Ambiguous loci of mutually nearest and mutually furthest points in Banach spaces

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Abstract

Let \(X\) be a real separable strictly convex Banach space and \(G\) a nonempty closed subset of \(X\). Let \(\mathcal{K}(X)\) (resp. \(\mathcal{K}^b(X)\)) denote the family of all nonempty boundedly compact (resp. compact) convex subsets of \(X\) endowed with the \(H\)-topology (resp. the Hausdorff distance), \(\mathcal{K}_G(X)\) (resp. \(\mathcal{K}_G^b(X)\)) the closure of the set \(\{A \in \mathcal{K}(X) : A \cap G = \emptyset\}\) (resp. \(\{A \in \mathcal{K}^b(X) : A \cap G = \emptyset\}\)), and \(\mathcal{V}(G)\) (resp. \(\mathcal{V}_G^b(G)\)) the family of \(A \in \mathcal{K}_G(X)\) (resp. \(A \in \mathcal{K}_G^b(X)\)) such that the minimization problem \(\min(A, G)\) fails to be well-posed. It is proved that for most (in the sense of the Baire category) closed subsets (resp. bounded closed subsets) \(G\) of \(X\), \(\mathcal{V}(G)\) (resp. \(\mathcal{V}_G^b(G)\)) is everywhere uncountable in \(\mathcal{K}_G(X)\) (resp. \(\mathcal{K}_G^b(X)\)). A similar result for the mutually furthest point problem is also given.

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1. Introduction

Let \(X\) be a real Banach space. We shall use the following notations for various families of subsets of \(X\).

\(\mathcal{K}(X)\)—the family of all nonempty closed subsets,

\(\mathcal{K}^b(X)\)—the family of all nonempty closed bounded subsets,

\(\mathcal{V}(G)\) (resp. \(\mathcal{V}_G^b(G)\)) is everywhere uncountable in \(\mathcal{K}_G(X)\) (resp. \(\mathcal{K}_G^b(X)\)). A similar result for the mutually furthest point problem is also given.

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\(\mathcal{C}(X)\)—the family of all nonempty closed convex subsets,
\(\mathcal{C}^b(X)\)—the family of all nonempty closed bounded convex subsets,
\(\mathcal{K}(X)\)—the family of all nonempty convex boundedly compact subsets,
\(\mathcal{K}^b(X)\)—the family of all nonempty convex compact subsets.

Let \(G, A \in \mathcal{A}(X)\). Define
\[
\lambda_{AG} := \inf \{ \|z - x\| : x \in A, z \in G \}
\]
and, if \(G\) and \(A\) are additionally bounded
\[
\mu_{AG} := \sup \{ \|z - x\| : x \in A, z \in G \}.
\]

Given a nonempty closed (resp. closed bounded) subset \(G\) of \(X\), according to [8], for \(A \in \mathcal{A}(X)\) (resp. \(A \in \mathcal{A}^b(X)\)), a pair of \((x_0, z_0)\) with \(x_0 \in A, z_0 \in G\) is called a solution of the minimization (resp. maximization) problem, denoted by \(\min(A, G)\) (resp. \(\max(A, G)\)), if \(\|x_0 - z_0\| = \lambda_{AG}\) (resp. \(\|x_0 - z_0\| = \mu_{AG}\)). Moreover, any sequence \(\{(x_n, z_n)\}\) with \(x_n \in A, z_n \in G\), such that \(\lim_{n\to\infty} \|x_n - z_n\| = \lambda_{AG}\) (resp. \(\lim_{n\to\infty} \|x_n - z_n\| = \mu_{AG}\)) is called a minimizing (resp. maximizing) sequence for \(\min(A, G)\) (resp. \(\max(A, G)\)). A minimization (resp. maximization) problem is said to be well-posed if it has a unique solution and every minimizing (resp. maximizing) sequence converges strongly to the solution.

Recall that the Hausdorff distance on the space \(\mathcal{A}^b(X)\) is defined by
\[
H(A, B) = \max \left\{ \sup_{a \in A} \inf_{b \in B} \|a - b\|, \sup_{b \in B} \inf_{a \in A} \|a - b\| \right\}, \quad A, B \in \mathcal{A}^b(X).
\]

It is known that \((\mathcal{A}^b(X), H)\) is a complete metric space.

