On Mutually Nearest and Mutually Furthest Points in Reflexive Banach Spaces

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Communicated by Frank Deutsch
Received June 2, 1998; accepted in revised form June 17, 1999

Let $G$ be a nonempty closed (resp. bounded closed) subset in a reflexive strictly convex Kadec Banach space $X$. Let $\mathcal{K}(X)$ denote the space of all nonempty compact convex subsets of $X$ endowed with the Hausdorff distance. Moreover, let $\mathcal{K}^G(X)$ denote the closure of the set $\{A \in \mathcal{K}(X) : A \cap G = \emptyset\}$. A minimization problem $\min(A, G)$ (resp. maximization problem $\max(A, G)$) is said to be well posed if it has a unique solution $(x_0, z_0)$ and every minimizing (resp. maximizing) sequence converges strongly to $(x_0, z_0)$. We prove that the set of all $A \in \mathcal{K}^G(X)$ (resp. $A \in \mathcal{K}(X)$) such that the minimization (resp. maximization) problem $\min(A, G)$ (resp. $\max(A, G)$) is well posed contains a dense $G$-subset of $\mathcal{K}^G(X)$ (resp. $\mathcal{K}(X)$), extending the results in uniformly convex Banach spaces due to Blasi, Myjak and Papini.

1. INTRODUCTION

Let $X$ be a real Banach space. We denote by $\mathcal{B}(X)$ the space of all nonempty closed bounded subsets of $X$. For a closed subset $G$ of $X$ and $A \in \mathcal{B}(X)$, we set

$$\lambda_{AG} = \inf\{\|z - x\| : x \in A, z \in G\},$$

and for $G \in \mathcal{B}(X)$, we set

$$\mu_{AG} = \sup\{\|z - x\| : x \in A, z \in G\}.$$

Given a nonempty closed subset $G$ of $X$ (resp. $G \in \mathcal{B}(X)$), according to [9], a pair $(x_0, z_0)$ with $x_0 \in A$, $z_0 \in G$ is called a solution of the minimization (resp. maximization) problem, denoted by $\min(A, G)$ (resp. $\max(A, G)$), if $\|x_0 - z_0\| = \lambda_{AG}$ (resp. $\|x_0 - z_0\| = \mu_{AG}$). Moreover, any sequence $\{(x_n, z_n)\}_n$,
\[ x_n \in A, z_n \in G, \text{ such that } \lim_{n \to \infty} \| x_n - z_n \| = \lambda_{AG} \text{ (resp. } \lim_{n \to \infty} \| x_n - z_n \| = \mu_{AG}) \text{ is called a minimizing (resp. maximizing) sequence. A minimization (resp. maximization) problem is said to be well posed if it has a unique solution } (x_0, z_0), \text{ and every minimizing (resp. maximizing) sequence converges strongly to } (x_0, z_0). \]

Set
\[ \mathcal{C}(X) = \{ A \in \mathcal{B}(X) : A \text{ is convex} \}, \]
and let \( \mathcal{C}(X) \) be endowed with the Hausdorff distance \( h \) defined as follows:
\[ h(A, B) = \max \left\{ \sup_{a \in A} \inf_{b \in B} \| a - b \|, \sup_{b \in B} \inf_{a \in A} \| a - b \| \right\}, \quad \forall A, B \in \mathcal{C}(X). \]

As is well known, under such metric, \( \mathcal{C}(X) \) is complete.

In [9], the authors considered the well posedness of the minimization and maximization problems. If \( X \) is a uniformly convex Banach space they proved that the set of all \( A \in \mathcal{C}_G(X) \) (resp. \( A \in \mathcal{G}(X) \)), such that the minimization (resp. maximization) problem \( \text{min}(A, G) \) (resp. \( \text{max}(A, G) \)) is well posed, is a dense \( G \) -subset of \( \mathcal{C}_G(X) \) (resp. \( \mathcal{G}(X) \)), where \( \mathcal{C}_G(X) \) is the closure of the set \( \{ A \in \mathcal{C}(X) : \lambda_{AG} > 0 \} \).

Furthermore, let
\[ \mathcal{K}(X) = \{ A \in \mathcal{C}(X) : A \text{ is compact} \} \]
and \( \mathcal{K}_G(X) = \mathcal{K}(X) \cap \mathcal{C}_G(X) \). Clearly, \( X \) can be embedded as a subset of \( \mathcal{K}(X) \) in a natural way that, for any \( x \in X \), \( A_x \in \mathcal{K}(X) \) is defined by
\[ A_x = \{ x \}. \]

It is our purpose in the present note to extend the results, with a completely different approach, to a reflexive strictly convex Kadec Banach space. We prove that if \( X \) is a reflexive strictly convex Kadec Banach space, then the set of all \( A \in \mathcal{K}_G(X) \) (resp. \( A \in \mathcal{K}(X) \)), such that the minimization problem \( \text{min}(A, G) \) (resp. maximization problem \( \text{max}(A, G) \)) is well posed, contains a dense \( G \) -subset of \( \mathcal{K}_G(X) \) (resp. \( \mathcal{K}(X) \)).

