Developable algebraic surfaces

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Abstract An algebraic surface can be defined by an implicit polynomial equation $F(x,y,z)=0$. In this paper, general characterizations of developable algebraic surfaces of arbitrary degree are presented. Using the shift operators of the subscripts of Bézier ordinates, the uniform apparent discriminants of developable algebraic surfaces to their Bézier ordinates are given directly. To degree 2 algebraic surfaces which are widely used in computer aided geometric design and graphics, all possible developable surface types are obtained. For more conveniently applying algebraic surfaces of high degree to computer aided geometric design, the notion of ε-quasi-developable surfaces is introduced and an example of using a quasi-developable algebraic surface of degree 3 to interpolate three curves of degree 2 is given.

Keywords: implicit surfaces, algebraic surfaces, developable surfaces, shift operators.

Developable surfaces are surfaces that can be unfolded into planes without stretch or tearing. They are studied as important objects all through in computer aided geometrical design. During design and manufacture of skins of many vehicles such as aircraft skins, ship hulls and automobile developable surfaces are a kind of surfaces that are useful simple and very important. Especially, skins of some industry products made of paper, leather, plywood, fabric, sheet-metal and plate-metal consist of pieces of developable surfaces. Therefore appealing to studying of developable surfaces is more and more urgent. In classical differential geometry characterizations and properties of developable surfaces have already been studied sufficiently. Developable surfaces are classified into three types that is, cylindrical surfaces, conical surfaces and tangent surfaces of curves. However, how to represent, calculate and render developable surfaces conveniently and efficiently in computer aided design systems is a problem that has not been solved thoroughly yet. There are two classes of approaches to studying developable surfaces. One approach based on the dual representation of surfaces and approaches based on the ordinary representation of surfaces. However, there are inherent difficulties with the former class of approaches. For example, designing a surface by its dual representation is not very intuitive, it may be difficult to avoid singularities and points at infinity. We can only limit our study to ruled surfaces and putting such surfaces together is not free and natural enough. There are mostly Aumann[10] and Lang and Röschel[11] who study the latter class of approaches. Aumann[10] studied the developable characterization of the ruled surface formed by two Bézier curve called the designing curve and the following curve on two parallel planes respectively. Lang and Röschel[11] presented the developable characterization of the rational Bézier surface of degree $1 \times n$ Chen and Wang presented the developable characterization of Bézier function surfaces of arbitrary degree on the rectangular and triangular domains[7, 8]. In this paper we will study the developable characterization of implicitly defined algebraic surfaces.

It is a method in common use in computer aided geometric design that algebraic curves or surfaces are represented by implicit polynomial equations. In $R^3$ an implicitly defined algebraic surface of degree $n$ is defined by $F(x,y,z)=0$ where $F$ is a polynomial of degree $n$ with real coefficient about $x,y,z$. The algebraic surface appearing later in this paper refers to the surface of $F$ type if we do not give an explanation in particular. There are several obvious advantages with this kind of representation compared to the parametric representation. Firstly, for its representation by polynomial equations computing is more efficient than that of representation by general analytic functions or rational polynomial functions. Secondly, algebraic surfaces naturally classify points
in 3D space into three categories points inside a solid points on the boundary and points outside the solid. Thus point classification becomes quite easy which is useful for geometric operations such as intersection and offset. Thirdly it has more degree of freedom than polynomial or rational polynomial parametric surfaces with the same algebraic degree. So it can represent more complex skins of products with lower algebraic degree. Lastly and the most importantly algebraic surfaces are closed under geometric operations such as intersection union convolution offset and blending. So the result surfaces after these operations can be represented exactly other than approximately which is important for geometric design. But the rational polynomial parametric surfaces we use usually are not closed.