De Blasi et al. [8] considered the well-posedness of the minimization and maximization problems. They proved that if \(X\) is a uniformly convex Banach space, then the set of \(A \in \mathcal{C}^b(X)\) (resp. \(A \in \mathcal{C}^b(X)\)) such that the minimization problem \(\min(A, G)\) (resp. maximization problem \(\max(A, G)\)) is well-posed, is a dense \(G/SO\)-subset of \(\mathcal{C}^b(X)\) (resp. \(\mathcal{C}^b(X)\)), where \(\mathcal{C}^b(X)\) stands for the closure of the set \(\{A \in \mathcal{C}^b(X) : \lambda_{AG} > 0\}\). Recently, the first author of the present paper extended the above result to the framework of reflexive locally uniformly convex Banach spaces for the class \(\mathcal{K}(X)\) in [13]. These are positive results on the well-posedness. For some motivating ideas and related works see also [1,2,6,7,9–12,14,15].

In the present paper, in spirit of some works on the nearest point and furthest problems from [3–5,16], we will establish some negative results on the well-posedness. More precisely, let \(\mathcal{V}(G)\) (resp. \(\mathcal{M}(G)\)) denote the set of all \(A \in \mathcal{K}^b_G(X)\) (resp. \(A \in \mathcal{K}^b(X)\)) such that the minimization problem \(\min(A, G)\) (resp. maximization problem \(\max(A, G)\)) fails to be well-posed, that is,
\[
\mathcal{V}(G) = \{A \in \mathcal{K}^b_G(X) : \min(A, G)\text{ is not well-posed}\},
\]
where \(\mathcal{K}^b_G(X) = \{A \in \mathcal{K}(X) : \lambda_{AG} > 0\}\) and
\[
\mathcal{M}(G) = \{A \in \mathcal{K}^b(X) : \max(A, G)\text{ is not well-posed}\}.
\]
We shall prove that if $X$ is a strictly convex and separable Banach space, then the family of nonempty closed subsets (resp. bounded closed subsets) $G \in \mathcal{F}(X)$ such that $\mathcal{V}(G)$ (resp. $\mathcal{M}(G)$) is everywhere uncountable in $\mathcal{K}_G(X)$ (resp. $\mathcal{K}_b(X)$) is residual in $\mathcal{A}(X)$ (resp. $\mathcal{A}_b(X)$), where $\mathcal{A}(X)$ is endowed with the $H$-topology while $\mathcal{A}_b(X)$ with the Hausdorff metric. Our results generalize some results on the nearest and furthest point problems from [4,5,16]. Moreover, it should be remarked that all known results on ambiguous loci in best approximation theory were established for some families of bounded subsets of $X$. Our results mutually nearest point problems obtained in the present paper are for some families of unbounded subsets of $X$.

We conclude this section with some notation. If $X$ is a Banach space and if $A \subset X$, then $\overline{A}$ stands for the closure of $A$, diam$(A)$ for the diameter of $A$, and $\overline{c}A$ for the closed convex hull of $A$. We use $B(x, r)$ to denote the closed ball with centre $x$ and radius $r$ in $X$, in particular, $B$ stands for $B(0, 1)$. We also need the following definition of everywhere uncountability in a topological space, see [3–5] for the definition in a metric space.

**Definition 1.1.** Let $S$ be a subset of the topological space $E$. $S$ is called **everywhere uncountable in $E$** if, for every $x \in E$ and its neighbourhood $U(x)$, the set of the intersection, $S \cap U(x)$, is nonempty and uncountable.

Note that if $A$ is everywhere uncountable, then it is dense in $E$; the converse is however not true, in general.

**2. Ambiguous loci of mutually nearest points**

We begin with the following definition. Recall that the Hausdorff metric can also be redefined by

$$H(A, B) = \max\{e(A, B), e(B, A)\}, \quad A, B \in \mathcal{A}_b(X), \quad (2.1)$$

where $e(A, B)$ is defined by

$$e(A, B) = \inf\{\varepsilon > 0 : B \subseteq A + \varepsilon B\}. \quad (2.2)$$

Furthermore, for any $\rho > 0$, $A, B \in \mathcal{A}(X)$, define

$$H_\rho(A, B) = \max\{e(A, B \cap \rho B), e(B, A \cap \rho B)\}. \quad (2.3)$$

Then it is easy to see that, for any $\rho > 0$ and $A, B \in \mathcal{A}(X)$,

$$H_\rho(A, B) \leq H(A, B) \quad (2.4)$$

if $H(A, B)$ is allowed to be $+\infty$.