It should be noted that the problems considered here are in the spirit of Stechkin [27]. Some further developments of Stechkin’s ideas can be founded in [2–6, 8, 11–17, 20, 24, 26] and in the monograph [10], while some generic results in spaces of convex sets and bounded sets can be founded in [2, 3, 7, 19, 21].

In sequel, let \( X^* \) denote the dual of \( X \). We use \( B(x, r) \) to denote the closed ball with center at \( x \) and radius \( r \). As usual, if \( A \subseteq X \), by \( A \) and \( \text{diam } A \) we mean the closure and the diameter of \( A \), respectively, while \( \overline{A} \) stands for the closed convex hull of \( A \).
DEFINITION 1.1. Let $D$ be an open subset of $X$. A real-valued function $f$ on $D$ is said to be Frechet differentiable at $x \in D$ if there exists an $x^* \in X^*$ such that

$$\lim_{y \to x} \frac{f(y) - f(x) - \langle x^*, y - x \rangle}{\|y - x\|} = 0.$$ 

$x^*$ is called the Frechet differential at $x$ which is denoted by $Df(x)$.

The following proposition on the Frechet differentiability of Lipschitz functions due to [24] is useful.

PROPOSITION 1.1. Let $f$ be a locally Lipschitz continuous function on an open set $D$ of a Banach space with equivalent Frechet differentiable norm (in particular, $X$ reflexive will do). Then $f$ is Frechet differentiable on a dense subset of $D$.

DEFINITION 1.2. A Banach space $X$ is said to be (sequentially) Kadec provided that for each sequence $\{x_n\} \subset X$ which converges weakly to $x$ with $\lim_{n \to \infty} \|x_n\| = \|x\|$ we have $\lim_{n \to \infty} \|x_n - x\| = 0$.

DEFINITION 1.3. A Banach space $X$ is said to be strongly convex provided it is reflexive, Kadec and strictly convex.

We also need a result concerning the characterization of strongly convex spaces, which is due to Konjagin [15], see also Borwein and Fitzpatrick [5].

PROPOSITION 1.2. A Banach space $X$ is strongly convex if and only if for every closed nonempty subset $G$ of $X$ there is a dense set of points $X \setminus G$ possessing unique nearest points.

2. MINIMIZATION PROBLEMS

Let $x \in X$, $A \in \mathcal{K}(X)$ and $G$ be a closed subset of $X$. We set

$$d_G(x) = \inf_{z \in G} \|x - z\|,$$

$$d_G(A) = \inf_{x \in A} d_G(x) = \lambda(G),$$

and

$$P_G(A) = \{x \in A : d_G(x) = d_G(A)\}.$$
Then
\[ |d_G(A) - d_G(B)| \leq h(A, B), \quad \forall A, B \in \mathcal{X}(X). \]

For \( A \in \mathcal{X}(X) \), let \( f_A \) be the functional on \( X \) defined as follows:
\[ f_A(x) = d_G(A + x), \quad \forall x \in X. \]

Then \( f_A \) is 1-Lipschitz and satisfies \( f_A(x) = f_A(x) + x(0) \).

**Lemma 2.1.** Suppose that \( f_A \) is Frechet differentiable at \( x = 0 \) with \( Df_A(0) = x^* \). Then \( \|x^*\| = 1 \) and for any \( x_0 \in P_{\mathcal{A}}(G), \{z_n\} \subset G \) with \( \lim_{n \to \infty} \|x_0 - z_n\| = d_G(x_0) \), we have
\[ d_G(x_0) = \lim_{n \to \infty} \langle x^*, x_0 - z_n \rangle. \]

**Proof.** Let \( x_0, \{z_n\} \) satisfy the assumptions of the lemma. Then for each \( t > 0, \)
\[ f_A(t(z_n - x_0)) - f_A(0) = d_G(A + t(z_n - x_0)) - d_G(A) \]
\[ \leq \|x_0 + t(z_n - x_0) - z_n\| - d_G(A) \]
\[ = (1 - t) \|x_0 - z_n\| - d_G(A) \]
\[ = -t \|x_0 - z_n\| + \|x_0 - z_n\| - d_G(A). \]

Let \( t_n = 2^{-n} + \|x_0 - z_n\| - d_G(A) \). Then from the Frechet differentiability of \( f_A(x) \) at \( x = 0 \), we have that
\[ \lim_{n \to \infty} \left[ \frac{f_A(t_n(z_n - x_0)) - f_A(0) - \langle x^*, z_n - x_0 \rangle}{t_n} \right] = 0, \]
so that
\[ \liminf_{n \to \infty} \left[ -\|x_0 - z_n\| + \langle x^*, x_0 - z_n \rangle \right] \geq 0 \]
and
\[ d_G(A) = \lim_{n \to \infty} \|x_0 - z_n\| \leq \liminf_{n \to \infty} \langle x^*, x_0 - z_n \rangle. \]

Note that \( \|x^*\| \leq 1 \) since \( f_A \) is 1-Lipschitz. It follows that
\[ \lim_{n \to \infty} \|x_0 - z_n\| \geq \lim_{n \to \infty} \|x^*\| \|x_0 - z_n\| \geq \limsup_{n \to \infty} \langle x^*, x_0 - z_n \rangle. \]
Comparison of the last two inequalities shows the desired results, proving the lemma.