In this paper we first define the algebraic surface patch using Bernstein-Bézier representation. Then general characterizations of developable algebraic surfaces of arbitrary degree are derived. The characterizations of developable algebraic surfaces of degree 2 and its some applications are also presented. Considering there are many skins of products which consist of the material with a little retractability it is unnecessary to limit skin surfaces to exact developable surfaces in many industry applications for example clothing making with fabric and manufacture with plastic film. And for more conveniently using algebraic surfaces of high degree in computer aided geometric design we introduce the notion of the ε-quasi-developable algebraic surfaces and give an example of using a quasi-developable algebraic surface of degree 3 to interpolate three curves of degree 2. However after all studying of developable algebraic surfaces is just at its beginning. For examples it is significant to make studies of classifying developable algebraic surfaces of degree 2 smoothly jointing several developable algebraic surfaces and constructing developable algebraic spline surfaces.

1 Algebraic surface patches

There are many types of the representations of algebraic surfaces. By defining each algebraic surface patch in a tetrahedron and adopting Bernstein-Bézier representations of polynomial piecewise algebraic surfaces are able to model a wide variety of geometric object. And a crucial property of this algebraic surface patch formulation is that it inherits most of the tools of Bézier curves and surfaces such as imposing cross-boundary derivative continuity and operator representations of surfaces. So we use this representation in this paper.

**Definition 1.** Let \( V \) be a given tetrahedron with vertices \( T_1, T_2, T_3, T_4 \). For a point \( p \in V \) let \( s_{i+k+l} \) be barycentric coordinates of \( p \) with respect to \( V \). Let \( I_n \) be the nonnegative integer set. To each set \( f_i = \sum f_{i+k+l} \in I_n \) \( i+j+k+l = n \) the algebraic surface patch of degree \( n \) in \( V \) is defined by

\[
F = B\sum_{i+j+k+l=n} f_{i+k+l} s_{i+k+l} = 0 \quad s + t + u + v = 1
\]

where

\[
B_{i+k+l}^{n} s_{i+k+l} = \binom{n}{i} \binom{n}{j} \binom{n}{k} \binom{n}{l}
\]

are Bernstein bases.

If we define shift operators \( E_r \) \( r = 1 \) \( 2 \) \( 3 \) \( 4 \) on the subscripts of the Bézier ordinates \( f_{i+k+l} \) as

\[
E_1 f_{i+k+l} = f_{i+1+k+l} \quad E_2 f_{i+k+l} = f_{i+k+1+l} \quad E_3 f_{i+k+l} = f_{i+k+l+1} \quad E_4 f_{i+k+l} = f_{i+k+l-1}
\]

then the algebraic surface patch of degree \( n \) in \( V \) can be represented as

\[
F = E_1 sE_1 + tE_2 + uE_3 + vE_4 f_{0000} = 0
\]

For more convenience the partial derivatives are given by an operator representation here. Substituting \( v = 1 - s - t - u \) into \( F \) and applying the chain rule the derivatives of degree 1 of \( F \) about \( s \) \( t \) \( u \) are

\[
F_r = n \sum_{i+j+k+l=n-1} a_{i+k+l}^{r} B_{i+k+l}^{n} s_{i+k+l} \quad a_{i+k+l}^{r} = \frac{E_r - E_s f_{i+k+l}}{s + t + u + v}
\]

Ulteriorly the derivatives of degree 2 of \( F \) about \( t \) \( u \) are

\[
F_{tu} = n \sum_{i+j+k+l=n-2} b_{i+k+l}^{tu} B_{i+k+l}^{n} s_{i+k+l} \quad b_{i+k+l}^{tu} = \frac{E_t - E_u f_{i+k+l}}{s + t + u + v}
\]

2 Developable algebraic surfaces

A developable surface is developable if and only if its Gaussian curvature \( K = \delta \frac{18}{170} \). So it is the key work to derive the expressions of the Gaussian curvatures \( K \) of surfaces.
Lemma 1. For an algebraic surface \( F_0 \subseteq \mathbb{R}^3 \) \( z = 0 \) the Gaussian curvature is
\[
K = \frac{-H}{F_x^2 + F_y^2 + F_z^2}
\]
where
\[
H = \begin{vmatrix}
F_{xx} & F_{xy} & F_{xz} & F_x \\
F_{yx} & F_{yy} & F_{yz} & F_y \\
F_{zx} & F_{zy} & F_{zz} & F_z \\
F_x & F_y & F_z & 0
\end{vmatrix}
\]