**Definition 2.1.** The $H_\rho$-topology on $\mathcal{A}(X)$ is defined as follows: for any $A \in \mathcal{A}(X)$, the neighbourhood basis of $A$ is the family of all subsets $U_\gamma(A, r)$ with $\gamma > 0$ and $r > 0$ defined by

$$U_\gamma(A, r) = \{B \in \mathcal{A}(X) : H_\gamma(A, B) < r\}. \quad (2.5)$$
Throughout the whole paper, we always endow $\mathcal{A}(X)$ and $\mathcal{A}^b(X)$ with the $H_p$-topology and Hausdorff metric, respectively. Thus, $\mathcal{H}(X)$ is a topological subspace of $\mathcal{A}(X)$ while $\mathcal{H}^b(X)$ is a metric subspace of $\mathcal{A}^b(X)$. For $F, G \in \mathcal{A}(X)$, let
\[
\mathcal{A}_F(X) := \{G \in \mathcal{A}(X) : \lambda_{FG} > 0\},
\] (2.6)
where the closure is taken in the topological space $\mathcal{A}(X)$. Recall that the set $\mathcal{U}(G)$ is defined by
\[
\mathcal{U}(G) = \{A \in \mathcal{H}(X) : \min(A, G) \text{ is not well-posed}\},
\]
where $\mathcal{H}(X) = \mathcal{H}(X) \cap \mathcal{A}(X) = \{A \in \mathcal{H}(X) : \lambda_{AG} > 0\}$.

**Theorem 2.1.** Suppose that $X$ is a separable strictly convex Banach space. Then the set
\[
\mathcal{A}^\gamma(X) := \{G \in \mathcal{A}(X) : \mathcal{U}(G) \text{ is everywhere uncountable in } \mathcal{H}(X)\}
\]
is residual in $\mathcal{A}(X)$.

**Proof.** For $r > 0$, $\gamma > 0$ and $F \in \mathcal{H}(X)$, define
\[
\mathcal{A}_{r,\gamma, F}(X) := \{G \in \mathcal{A}_F(X) : \mathcal{U}(G) \cap U_\gamma(F, r) \text{ is countable in } \mathcal{H}(X)\},
\] (2.7)
where $U_\gamma(F, r)$ is defined by $U_\gamma(F, r) = \{B \in \mathcal{A}(X) : H_\gamma(F, B) < r\}$. We will prove that $\mathcal{A}_{r,\gamma, F}$ is nowhere dense in $\mathcal{A}(X)$. For the purpose, let $G \in \mathcal{A}_{r,\gamma, F}$. Since $X$ is strictly convex and $F$ is boundedly compact, without loss of generality, we may assume that $\min(F, G)$ has a unique solution $(f_F, g_F)$. We may also assume $\lambda_{FG} = 0$. If, otherwise, $\lambda_{FG} > 0$, we can take $\tilde{G}$ which is near $G$, such that $\min(F, \tilde{G})$ has a unique solution and $\lambda_{F,\tilde{G}} > 0$. It is sufficient to show that, for any $\rho > 0$, $\epsilon > 0$, there exist $Y \in U_\rho(G, \epsilon) \cap \mathcal{A}_{r,\gamma, F}$ and $\rho' > 0$, $\epsilon' > 0$ such that $U_{\rho'}(Y, \epsilon') \cap \mathcal{A}_{r,\gamma, F} = \emptyset$.

Without loss of generality, let $\rho > 0$ and let $\epsilon$ satisfy that $0 < \epsilon < \min\{\rho, \lambda_{FG}\}$. Let
\[
g_1 = f_F + \left(1 - \frac{\epsilon}{2\lambda_{FG}}\right)(g_F - f_F).
\] (2.8)
It is immediately clear that
\[
d(g_1, F) = \|f_F - g_1\| = \lambda_{FG} - \epsilon/2 > \epsilon/2
\] (2.9)
and
\[
d(g_1, G) = \|g_F - g_1\| = \epsilon/2.
\] (2.10)
Let next $g_2 \in X$ be such that
\[
\|g_1 - g_2\| = \epsilon/2 \quad \text{and} \quad d(g_2, F) = d(g_1, F),
\] (2.11)
where $d(g_1, F)$ denote the distance from $g_1$ to $F$. Note that $g_1, g_2, g_F$ are not colinear and $X$ is strictly convex. Then, by (2.10) and (2.11),
\[
d(g_2, G) \leq \|g_2 - g_F\| < \|g_2 - g_1\| + \|g_1 - g_F\| = \epsilon.
\] (2.12)
Put
\[
Y := G \cup \{g_1, g_2\}.
\]
We now estimate by (2.10) and (2.12)