**Lemma 2.2.** The set-valued map \( P_A(G) \) with respect to \( A \) is upper semi-continuous in the sense that for each \( A_0 \in \mathcal{K}_G(X) \) and any open set \( U \) with \( P_A(G) \subseteq U \), there exists \( \delta > 0 \) such that for any \( A \in \mathcal{K}_G(X) \) with \( h(A, A_0) < \delta \), \( P_A(G) \subseteq U \).

**Proof.** Suppose on the contrary that there exist \( \{ A_n \} \subset \mathcal{K}_G(X) \) and \( A \in \mathcal{K}_G(X) \) with \( \lim_{n \to \infty} h(A_n, A) = 0 \), such that \( P_{A_n}(G) \not\subseteq U \) for some open subset \( U \) with \( P_A(G) \subseteq U \) and each \( n \). Let \( x_n \in P_{A_n}(G) \setminus U \) for any \( n \). Note that \( \bigcup_n A_n \) is relatively compact and \( \{ x_n \} \subset \bigcup_n A_n \). It follows that there exists a subsequence, denoted by itself, such that \( \lim_{n \to \infty} |x_n - x_0| = 0 \) for some \( x_0 \in X \). Clearly, \( x_0 \notin U \). However, by \( \lim_{n \to \infty} h(A_n, A) = 0 \), there exists \( \{ a_n \} \subset A \) such that \( \lim_{n \to \infty} |x_n - a_n| = 0 \) so that

\[
\limsup_{n \to \infty} |a_n - x_0| \leq \lim_{n \to \infty} |x_n - a_n| + \lim_{n \to \infty} |x_n - x_0| = 0
\]

and \( x_0 \in A \). Furthermore, for each \( n \),

\[
\inf_{z \in G} |z - x_0| \leq \inf_{z \in G} |z - x_n| + |x_n - x_0|
\]

\[
\leq d_G(A) + h(A_n, A) + |x_n - x_0|,
\]

which shows that \( x_0 \in P_A(G) \), contradicting that \( x_0 \notin U \). The proof is complete.

Let

\[
L_a(G) = \left\{ A \in \mathcal{K}_G(X) : \inf \left\{ \langle x^*, x - z \rangle : z \in G \land B(x, d_G(x) + \delta) \right\} > (1 - 2^{-n}) d_G(A), \text{ for some } \delta > 0, x^* \in X^* \text{ with } \|x^*\| = 1 \right\}
\]

Also let

\[
L(G) = \bigcap_n L_n(G).
\]

**Lemma 2.3.** Suppose that \( X \) is reflexive. Then \( L(G) \) is a dense \( G_d \)-subset of \( \mathcal{K}_G(X) \).
Proof. To show that $L(G)$ is a $G^*$-subset of $\mathcal{K}G(X)$, we only need prove that $L_n(G)$ is open for each $n$. Let $A \in L_n(G)$. Then there exist $x^* \in X^*$ with $\|x^*\| = 1$ and $\delta > 0$ such that

$$
\beta = \inf \{ \langle x^*, x - z \rangle : x \in P_d(G), z \in G \cap B(x, d_G(x) + \delta) \}
$$

$$
-(1 - 2^{-n}) d_G(A) > 0.
$$

Let $\lambda > 0$ be such that $\lambda < \min \{ (\delta/2), (\beta/2) \}$. It follows from Lemma 2.2 that there exists $0 < \epsilon < \lambda$ such that for any $F \in \mathcal{K}G(X)$ with $h(F, A) < \epsilon$ and each $y \in P_d(G)$ there exists $x \in P_d(G)$ satisfying $\|y - x\| < \lambda$. For $\delta^* = \delta - 2\lambda$ we have

$$
H = G \cap B(y, d_G(y) + \delta^*) \subset G \cap B(x, d_G(x) + \delta).
$$

Thus if $z \in H$,

$$
\langle x^*, x - z \rangle \geq \beta + (1 - 2^{-n}) d_G(A)
$$

and

$$
\langle x^*, y - z \rangle \geq \beta + (1 - 2^{-n}) d_G(F) - \lambda.
$$

Then

$$
\inf \{ \langle x^*, y - z \rangle : z \in H, y \in P_d(G) \} > (1 - 2^{-n}) d_G(F)
$$

and $F \in L_n(G)$ for all $F \in \mathcal{K}G(X)$ with $h(F, A) < \epsilon$, which implies that $L_n(G)$ is open in $\mathcal{K}G(X)$.

In order to prove the density of $L(G)$ in $\mathcal{K}G(X)$, from Proposition 1.1, it suffices to prove that if $f_d(x)$ is Frechet differentiable at $x = 0$ then $A \in L(G)$.