Proof. Let \( d\mathbf{X} = dxdydz \) and \( \mathbf{G} = F_x F_y F_z F_{xy} F_{xz} F_{yz} \). Then the normal of the surface is \( \mathbf{N} = \mathbf{G} \mathbf{G}^{-1} \) and \( d\mathbf{N} = d\mathbf{G} \mathbf{N} \mathbf{G}^{-1} \). From the Rodrigues formula\(^{19}\) we have \( d\mathbf{N} + k d\mathbf{X} = 0 \) where \( k \) is the principal curvature of the surface. Substituting \( d\mathbf{X} d\mathbf{N} \) into it\(\) after some calculations we have
\[
\sum a_{ij} - k \mathbf{I} dxdydz = 0
\]
where \( \mathbf{I} \) is the third order unit matrix and \( a_{ij} \) is a matrix of order 3 whose elements are expressions of partial derivatives of degree 1 and 2 of the surface
\[
a_{ij} = F_{ij} \mathbf{G} \mathbf{G}^{-1} \sum_{k=1}^3 F_{ik} F_{kj} \mathbf{G} \mathbf{G}^{-1} \mathbf{G}^3
\]

Lemma 3.\(\)
\[
\sum_{i+j+k+1 = n} x^{i} y^{j} z^{k} \frac{\partial^n}{\partial x^i \partial y^j \partial z^k} \sum_{i+j+k+1 = n} x^{i} y^{j} z^{k} \frac{\partial^n}{\partial x^i \partial y^j \partial z^k} \sum_{i+j+k+1 = n} x^{i} y^{j} z^{k} \frac{\partial^n}{\partial x^i \partial y^j \partial z^k} \sum_{i+j+k+1 = n} x^{i} y^{j} z^{k} \frac{\partial^n}{\partial x^i \partial y^j \partial z^k} \sum_{i+j+k+1 = n} x^{i} y^{j} z^{k} \frac{\partial^n}{\partial x^i \partial y^j \partial z^k}
\]

Applying Lemma 3 we can derive the necessary and sufficient condition that the algebraic surface \( \mathbb{R}^3 \) of degree \( n \) is developable. Note that there is a linear relation between the partial derivatives of a function under barycentric coordinates and \( x y z \) coordinates. Substituting 2a and 2b into 5 we obtain
\[
H = \begin{vmatrix}
4n - 6 \\
-2n - 2 \mathbf{n} - 1 \mathbf{n} - 1 \mathbf{n}
\end{vmatrix}
\]
where
\[
A_1 = A_2 \begin{vmatrix}
12n \mathbf{n} + 12 \mathbf{n} + 12 \mathbf{n} + 12 \mathbf{n}
\end{vmatrix}
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A_2 = \begin{vmatrix}
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\end{vmatrix}
\]
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A_3 = 2A_1 \begin{vmatrix}
12n \mathbf{n} + 12 \mathbf{n} + 12 \mathbf{n} + 12 \mathbf{n}
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A_4 = A_1 \begin{vmatrix}
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A_5 = A_1 \begin{vmatrix}
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A_9 = A_1 \begin{vmatrix}
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Theorem 1. The algebraic surface $\{1\}$ of degree $n$ is developable if and only if for all $i,k,l \geq 0$ the following $4n - 3 \choose 3$ equations hold:

$$H_{ijkl} = \sum_{i_1 + j_1 + k_1 + l_1 = n-2} i \cdot j_1 j_2 j_3 j_4 \cdot k_1 k_2 k_3 k_4 \cdot l_1 l_2 l_3 l_4 = 0$$

where $A_I$ is defined by $7$ and $8$.