\[ H(G, Y) = \max \left\{ \sup_{g \in G} d(g, Y), \sup_{y \in Y} d(y, G) \right\} \]

\[ = \sup_{y \in Y} d(y, G) \]

\[ = \max \{ d(g_1, G), d(g_2, G) \} < \varepsilon. \]

This with (2.4) implies that \( Y \in U_\rho(G, \varepsilon). \)

Let now \( f_1 := f_\rho \) and \( f_2 \in F \) be the unique best approximation to \( g_2 \) from \( F \). Set

\[ \tau := \frac{\varepsilon}{8d(g_1, F)}, \quad (2.13) \]

\[ u_\delta^i := (1 - \delta)f_i + \delta g_i, \quad 0 \leq \delta \leq 1, \quad i = 1, 2 \quad (2.14) \]

and

\[ h := \frac{1}{2} \min_{\delta \in [\tau/2, \tau]} \min \{ d(g_1, F_2^\delta) - d(g_2, F_2^\delta), d(g_2, F_1^\delta) - d(g_1, F_1^\delta) \}, \quad (2.15) \]

where

\[ F_1^\delta := \text{co}(F \cup \{ u_\delta^i \}), \quad i = 1, 2. \quad (2.16) \]

Then \( F_1^\delta \) is boundedly compact. We now show that \( h > 0 \). Assume the contrary that \( h \leq 0 \). With no loss of generality, we may assume that

\[ d(g_1, F_2^\delta) \leq d(g_2, F_2^\delta). \quad (2.17) \]

Let \( \bar{f} \) be the unique point in \( F_2^\delta \) such that \( \|g_1 - \bar{f}\| = d(g_1, F_2^\delta) \). Write \( \bar{f} = (1 - \bar{\tau})f + \bar{\tau}u_2^\delta \) for some \( \bar{\tau} \in [0, 1] \) and \( f \in F \) to get

\[ d(\bar{f}, F) \leq \|\bar{f} - (1 - \bar{\tau})f - \bar{\tau}f_2\| = \bar{\tau}\|u_2^\delta - f_2\| = \bar{\tau}\|u_2^\delta - f_2\| = \bar{\tau}d(g_2, F) \]

\[ \leq \bar{\tau}d(u_2^\delta, F). \quad (2.18) \]

It turns out, by (2.17) and (2.11), that

\[ d(g_1, F) \leq \|g_1 - \bar{f}\| + d(\bar{f}, F) \]

\[ = d(g_1, F_2^\delta) + d(\bar{f}, F) \]

\[ \leq d(g_2, F_2^\delta) + d(\bar{f}, F) \]

\[ \leq \|g_2 - u_2^\delta\| + \bar{\tau}d(u_2^\delta, F) \]

\[ \leq \|g_2 - u_2^\delta\| + d(u_2^\delta, F) \]

\[ = d(g_2, F) = d(g_1, F). \]

So we must have \( \bar{\tau} = 1 \) and \( \bar{f} = u_2^\delta \), resulting by (2.17) again in that

\[ \|g_1 - u_2^\delta\| = d(g_1, F_2^\delta) \leq \|g_2 - u_2^\delta\|. \quad (2.19) \]
Noting by (2.11)
\[ \|g_1 - f_2\| \geq d(g_1, F) = d(g_2, F) = \|g_2 - f_2\| \] (2.20)
we obtain by (2.19) that
\[
\|g_1 - f_2\| = \|g_1 - u^g_2 + u^g_2 - f_2\| \\
\leq \|g_1 - u^g_2\| + \|u^g_2 - f_2\| \\
\leq \|g_2 - u^g_2\| + \|u^g_2 - f_2\| \\
= \|g_2 - f_2\| \\
\leq \|g_1 - f_2\|.
\] (2.21)
Hence we have
\[
\|g_1 - u^g_2 + u^g_2 - f_2\| = \|g_1 - u^g_2\| + \|u^g_2 - f_2\|
\]
and, since \(X\) is strictly convex
\[
g_1 - u^g_2 = \lambda (u^g_2 - f_2)
\] (2.22)
for some \(\lambda > 0\). This contradicts the fact that \(g_1, f_2, u^g_2\) are not colinear.

