\sum_{i=1}^{k} a_i = \sum_{j=1}^{N} b_j.
\]

**References**