From $7$ and $8$ we know that $A_I$ is defined by $a_{ijkl}^r$ and $b_{ijkl}^p$ ultimately. And from $2a$ and $2b$ we know $a_{ijkl}^r$ and $b_{ijkl}^p$ only depend on $f_{ijkl}$. So the developable characterization of the algebraic surface $\{1\}$ of degree $n$ can be determined only by the $4n - 3 \choose 3$ equations about $f_{ijkl}$.

When $n = 1$ we have $4n - 6 = -2 < 0$. So the developable conditions $9$ of the algebraic surface $\{1\}$ hold. In fact, when $n = 1$, directly from Definition 1 we know that it is a plane patch which is the trivial case of developable surfaces.

Corollary 1. When $n = 1$ the algebraic surface $\{1\}$ is always developable.

3. Developable algebraic surfaces of degree 2

From Theorem 1 in the above section we can get the developable characterization of algebraic surface $\{1\}$ of degree 2 which is often used in computer aided geometric design.

Theorem 2. The algebraic surface $\{1\}$ of degree 2 is developable if and only if the following 10 equations hold:

$$H_{ijkl} = \sum_{i_1 + j_1 + k_1 + l_1 = n-2} i \cdot j_1 j_2 j_3 j_4 \cdot k_1 k_2 k_3 k_4 \cdot l_1 l_2 l_3 l_4 = 0$$

where $A_I$ is defined by $7$ and $8$ with $i_1 = j_1 = k_1 = l_1 = i_2 = j_2 = k_2 = l_2 = 0$ $i_4 = i - i_3$ $j_4 = j - j_3$ $k_4 = k - k_3$ $l_4 = l - l_3$ $i j k l \geq 0$ $i + j + k + l = 2$.

According to Theorem 2 and with the help of some mathematical software such as Maple and Mathematica we can obtain 23 solutions of the set of 10 equations about $f_{ijkl}$ in which there are 7 $11$ and 1 solution with $5 \ 6 \ 7$ and 8 free parameters respectively. Substituting one of solutions into algebraic surface $\{1\}$ we can get one type of developable algebraic surfaces of degree 2. In the following we give four examples for every one of solutions with $5 \ 6 \ 7$ and 8 free parameters.

Example 1. One type of developable algebraic surfaces of degree 2 with 5 free parameters is

$$f_{0101} + f_{0020} - 2f_{0011}a^2 + 2f_{1010} - f_{0011}b^2 + f_{0110} - f_{0011}c + f_{0001}d + f_{0011}e = 0$$

is the coefficients of the power basis

$$s^2t^2 | u^2 | v^2 | w^2 | x^2 | y^2 | z^2$$

where $a = f_{0011}$ $b = f_{1010}$ $c = f_{0001}d = f_{0101}e = f_{0101}f$.

Example 2. One type of developable algebraic surfaces of degree 2 with 6 free parameters is

$$f_{0200} + f_{1001} - 2f_{1100}a^2 + 2f_{0110} - f_{1001}b^2 + f_{1010} - f_{1100}c + f_{1001}d + f_{1010}e = 0$$

is the coefficients of the above basis $11$ are

$$0 \ a + c - 2b^2 a - a - d - a $$

where $a = f_{1001}b = f_{1100}c = f_{0200}d = f_{1010}e = f_{0001}g = f_{0101}h$.

Example 3. The coefficients of the $11$ in one type of developable algebraic surfaces of degree 2 with 7 free parameters are

$$-2e + b + g$$

$$b^2 g + h^2 b^2 - 2h^2 e + 4heb - 2hbg - 2hbg^2 - 2e b - 2h e$$

$$-2d + c + b - 2a - 2a + b + 2a + b$$

$$eh - he + bd + ha - ab - dh + hb - b^2$$

where $a = f_{1010}b = f_{0002}c = f_{0020}d = f_{1001}e = f_{1001}g = f_{2000}h = f_{0101}$.