Now let
\[
\varepsilon' := \min \left\{ \frac{\varepsilon}{8}, h \right\} \quad \text{and} \quad \rho' = 2 \left( \max_{f \in F} \|f\| + \max_{i=1,2} \left( \|f_i\| + \|g_i\| \right) + 1 \right).
\] (2.23)
For any \(Z \in U_{\rho'}(Y, \varepsilon')\), we get
\[
d(g_i, Z) < H_{\rho'}(Y, Z) < \varepsilon', \quad i = 1, 2
\] (2.24)
as \(g_i \in Y \cap \rho' \mathcal{B}\). Set
\[
Z_i := \mathcal{B}(g_i, \varepsilon') \cap Z, \quad i = 1, 2.
\] (2.25)
Then, by (2.24) and (2.25), \(Z_i \neq \emptyset\) for \(i = 1, 2\) and by (2.11) and (2.23), \(Z_1 \cap Z_2 = \emptyset\). Let
\[
A^g_i := \text{co}(F \cup \{(1 - t)u^g_1 + tu^g_2\}), \quad 0 \leq t \leq 1, \quad \tau/2 \leq \delta \leq \tau.
\] (2.26)
Then, for each \(0 \leq t \leq 1, \quad \tau/2 \leq \delta \leq \tau, \quad A^g_i\) is boundedly compact. We next show that
\[
\lambda_{A^g_{i|Z}} = \lambda_{A^g_{i|Z_1 \cup Z_2}}, \quad 0 \leq t \leq 1 \text{ and } \tau/2 \leq \delta \leq \tau.
\] (2.27)
Indeed, for each \(a := (1 - s)f + s[(1 - t)u^g_1 + tu^g_2] \in A^g_{i|Z}\), where \(0 \leq s \leq 1, \quad 0 \leq t \leq 1, \quad \tau/2 \leq \delta \leq \tau, \quad f \in F\), we have
\[
d(a, F) \leq \|a - (1 - s)f - s[(1 - t)f_1 + tf_2]\| \leq \max \{\|f_1 - u^g_1\|, \|f_2 - u^g_2\|\}
\]
\[
\leq \delta d(g_1, F).
\] (2.28)
Similarly, for \(i = 1, 2\) and each \(b := (1 - s)f + su^g_i \in F^g_{i|Z}\),
\[
d(b, F) \leq \|b - [(1 - s)f + sf_i]\| = s\|f_i - u^g_i\| = sd(u^g_i, F) \leq \delta d(g_i, F).
\] (2.29)
It follows from the inclusions $F \subseteq A_i^\delta$, $F_i^\delta \subseteq F_i^\delta$ and (2.27), (2.28) that, for $i = 1, 2$

$$H(A_i^\delta, F_i^\delta) = \max \left\{ \sup_{a \in A_i^\delta} d(a, F_i^\delta), \sup_{b \in F_i^\delta} d(b, A_i^\delta) \right\}$$

$$\leq \max \left\{ \sup_{a \in A_i^\delta} d(a, F), \sup_{b \in F_i^\delta} d(b, F) \right\}$$

$$\leq \delta \cdot \max \{d(g_1, F), d(g_2, F)\}$$

$$= \delta d(g_1, F). \quad (2.30)$$

For $i = 1, 2$, by the definition of $F_i^\delta$, we obtain that

$$d(g_i, F_i^\delta) = \|g_i - u_i^\delta\| = (1 - \delta) d(g_i, F). \quad (2.31)$$

Hence, from (2.24), (2.30) and (2.31), we have that, for $i = 1, 2$

$$\lambda_{A_i^\delta Z_i} \leq d(g_i, A_i^\delta) + d(g_i, Z_i)$$

$$\leq H(A_i^\delta, F_i^\delta) + d(g_i, F_i^\delta) + \epsilon'$$

$$\leq \delta d(g_1, F) + (1 - \delta) d(g_i, F) + \epsilon'$$

$$= \lambda_{FG} - \epsilon/2 + \epsilon', \quad (2.32)$$

where the last equality is because of (2.9) and (2.11).