Suppose on the contrary that for some $n$ there exist $\{x_m\} \subset P_d(G)$ and $\{z_m\} \subset G \cap B(x_m, d_G(x_m) + 2^{-m})$ such that

$$
\langle x^*, x_m - z_m \rangle \leq (1 - 2^{-n}) d_G(A), \quad \forall m,
$$

where $x^* = Df_d(0)$. With no loss of generality, we assume that

$$
\lim_{m \to \infty} \|x_m - x_0\| = 0 \quad \forall x_0 \in P_d(G).
$$

Observe that $\lim_{m \to \infty} \|x_m - z_m\| = d_G(A)$. Then $\lim_{m \to \infty} \|x_0 - z_m\| = d_G(A)$. Thus Lemma 2.1 implies that

$$
\lim_{m \to \infty} \langle x^*, x_0 - z_m \rangle = d_G(A)
$$

so that

$$
\lim_{m \to \infty} \langle x^*, x_m - z_m \rangle = d_G(A)$$
which contradicts that
\[ \langle x^*, x_m - z_m \rangle \leq (1 - 2^{-m}) d_G(A), \quad \forall m. \]

This completes the proof.

**Lemma 2.4.** Suppose \( X \) is a reflexive Kadec Banach space. Let \( A \in L(G) \). Then any minimizing sequence \( \{ (x_n, z_n) \} \) with \( x_n \in A, \ z_n \in G \) has a subsequence which converges strongly to a solution of the minimization problem \( \min(A, G) \).

**Proof.** Let \( A \in L(G) \). Then \( A \in L_m(G) \) for any \( m = 1, 2, \ldots \). By the definition of \( L_m(G) \), there exist \( \delta_m > 0, x_m^* \in X^*, \| x_m^* \| = 1 \) such that
\[
\inf \{ \langle x_m^*, x - z \rangle : z \in G \cap B(x, d_G(x) + \delta_m), x \in P_A(G) \} > (1 - 2^{-m}) d_G(A).
\]

Let \( \{ (x_n, z_n) \} \) with \( x_n \in A, \ z_n \in G \) be any minimizing sequence. With no loss of generality, we assume that \( x_n \to x_0 \) strongly and \( z_n \to z_0 \) weakly as \( n \to \infty \) for some \( x_0 \in P_A(G), \ z_0 \in X \), since \( A \) is compact and \( X \) is reflexive.

Then we have that
\[
\| x_0 - z_0 \| \leq \liminf_{n \to \infty} \| x_n - z_n \| = d_G(A).
\]

We also assume that \( \delta_n \leq \delta_m \) if \( m < n \) and \( z_n \in G \cap B(x_0, d_G(x_0) + \delta_n) \) for all \( n > m \). Thus,
\[
\langle x_m^*, x_0 - z_n \rangle > (1 - 2^{-m}) d_G(A), \quad \forall n > m
\]
and
\[
\langle x_m^*, x_0 - z_0 \rangle > (1 - 2^{-m}) d_G(A), \quad \forall m.
\]

Hence we have
\[
\| x_0 - z_0 \| \geq \limsup_{m \to \infty} \langle x_m^*, x_0 - z_0 \rangle \geq d_G(A).
\]

This shows that \( \| x_0 - z_0 \| = d_G(A) \). Now the fact that \( X \) is Kadec implies that \( \lim_{n \to \infty} \| z_n - z_0 \| = 0 \) and \( z_0 \in G \). Clearly, \( (x_0, z_0) \) is a solution of the minimization problem \( \min(A, G) \) and completes the proof.

Let
\[
Q_n(G) = \left\{ A \in \mathcal{K}_G(X) : \text{diam } P_A(G) < \frac{1}{n} \right\}
\]
and let
\[ Q(G) = \bigcap_{n} Q_{n}(G). \]

**Lemma 2.5.** Suppose that \( X \) is reflexive Kadec Banach space. Then \( Q(G) \) is a dense \( G_{\delta} \)-subset of \( \mathcal{K}(X) \).

**Proof.** Given \( n \) and \( A \in Q_{n}(X) \), we define
\[ c = \frac{1}{n} - \operatorname{diam} P_{A}(G) \]
and
\[ U = \{ x \in X : d_{P_{A}(G)}(x) < \frac{c}{3} \}. \]
Then
\[ \operatorname{diam} U < \operatorname{diam} P_{A}(G) + \frac{2c}{3} < \frac{1}{n}. \]

It follows from Lemma 2.2 that there exists \( \lambda > 0 \) such that \( P_{F}(G) \subseteq U \) for any \( F \in \mathcal{K}(X) \) with \( h(F, A) < \lambda \). This shows \( \operatorname{diam} P_{F}(G) < \frac{1}{n} \) for any \( F \in \mathcal{K}(X) \) with \( h(F, A) < \lambda \) so that \( Q_{n}(G) \) is open and \( Q(G) \) is a \( G_{\delta} \)-subset of \( \mathcal{K}(X) \).