Example 4. The coefficients of the $11$ in the type of developable algebraic surfaces of degree 2 with 8 free parameters are
\[-2g + b + h \bar{w}_1 \bar{w}_2 - g - p + b + a \bar{w}_1 \]
\[d - g + b - c - p - c + e + b \bar{w}_1 \bar{w}_2 - b \bar{w}_1 \bar{w}_2 + b + c \bar{w}_1 \bar{w}_2\]

where

\[w_1 = -2bh - b^2 d - bhc + b^2 h + cph\]
\[+ gbc + gbd + p^2 d - a^2 c + a^2 b - dp^2\]
\[+ pad + ebd - hep - ebc + pac - gdp\]
\[+ gca - 2gap + gep - 2gcp + gca + 2g^2 p\]
\[+ g^2 e - bg^2 - bad + 2bpd \]
\[-ch + hb - g^2 + gc + dg - bd \]

\[w_2 = cpd - c^2 a + cad - ech - edb - bph\]
\[-2bhc + b^2 h + cph + 2g^2 c + d^2 b + gec\]
\[-d^2 p + ebp + c^2 h - 2dgc + gdp - gca\]
\[-2gcp - g^2 e - bg^2 - b^2 a + abg + pgb\]
\[+ dge + 2bac - bad \bar{w}_1 \bar{w}_2 \bar{w}_1 \bar{w}_2 ph - gp\]
\[+ g^2 - ga - hb + ab \bar{w}_1 \bar{w}_2 \bar{w}_1 \bar{w}_2 \]

\[a = f_{1100} b = f_{0002} c = f_{0010} d = f_{1010} \]
\[e = f_{0100} g = f_{1000} h = f_{2000} p = f_{0101} .\]

4 Applications of developable algebraic surfaces of degree 2

From the above section we know that there are several degrees of freedom in every one type of developable algebraic surfaces of degree 2. So we can control the shapes of the surfaces using the degrees of freedom to serve our own purposes. As an example we can use the type of developable algebraic surfaces in Example 1 in the above section to interpolate 4 points given since it has 4 degrees of freedom. After substituting the coordinates of the 4 points into the surface equation in Example 1 we get 4 linear equations about \(f_{ijkl}\). Solving the set of linear equations we get a developable algebraic surface of degree 2 that interpolates the 4 points given at once. To leave one degree of freedom to select a good surface we give an example to interpolate 3 points here.

Example 5. Seek a developable algebraic surface of degree 2 to interpolate 3 points \(p_1 = 0.16, 0.4, 0.5\) \(p_2 = 0.475, 0.1, 0.2\) \(p_3 = 0, 0.4, 0.1\) Without loss of generality here we let the 3 points in the tetrahedron have vertices \(0, 0, 0, 0\) \(1, 0, 0, 0\) \(0, 1, 0, 0\) \(0, 0, 1, 0\).

After solving the set of linear equations to the problem we can select one surface defined by \(u^2 - 2us - 3ut + u + 0.01 = 0\). Fig. 1 shows its parts with \(s \leq u \leq 0, 0.1\).

Fig. 1. A developable algebraic surface of degree 2 interpolating 3 points given contour map.

Lemma 4. Bezout’s theorem A space curve \(l\) of degree \(m\) either intersects a surface \(\Omega\) of degree \(n\) in exactly \(mn\) points properly counting complex infinite and multiple intersections or else the curve \(l\) lies completely on the surface \(\Omega\).

From Lemma 4 we know that if a straight line intersects an algebraic surface of degree 2 in 3 points the entire line must lie on the surface.

Example 6. Seek a developable algebraic surface of degree 2 to interpolate the line \(O_1O_2\) with \(O_1 = 0.0, 0, 0, 0\) \(O_2 = 1, 1, 1, 1\). Selecting 3 points in the line \(p_1 = 0, 0, 1, 0.1\) \(p_2 = 0.2, 0.2, 0.2\) \(p_3 = 0.3, 0.3, 0.3\) we can obtain a plane patch that interpolates the 3 points and also interpolates the line \(O_1O_2\) which is certainly developable.