Now, let $z \in Z$. Suppose that, for some $i = 1, 2$

$$d(z, A_i^\delta) \leq \lambda_{A_i^\delta Z_i}. \quad (2.33)$$

Observe that

$$\max_{a \in A_i^\delta} \|a\| \leq \max_{a \in F} \|a\|, \max_{i=1,2} (\|f_i\| + \|g_i\|) \quad (2.34)$$

and

$$\lambda_{A_i^\delta Z_i} \leq \lambda_{FZ_i} \leq \max_{a \in F} \|a - g_i\| + d(g_i, Z_i) \leq \max_{a \in F} \|a\| + \|g_i\| + 1. \quad (2.35)$$

It follows from (2.33) to (2.35) and (2.23) that

$$\|z\| \leq d(z, A_i^\delta) + \max_{a \in A_i^\delta} \|a\|$$

$$\leq \max_{a \in F} \|a\| + 1 + \max_{a \in F} \|a\|, \max_{i=1,2} (\|f_i\| + \|g_i\|) \quad$$

$$\leq 2 \left( \max_{a \in F} \|a\| + \max_{i=1,2} (\|f_i\| + \|g_i\|) \right) + 1$$

$$< \rho'.$$

Furthermore, since $\delta \leq \tau$, by (2.4), (2.28) and (2.13),

$$H_j(A_i^\delta, F) \leq H(A_i^\delta, F) \leq \sup \{d(a, F) : a \in A_i^\delta\} \leq \delta d(g_1, F) < \epsilon/8. \quad (2.36)$$
Consequently,
\[ \lambda_{G A_t^o} \geq \lambda_{F G} - H(A_t^o, F) \geq \lambda_{F G} - \varepsilon/8. \]  
(2.37)

Since \( H_{r'}(Y, Z) < \varepsilon' \) and \( \|z\| < \rho' \), there exists \( y \in Y \) such that \( \|z - y\| < \varepsilon' \) so that \( y \neq g_i \) and \( y \in G \). Thus, if \( \|z - g_i\| > \rho \), we have, by (2.37) and (2.32)
\[ d(z, A_t^o) \geq d(y, A_t^o) - \varepsilon' \geq \lambda_{G A_t^o} - \varepsilon' \geq \lambda_{G F} - \varepsilon/8 - \varepsilon' > \lambda_{G F} - \varepsilon/2 + \varepsilon' \geq \lambda_{A_t^o Z_i} \]
which contracts (2.33). This implies that \( z \in Z_i \), and hence (2.27) holds.

On the other hand, we have by definition of \( \varepsilon' \)
\[ \lambda_{F_t^1 Z_1} \leq d(g_1, F_1^o) + \varepsilon' \]
\[ \leq d(g_1, F_1^o) + \frac{1}{2} [d(g_2, F_1^o) - d(g_1, F_1^o)] \]
\[ = d(g_2, F_1^o) + \frac{1}{2} [d(g_1, F_1^o) - d(g_2, F_1^o)] \]
\[ \leq \lambda_{F_t^1 Z_2} + \varepsilon' + \frac{1}{2} [d(g_1, F_1^o) - d(g_2, F_1^o)] \]
\[ \leq \lambda_{F_t^1 Z_2}. \]  
(2.38)

Similarly we have
\[ \lambda_{F_t^2 Z_2} \leq \lambda_{F_t^2 Z_1}. \]  
(2.39)

For arbitrary fixed \( \delta \in [\tau/2, \tau] \), let
\[ r(t) := \lambda_{Z_t A_t^o} - \lambda_{Z_t A_t^o}, \quad 0 \leq t \leq 1. \]

Then, by (2.38) and (2.39)
\[ r(0) = \lambda_{F_t^1 Z_1} - \lambda_{F_t^1 Z_2} \leq 0, \]
\[ r(1) = \lambda_{F_t^2 Z_1} - \lambda_{F_t^2 Z_2} \geq 0. \]