Now let us prove that \( Q(G) \) is dense. From Lemma 2.3 and 2.4 it suffices to prove that for any \( A \in L(G) \) and a solution \( (x_{0}, z_{0}) \) of \( \min(A, G) \), the set \( A_{x} \) defined by
\[ A_{x} = \overline{co}(A \cup \{ x_{a} \}) \]
is in \( Q(G) \) for all \( 0 < x < 1 \), where \( x_{a} = ax_{0} + (1 - a) z_{0} \).

Observe that for each \( 0 < x < 1 \), if \( x \in A_{x} \), \( x \neq x_{a} \), then \( x = ta + (1 - t) x_{a} \) for some \( 0 < t \leq 1 \) and \( a \in A \). Set \( a_{0} = ta + (1 - t) x_{0} \). Then \( a_{0} \in A \) and
\[
\inf_{x \in G} \| z - x \| \geq \inf_{x \in G} \| z - a_{0} \| - \| a_{0} - x \|
\geq \| z_{0} - x_{0} \| - (1 - t) \| x_{0} - x_{a} \|
= (1 - (1 - t)(1 - a)) \| z_{0} - x_{0} \|
> a \| z_{0} - x_{0} \| = \| z_{0} - x_{a} \| \geq \lambda_{A_{x}, G}.
\]
This shows \( P_{A_{x}}(G) = x_{a} \) and proves the lemma.
Now we are ready to give the main theorem of this section.

**Theorem 2.1.** Suppose that $X$ is a strongly convex Banach space. Let $G$ be a closed subset of $X$. Then the set of all $A \in \mathcal{K}_G(X)$ such that the minimization problem $\min(A, G)$ is well posed contains a dense $G_δ$-subset of $\mathcal{K}_G(X)$.

**Proof.** It suffices to prove that $\min(A, G)$ is well posed if $A \in Q(G) \cap L(G)$, as $Q(G) \cap L(G)$ is a dense $G_δ$-subset of $\mathcal{K}_G(X)$.

We first show that $\min(A, G)$ has a unique solution. Suppose there is $A \in Q(G) \cap L(G)$ such that $\min(A, G)$ has two solutions $(x_0, z_0)$, $(x_1, z_1)$. Clearly $x_1 = x_0$ because $A \in Q(G)$. On the other hand, since $A \in L(G)$, for each $n$, there exists $x_n^* \in X$, $\|x_n^*\| = 1$ satisfying

$$\langle x_n^*, x_0 - z_i \rangle > (1 - 2^{-n}) d_δ(A), \quad i = 0, 1$$

so that

$$\|x_0 - z_0 + x_0 - z_1\| \geq \limsup_{n \to \infty} \langle x_n^*, x_0 - z_0 + x_0 - z_1 \rangle \geq 2d_δ(A).$$

Thus, using the strict convexity of $X$, we have $z_0 = z_1$, proving the uniqueness.

Now let $(x_n, z_n)$ with $x_n \in A$, $z_n \in G$ be any minimizing sequence. Then from the uniqueness and Lemma 2.4 it follows that $(x_n, z_n)$ converges strongly to the unique solution of the minimization problem $\min(A, G)$. The proof is complete.

**Remark 2.1.** Theorem 2.1 is a multivalued version of a theorem due to Lau [17].

Note that if $\min(A, G)$ has a unique solution $(x_0, z_0)$, then $x_0$ has a unique nearest point in $G$. This, with Proposition 1.2 and Theorem 2.1, make us prove the following theorem.

**Theorem 2.2.** Let $X$ be a Banach space. Then the following statements are equivalent:

(i) $X$ is strongly convex;

(ii) for every closed non-empty subset $G$ of $X$, the set of all $A \in \mathcal{K}_G(X)$ such that the minimization problem $\min(A, G)$ is well posed contains a dense $G_δ$-subset of $\mathcal{K}_G(X)$;

(iii) for every closed non-empty subset $G$ of $X$, the set of all $A \in \mathcal{K}_G(X)$ such that the minimization problem $\min(A, G)$ is well posed contains a dense subset of $\mathcal{K}_G(X)$. 

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Proof. By Theorem 2.1, it suffices to prove that (iii) implies (i). For any fixed \( x \in X \setminus G \) and any \( \varepsilon > 0 \), \( \varepsilon < d_G(x) \), let \( A_\varepsilon \) denote the closed ball at \( x \) with radius \( \varepsilon/2 \). From (iii) it follows that there exists \( B_\varepsilon \in \mathcal{K}(X) \) such that \( h(A_\varepsilon, B_\varepsilon) < (\varepsilon/2) \) and \( \min(B_\varepsilon, G) \) is well posed so that \( \min(B_\varepsilon, G) \) has a unique solution \((x', z')\). Thus,

\[
\|x' - x\| \leq h(A_\varepsilon, B_\varepsilon) + \frac{\varepsilon}{2} < \varepsilon
\]

and \( x' \) has a unique nearest point \( z' \) in \( G \). Using Proposition 1.2, we complete the proof.