5 Interpolating curves with developable algebraic surfaces

From Lemma 4 we can interpolate some points in a curve with a developable algebraic surface of degree \(n \geq 2\) to interpolate the curve. And we can let a surface interpolate points through solving a set of linear equations about \(f_{ijkl}\). On the other hand the developable conditions are also given by some constraint equations. Therefore we can put the equations to interpolate curves with a surface and solve the developable conditions of the surface together. Thus we have the result surface satisfies the conditions of both interpolation and developability there are fewer degrees of freedom in the obtained surface. So it is used more conveniently. For developable algebraic surfaces of degree 2 we have

Proposition 1. There always exists a developable algebraic surface of degree 2 that interpolates an algebraic plane curve.

Proof. An algebraic plane curve always lies on a right circular cone. And a right circular cone is a developable algebraic surface of degree 2. So the propo-
Example 7. Seek a developable algebraic surface of degree 2 to interpolate the plane curve \( s^2 - 2t = 0 \) \( u = 0 \).

There are two solutions for this problem. The surfaces of the first solution are parabolic cylinders\( s \parallel 1 + f_{0100}^2 u = \parallel 1 \) \( t - f_{0100}^2 u = \parallel 0 \). Fig. 2 \( a \) shows the surface with \( f_{0100}^2 = 1 \). The surfaces of the second solution are parabolic cylinders\( s - f_{0011}^2 t - f_{0011}^2 u = \parallel 0 \). Fig. 2 \( b \) shows the surface with \( f_{0011}^2 = 1 \) and \( f_{0100}^2 = -2 \).

Fig. 2. A developable algebraic surface of degree 2 interpolating a curve contour map\( a \) \( a \) With \( f_{0100}^2 = 1 \) \( b \) with \( f_{0011}^2 = 1 \) \( f_{0100}^2 = -2 \).

Now we study another problem that is to seek an algebraic surface of degree \( \parallel n > 3 \parallel \) that is developable and at the same time interpolates several curves such as 3 curves of degree 2. If there are only interpolating conditions\( \parallel \) using Lemma 4 we can verify the existence of algebraic surfaces of degree \( n \) that satisfy the interpolating conditions. However\( \parallel \) if there are both interpolating and developable conditions\( \parallel \) we cannot assure that the system has a solution in general. Moreover\( \parallel \) the amount \( H_{ijkl} \) in the developable conditions\( \parallel 9 \parallel \) is nonlinear polynomial about \( f_{ijkl} \). For algebraic surfaces of high degree\( \parallel \) the computing complexity is very high. To get over the limitation of applications of developable algebraic surfaces in geometric design\( \parallel \) we introduce the notion of quasi-developable surfaces as follows.

Definition 2. For an algebraic surface\( \parallel 1 \parallel \) of degree \( n \)\( \parallel \) if the sum of the square of all the \( H_{ijkl} \) in the developable conditions is less than a given nonnegative small real number \( \epsilon \)\( \parallel \) that is\( \parallel H_i = \sum_{i+j+k+l=0} H_{ijkl}^2 \leq \epsilon \parallel \) then we say the algebraic surface\( \parallel 1 \parallel \) is \( \epsilon \)-quasi-developable. The \( H_i \) is called the quasi-developable order of the surface. Here \( H_{ijkl} \) are defined by Theorem 1.

Theorem 3. For any given nonnegative real number \( \epsilon \)\( \parallel \) a developable algebraic surface\( \parallel 1 \parallel \) of degree \( n \) is \( \epsilon \)-quasi-developable. When \( \epsilon \) runs to zero\( \parallel \) the \( \epsilon \)-quasi-developable surfaces tend to become developable.

Proof. The former half part of the theorem is obviously true. From\( \parallel 3 \parallel \)\( \parallel 6 \parallel \) noting the property of “Convex hull” of polynomial parametric surfaces defined by Bernstein bases\( \parallel \) the latter half part can be obtained.