It follows from the continuity of \( r(\cdot) \) that there exists \( 0 \leq t_0 \leq 1 \) such that \( r(t_0) = 0 \); hence
\[ \lambda_{Z_t A_t^o} = \lambda_{Z_t A_t^o} \]  
(2.40)

for each \( \delta \in [\tau/2, \tau] \). Since \( Z_1 \) and \( Z_2 \) are closed and disjoint, it follows that the problem \( \min(A_t^o, Z) \) is not well-posed for each \( \delta \in [\tau/2, \tau] \). This means that \( A_t^o \in \mathcal{V}(Z) \) for each \( \delta \in [\tau/2, \tau] \). In addition, by (2.36) and (2.37), \( A_t^o \in U_{r}(F, r) \cap \mathcal{K}_G(X) \). This implies that \( \mathcal{V}(Z) \cap U_{r}(F, r) \) contains at least uncountable elements and consequently, \( Z \notin \mathcal{A}_{F, r, r} \). Therefore, \( \mathcal{A}_{F, r, r} \) is nowhere dense in \( \mathcal{A} \) as \( Z \in U_{r'}(Y, \varepsilon') \) is arbitrary.

Since the separability of \( X \) implies that \( \mathcal{K}(X) \) is separable, there exists a countable subset \( \mathcal{S} \) of \( \mathcal{K}(X) \) which is dense in \( \mathcal{K}(X) \). Let \( Q_+ \) be the set of positive rationals. Define
\[ \mathcal{A} := \bigcup_{F \in \mathcal{S}} \bigcup_{r, r' \in Q_+} \mathcal{A}_{F, r, r}. \]
Then $\tilde{A}$ is of the first Baire category in $\mathcal{A}(X)$. To complete the proof it remains to prove that $\mathcal{A}(X) \subseteq \mathcal{A}(X) \setminus \tilde{A}$. To this end, let $G \in \mathcal{A}(X) \setminus \tilde{A}$. For any $r > 0$, $\gamma > 0$ and any $F \in \mathcal{K}_G(X)$, there exist $\tilde{r}, \tilde{\gamma} \in Q_+$ and $\tilde{F} \in \mathcal{F}$ such that $\tilde{F} \in \mathcal{K}_G(X)$ and $U_{\tilde{r}}(\tilde{F}, \tilde{\gamma}) \subseteq U_{\gamma}(F, r)$. Note that $G \in \mathcal{A}(X) \setminus \mathcal{A}_F(\tilde{r}, \tilde{\gamma})$. Since $\tilde{F} \cap G = \emptyset$, it follows that $\forall (G) \cap U_{\tilde{r}}(\tilde{F}, \tilde{\gamma})$ is nonempty and uncountable and so is $\forall (G) \cap U_{\gamma}(F, r)$. This shows that $\forall (G)$ is everywhere uncountable in $\mathcal{K}_G(X)$ and the proof is complete.

Similarly, modifying slightly the proof of Theorem 2.1, we also have following theorems. Let $\mathcal{K}_b G(X)$ denote the intersection set of $\mathcal{K}_b(X)$ and $\mathcal{C}_G(X)$.

**Theorem 2.2.** Suppose that $X$ is a separable strictly convex Banach space. Then the set $\mathcal{A}_b^*(X) := \{ G \in \mathcal{A}_b(X) : \forall (G) \text{ is everywhere uncountable in } \mathcal{K}_b G(X) \}$ is residual in $\mathcal{A}_b(X)$.

**Theorem 2.3.** Suppose that $X$ is a separable strictly convex Banach space. Then the set $\mathcal{A}^*(X) := \{ G \in \mathcal{A}_b(X) : \forall (G) \text{ is everywhere uncountable in } \mathcal{K}_G(X) \}$ is residual in $\mathcal{A}_b(X)$.

### 3. Ambiguous loci of mutually furthest points

We study in this section the ambiguous loci of mutually furthest points. For $x \in X$ and $F \in \mathcal{A}_b(X)$, write $e(x, F) = \sup_{f \in F} \| x - f \|$. Recall that the set $\mathcal{M}(G)$ is defined by $\mathcal{M}(G) = \{ A \in \mathcal{K}_b(X) : \text{max}(A, G) \text{ is not well-posed} \}$.

**Theorem 3.1.** Assume that $X$ is a separable strictly convex Banach space. Then the set $\mathcal{A}_b^v(X) := \{ G \in \mathcal{A}_b(X) : \mathcal{M}(G) \text{ is everywhere uncountable in } \mathcal{K}_b(X) \}$ is residual in $\mathcal{A}_b(X)$.