Remark 2.2. Let \( X \) be a space of finite dimensions. It follows from Remark 3.4 in [9] that Theorem 2.1 and so Theorem 2.2 may not hold if \( \mathcal{K}(X) \) is replaced by \( \mathcal{K}(X) \).

3. MAXIMIZATION PROBLEMS

In order to establish the well posedness result of maximization problems we need some lemmas on furthest points.

Let \( E \) be a real Banach space and \( G \) be a bounded closed subset of \( E \). We set

\[
F_G(x) = \sup_{z \in G} \|x - z\|, \quad \forall x \in E.
\]

Thus \( z \in G \) is called a furthest point of \( x \) with respect to \( G \) if \( \|z - x\| = F_G(x) \). The set of all furthest point of \( x \) with respect to \( G \) is denoted by \( R_G(x) \), that is,

\[
R_G(x) = \{z \in G : \|z - x\| = F_G(x)\}.
\]

Lemma 3.1. Suppose that \( F_G(\cdot) \) is Frechet differentiable at \( x \in E \) with \( DF_G(x) = x^* \). Then \( \|x^*\| = 1 \), and for any \( \{z_n\} \subset G \) with \( \lim_{n \to \infty} \|x - z_n\| = F_G(x) \), we have

\[
\lim_{n} \langle x^*, x - z_n \rangle = F_G(x).
\]

Proof. Let \( \{z_n\} \subset G \) such that \( \lim_{n \to \infty} \|x - z_n\| = F_G(x) \). It follows that for \( \forall t < 0 \),

\[
F_G(x + t(z_n - x)) - F_G(x) \geq -t \|x - z_n\| + \|x - z_n\| - F_G(x).
\]
Taking $t_n < 0$, $t_n \to 0$ with $t_n > F(x) - \|x - z_n\|$, we have
\[
\lim_n \left( \frac{F(x + t_n(z_n - x)) - F(x)}{t_n} - \langle x^*, z_n - x \rangle \right) = 0.
\]
This implies that
\[
\liminf_n (- \|x - z_n\| - t_n + \langle x^*, x \rangle) \geq 0.
\]
Now $\|x^*\| \leq 1$ since $F(x)$ is 1-Lipschitz. It follows that
\[
F(x) \leq \liminf_n \langle x^*, x - z_n \rangle \\
\leq \limsup_n \langle x^*, x - z_n \rangle \\
\leq \lim_n \|x^*\| \|x - z_n\| \\
\leq \|x^*\| F(x) \leq F(x).
\]
This shows that $\|x^*\| = 1$ and
\[
\lim_n \langle x^*, x - z_n \rangle = F(x).
\]
The proof is complete.

For $y \in E$, define
\[
S = \overline{\text{span } G}, \quad E_y = S \oplus \text{span } \{ y \},
\]
and let $J(G)$ denote the set of all $y \in E$ such that $F(x)$ is Frechet differentiable at $y$ when $F(x)$ is restricted on the subspace $E_y$.

**Lemma 3.2.** $J(G)$ is a $G_\delta$-subset of $E$.

**Proof.** For any $y \in E$, let $J_y(G)$ denote the set of all points $x \in E_y$ such that $F(x)$ is Frechet differentiable at $x$ when $F(x)$ is restricted on the subspace $E_y$. Clearly, $J_y(G) \in J(G)$ for any $y \in E$. Then $J(G) = \bigcup_{y \in E} J_y(G)$ is a $G_\delta$-subset of $E$ from Proposition 1.25 of [23] or [20] since $F(x)$ is convex on $E$.

**Lemma 3.3.** Let $\mathcal{D}$ be a closed convex subset of $E$. Suppose that $S$ is reflexive and $S \subset \mathcal{D}$. Then $\mathcal{D} \cap J(G)$ is a dense $G_\delta$-subset of $\mathcal{D}$.
Proof. From Lemma 3.2, it suffices to prove that $\mathcal{D} \cap J(G)$ is dense in $\mathcal{D}$. Toward this end, for fixed $y \in \mathcal{D}$, set
\[ O = \{ xy + x : x \in \mathcal{S}, 0 < x < 1 \}. \]
Then $O \subset \mathcal{D}$ is open in $E_y$ and $E_y$ is reflexive. It follows from the convexity of the function $F_\phi$ and Proposition 1.1 (see also [23]) that $F_\phi(\cdot)$ is Fréchet differentiable on a dense subset of $E_y$ when $F_\phi(\cdot)$ is restricted on the subspace $E_y$, so that there exists $\{ x_n \} \subset O$ such that $F_\phi(\cdot)$ is Fréchet differentiable at $x_n$ and $x_n \to y$. Observe that $E_{x_n} = E_y$ for any $n$. It follows that $\mathcal{D} \cap J(G)$ is dense in $\mathcal{D}$. The proof is complete.