Algorithm 1. Constructing a quasi-developable algebraic surface\( \parallel 1 \parallel \) of degree \( n \) to interpolate given \( m_1 \) curves of degree \( n_1 \)\( \parallel \)

Suppose \( p = \parallel n + 3 \parallel n + 2 \parallel n + 1 \parallel 6 - 1 \geq m_1 \) \( n_1 + 1 \parallel q \parallel \) to assure that there exists a surface interpolating the given curves where \( q \) is the number of intersection points of the \( m_1 \) curves of degree \( n_1 \).\n
Step 1. Get the set \( S \) of intersection points of the \( m_1 \) curves of degree \( n_1 \).

Step 2. Get \( n_1 + 1 - SI \) sample points on the given \( m_1 \) curves\( \parallel \) where \( SI \parallel \) is the number of intersection points on each curve in the set \( S \).

Step 3. Set the four vertices of the tetrahedron \( V \) which defines the surface\( \parallel 1 \parallel \) such that \( V \) includes all of the intersection points.

Step 4. Compute the barycentric coordinates of all of the sample points and intersection points with respect to \( V \).

Step 5. Obtain the set of interpolation equations about \( f_{ijkl} \) by substituting all of the barycentric coordinates into Eq. 1\( \parallel 1 \parallel \).

Step 6. If \( p = m_1 \) \( n, n_1 + 1 \parallel q \parallel \) then go to Step 7\( \parallel \) otherwise solve the extreme value problem \( \sum_{i+j+k+l=0} H_{ijkl}^2 = \text{MIN} \) using Lagrange’ s method of multipliers under the constraint condition that the sum of all of free variables does not vanish\( \parallel \) i.e. \( \sum_{i+j+k+l=0} f_{ijkl} = \text{const} \neq 0 \). Here using the constraint condition is to refuse the trivial solution that all of \( f_{ijkl} \) vanish.

Step 7. Computing the value of \( H_i \).

Remark of Algorithm 1. From\( \parallel 7 \parallel \)\( \parallel 9 \parallel \) we know that \( H_{ijkl} \) are algebraic polynomial of \( f_{ijkl} \). And
after Step 5 in Algorithm 1 the degree of freedom of $f_{ijkl}$ is only $p \begin{pmatrix} m \end{pmatrix} n_1 + 1 - q$. So the computing complexity in Step 6 is down. When $p = m \begin{pmatrix} n \end{pmatrix} n_1 + 1 - q$ the result surface only satisfies the interpolating conditions. If the degree of surfaces is not high enough and many curves need to be interpolated the quasi-developable order $H_5$ of surfaces will be big.

**Example 8.** Construct a quasi-developable algebraic surface of degree 3 to interpolate a curved triangle formed by 3 plane curves of degree 2. These curves are $t = u - 1$, $s = 0$, $u - s - 1$, $s + t - 1$, $u = 0$.

To solve the problem, we select 7 sample points on each given curve and let two of selected points be intersection points of the 3 curves. Thus there are 18 points which need to be interpolated at the same time in the curved triangle in fact. By Lemma 4 a surface interpolates these 3 curves if it interpolates all of the 18 points. Thus 18 degrees of freedom are required to force a surface to interpolate the 3 curves. There are $18 = 20 - 1$ degrees of freedom in an algebraic surface of degree 3 so the remained degree of freedom can be used to satisfy developable conditions. Applying Algorithm 1 we can obtain the quasi-developable order $H_5 = 0.5e-13$. Fig. 3 a shows the resulting surface. The curves formed by its Gaussian curvature at $u = 0.1$ and $u = 0.5$ are shown in Fig. 3 b and c respectively. They show the surface is close to a developable one. In addition it is well known that the Gaussian image of a developable surface is a curve on the unit sphere according to differential geometry [439]. The Gaussian image of the surface is shown in Fig. 4 which is a long and narrow curved strip. It also shows the result that quasi-developable surface is close to a developable surface.

Fig. 4. The Gaussian image of the surface in Fig. a.

**References**