**Proof.** Since the proof is similar to that of Theorem 2.1, we only sketch it here. For $F \in \mathcal{K}_b(X)$ and $r > 0$, define $\mathcal{A}_{F,r} := \{ G \in \mathcal{A}_b(X) : \mathcal{M}(G) \cap U(F, r) \text{ is countable in } \mathcal{K}_b(X) \}$, where $U(F, r) = \{ B \in \mathcal{A}_b(X) : h(F, B) < r \}$. We will show that $\mathcal{A}_{F,r}$ is nowhere dense in $\mathcal{A}_b(X)$. In fact, let $G \in \mathcal{A}_{F,r}$. We may assume $\mu_{FG} > 0$ and $\max(F, G)$ has a unique solution $(f_F, g_F)$. Let $0 < \varepsilon < \min \{ r, \mu_{FG} \}$ and $g_1 := f_F + \left( 1 + \frac{\varepsilon}{2\mu_{FG}} \right) (g_F - f_F)$. 


It is clear that \( e(g_1, F) = \| f_F - g_1 \| = \mu_{FG} + \varepsilon/2 \). Let \( g_2 \in X \) be such that 
\[
\| g_1 - g_2 \| = \varepsilon/2 \quad \text{and} \quad e(g_2, F) = e(g_1, F)
\]
and put
\[
Y := G \cup \{ g_1, g_2 \}.
\]
Then we have that \( H(G, Y) \leq \varepsilon \) since \( \| g_i - g_F \| \leq \varepsilon \) for \( i = 1, 2 \). Let \( f_1 := f_F \) and \( f_2 \in F \) be such that \( \| f_2 - g_2 \| = e(g_2, F) \). Set
\[
\tau := \frac{\varepsilon}{8e(g_1, F)},
\]
\[
u_i^\delta := (1 + \delta)f_i - \delta g_i, \quad 0 \leq \delta \leq 1, \quad i = 1, 2
\]
and
\[
h := \frac{1}{2} \min_{\delta \in [\tau/2, \tau]} \min \{ e(g_2, F_2^\delta) - e(g_1, F_1^\delta), e(g_1, F_1) - e(g_2, F_2^\delta) \},
\]
where
\[
F_i^\delta = \text{co}(F \cup \{ u_i^\delta \}), \quad i = 1, 2.
\]
Then \( h > 0 \). Now let
\[
\rho := \min \left\{ \frac{\varepsilon}{8}, h \right\}
\]
and for \( Z \in U(Y, \rho) \), set
\[
Z_i := B(g_i, \rho) \cap Z, \quad i = 1, 2.
\]
Then \( Z_i \neq \emptyset \) for \( i = 1, 2 \) and \( Z_1 \cap Z_2 = \emptyset \). Let
\[
A_i^\delta := \text{co}(F \cup \{(1 - t)u_i^\delta + tu_i^\delta \}), \quad 0 \leq t \leq 1, \quad \tau/2 \leq \delta \leq \tau.
\]
Then we have that for each \( \delta \in [\tau/2, \tau] \) there exists \( 0 \leq t_\delta \leq 1 \) such that the problem
\[
\max(A_i^\delta, Z) \text{ is not well-posed}. \quad (3.1)
\]
This implies that \( Z \notin \mathcal{A}_{F, r} \) and \( \mathcal{A}_{F, r} \) is nowhere dense in \( \mathcal{A}^b(X) \) since \( Z \in U(Y, \rho) \) is arbitrary.

Now let \( \mathcal{S} \) be a countable dense subset for \( \mathcal{A}^b(X) \) and let \( Q_+ \) be the set of positive rationals. Define
\[
\mathcal{J} := \bigcup_{F \in \mathcal{S}} \bigcup_{r \in Q_+} \mathcal{A}_{F, r}.
\]
Then \( \mathcal{J} \) is of the first Baire category in \( \mathcal{A}^b(X) \) and \( \mathcal{A}_+(X) \supseteq \mathcal{A}^b(X) \backslash \mathcal{J} \). Hence \( \mathcal{A}_+(X) \) is residual in \( \mathcal{A}^b(X) \) and the proof is complete. \( \Box \)

References