Now we suppose $\mathcal{X}(X)$ to be endowed with the addition and multiplication as follows:
\[ A + B = \{ a + b : a \in A, b \in B \}, \quad \forall A, B \in \mathcal{X}(X), \]
\[ \lambda A = \{ \lambda a : a \in A \}, \quad \forall A \in \mathcal{X}(X), \quad \lambda \geq 0. \]
Then it follows from the proof of Theorem 2 in [25] that

**Lemma 3.4.** Suppose that $X$ is a reflexive Banach space. Then there exists a Banach space $(E, \| \cdot \|_E)$ such that $\mathcal{X}(X)$ is embedded as a convex cone in such a way that

(i) the embedding is isometric, that is, $\forall A, B \in \mathcal{X}(X)$, $h(A, B) = \| A - B \|_E$;

(ii) addition in $E$ induces addition in $\mathcal{X}(X)$;

(iii) multiplication by nonnegative scalars in $E$ induces the corresponding operation in $\mathcal{X}(X)$;

(iv) linear operation in $E$ induces linear operation in $X$.

Thus, from $X \subset E$, for $G \in \mathcal{B}(X)$, $A \in \mathcal{X}(X) \subset E$, we have
\[ R_\phi(A) = \{ z \in G : \| A - z \|_E = F_\phi(A) \} = \{ z \in G : \sup_{x \in A} \| x - z \| = \mu_{AG} \}. \]

**Lemma 3.5.** Suppose that $X$ is reflexive Kadec Banach space. Let $E$ be given by Lemma 3.4 and $G \in \mathcal{B}(X)$. Then for $A \in J(G)$ any sequence $\{ z_n \} \subset G$ with $\lim_{n \to \infty} \sup_{x \in A} \| x - z_n \| = \mu_{AG}$ has a subsequence which converges strongly to an element of $R_\phi(A)$. 

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Proof. Let \( A \in J(G) \) and let \( \{z_n\} \subseteq G \) such that \( \lim_{n \to \infty} \sup_{x \in A} \|x - z_n\| = \mu_{A,G} \). Using Lemma 3.1 and Lemma 3.4, there exists \( x^*_k \in E^* \) such that \( \|x^*_k\| = 1 \) and

\[
\lim_n \langle x^*_k, A - z_n \rangle = F_G(A).
\]

By the reflexivity of \( X \), there exists a subsequence \( z_{n_k} \), denoted by itself, which converges weakly to \( z \in X \). Thus,

\[
\|A - z\|_{E^*} \geq \langle x^*_k, A - z \rangle = \lim_n \langle x^*_k, A - z_n \rangle = F_G(A).
\]

Note that

\[
\|A - z\|_{E^*} \leq \lim_n \|A - z_n\|_{E^*} \leq F_G(A).
\]

Then

\[
\lim_n \|A - z_n\|_{E^*} = \|A - z\|_{E^*}.
\]

Since \( A \) is compact, we take \( a_0 \in A \) and \( x^* \in X^* \), \( \|x^*\| \leq 1 \) such that

\[
\|a_0 - z\| = \sup_{a \in A} \|a - z\| = F_G(A)
\]

and

\[
\langle x^*, a_0 - z \rangle = \|a_0 - z\| = F_G(A).
\]

From the fact that \( \{x_n\} \) converges weakly to \( z \), we have

\[
\|a_0 - z\| = \langle x^*, a_0 - z \rangle = \lim_n \langle x^*, a_0 - z_n \rangle
\leq \liminf_n \|a_0 - z_n\| \leq \limsup_n \|a_0 - z_n\|
\leq \sup_{a \in A, x \in G} \|a - x\| = F_G(A),
\]

so that

\[
\lim_n \|a_0 - z_n\| = \|a_0 - z\|.
\]

Then the fact that \( X \) is Kadec shows \( \lim_{n \to \infty} \|z_n - z\| = 0 \) and \( z \in G \), proving the lemma.
Let

\[ V_n = \left\{ A \in \mathcal{K}(X) : \operatorname{diam} R_A(G) < \frac{1}{n} \right\} \]

and let

\[ V(G) = \bigcap_n V_n \]

where \( R_A(G) = \{ x \in A : \sup_{z \in G} \| z - x \| = \mu_A(G) \} \).

**Lemma 3.6.** Suppose that \( X \) is reflexive Kadec Banach space. Then \( V(G) \) is a dense \( G_\delta \)-subset of \( \mathcal{K}(X) \).

**Proof.** Exactly as in the proof of Lemma 2.5 we can obtain that \( V(G) \) is a \( G_\delta \)-subset of \( \mathcal{K}(X) \). To prove the density, for any \( A \in J(G) \), by Lemma 3.5, we may take \((x_0, z_0)\) to be a solution of \( \max(A, G) \) with \( x_0 \in A \), \( z_0 \in G \), and let \( x_\alpha = \alpha x_0 + (1 - \alpha) z_0 \) for \( \alpha > 1 \). We define \( A_\alpha = \overline{c}o(A \cup \{ x_\alpha \}) \).

Thus, using Lemma 3.3, the proof will be completed if we can prove that \( A_\alpha \in V(G) \) for all \( \alpha > 1 \).

Now for any \( x \in A_\alpha \) if \( x \neq x_\alpha \) then \( x = t x_\alpha + (1 - t) a \) for some \( a \in A \) and \( 0 \leq t < 1 \). Thus we have

\[
\sup_{z \in G} \| z - x \| \leq t \sup_{z \in G} \| z - x_\alpha \| + (1 - t) \sup_{z \in G} \| z - a \|
\]

\[
\leq t \left( \sup_{z \in G} \| z - x_0 \| + \| x_0 - x_\alpha \| \right) + (1 - t) \| z_0 - x_0 \|
\]

\[
= t \left( \| z_0 - x_0 \| + (\alpha - 1) \| z_0 - x_\alpha \| \right) + (1 - t) \| z_0 - x_0 \|
\]

\[
= (t \alpha + 1 - t) \| z_0 - x_0 \| < \| z_0 - x_\alpha \|
\]

This implies that \( R_A(G) = x_\alpha \) and \( A_\alpha \in V(G) \) for all \( \alpha > 1 \).

The main theorem of this section is stated as follows:

**Theorem 3.1.** Suppose that \( X \) is a strongly convex Banach space and \( G \in \mathcal{B}(X) \). Then the set of all \( A \in \mathcal{K}(X) \) such that the maximization problem \( \max(A, G) \) is well posed contains a dense \( G_\delta \)-subset of \( \mathcal{K}(X) \).

**Proof.** Note that for any \( A \in J(G) \cap \mathcal{K}(X) \), \( R_A(G) = \{ z_0 \} \) is a singleton.

In fact, suppose that \( R_A(G) \) contains at least two distinct elements \( x_0, x_1 \in G \). Then by Lemma 3.1 there exists \( x^* \in E^* \) satisfying

\[
\langle x^*, A - x_0 \rangle = \langle x^*, A - x_1 \rangle = F_G(A).
\]
Hence
\[ \|A - x_0 + A - x_1\|_E = 2F_g(A).\]

Take \(a_0 \in A\) such that
\[ \|a_0 - \frac{1}{2}(x_0 + x_1)\| = h(A, \frac{1}{2}(x_0 + x_1)) = \|A - \frac{1}{2}(x_0 + x_1)\|_E.\]

Then
\[ \|a_0 - x_0 + a_0 - x_1\| = \|A - x_0 + A - x_1\|_E \]

and
\[ 2F_g(A) = \|a_0 - x_0 + a_0 - x_1\| \leq \|a_0 - x_0\| + \|a_0 - x_1\| \leq 2F_g(A).\]

This implies that
\[ \|a_0 - x_0 + a_0 - x_1\| = \|a_0 - x_0\| + \|a_0 - x_1\|.\]

It follows from the strict convexity of \(X\) that \(x_0 = x_1\), which is a contradiction. So \(R_g(A)\) is a singleton.

Note that for any \(A \in \mathcal{J}(G) \cap \mathcal{V}(G)\), the maximization problem \(\max(A, G)\) has a unique solution. Now let \((x_n, z_n)\) with \(x_n \in A, z_n \in G\) be any maximizing sequence. Then, using Lemma 3.5 and the compactness of \(A\), we have that \((x_n, z_n)\) converges strongly to the unique solution and complete the proof by Lemma 3.3 and 3.6.

**Remark 3.1.** Theorem 3.1 is a multivalued version of results due to Asplund [1], Panda & Kapoor [22], Zhivkov [28] and Fitzpatrick [13].

**Remark 3.2.** Note that if \(\max(A, G)\) has a unique solution \((x_0, z_0)\) then \(x_0\) has a unique furthest point in \(G\), which implies that there is a dense set of \(X\) possessing unique furthest points in \(G\) provided that the result of Theorem 3.1 holds. This enables us to construct some counterexamples to which Theorem 3.1 fails if \(X\) is not strongly convex. In fact, in this case, either \(X\) is not both reflexive and strictly convex, or \(X\) is not Kadec. In the first case Example 5.3 in [13] and Remark 4.4 in [9] apply. In the second case, let \(X\) be the renormed space \(l_2 \oplus \mathbb{R}\) in [12] by taking
\[ \|(x, r)\| = \max\{\|x\|, |r|\} + \left[ r^2 + \sum_{n} 2^{-2n}x_n^2 \right]^{1/2} \]

for \((x, r) \in X\). Let
\[ G = \{(e_n, 2 - n^{-1}) : n = 2, 3, \ldots\} \]
and
\[ U = \{(u, r) : \|u\| < \frac{1}{2}, |r| < \frac{1}{4}\}. \]
Then, for \((u, r) \in U\),
\[ F_G(u, r) = 2 - r + \left( (2 - r)^2 + \sum_n 2^{-2n} u_n^2 \right)^{1/2}. \]
However, for each \((e_n, 2 - n^{-1}) \in G\)
\[ \|(u, r) - (e_n, 2 - n^{-1})\| > F_G(u, r), \]
which shows no points in the set \(U\) has a furthest point in \(G\). Hence Theorem 3.1 fails. Obviously, \(X\) is both reflexive and strictly convex.

REFERENCES