The central limit theorem for the independence number for minimal spanning trees in the unit square

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Assume that $P_1$ is a Poisson point process of intensity 1 in $R^2$. Let $T_n$ be a minimal spanning tree (MST) on $P_1 \cap [-n^{1/2}/2, n^{1/2}/2]^2$, and let $N(T_n)$ be the independence number of $T_n$, i.e., the size of largest independent sets. In this paper, we prove a central limit theorem for $N(T_n)$.

1. Introduction

Beardwood, Halton & Hammersley (1959) showed that the length $L_n$ of the travelling salesman tour on the $n$ random points in $R^d$ satisfies a law of large numbers. Furthermore, Steele (1988) and Redmond & Yukich (1994) found that this type of asymptotics is commonly observed in several combinatorial graphs including the minimal matching and the minimal spanning tree. The reader may look at Steele (1997) and Yukich (1998) for the grand view of this kind of the law of large numbers.

The subject of this paper is on the central limit theorem. The possibility of a central limit theorem for the travelling salesman tour was already suggested in Beardwood, Halton & Hammersley (1959); the BHH conjecture. However, until now only a very few attempts have been tried and among these trials Avram & Bertsimas (1993) and Kesten & Lee (1996) are worth to look carefully.

The remarkable observation of Avram & Bertsimas (1993) is that conditioned on certain event $G_n$, the length $L_n$ of several combinatorial graphs
on the $n$ random points in $\mathbb{R}^d$ can be written as the sum of locally dependent random variables and that the probability of $G_n^c$ is negligible. So, they found the way to represent the random variable $L_n$ as the sum of locally dependent random variables and they showed in this setting how to apply Stein's method which is the main theme of this special volume. By using Stein's method they proved the central limit theorem for the length $L_n$ of several combinatorial graphs including the nearest neighbor graph. This is the first real progress in the direction toward the possible full justification of the BHH conjecture.

However, the method of Avram & Bertsimas (1993) does not work for the length $L_n$ of the more complicated combinatorial graphs. Kesten & Lee (1996) provided a new idea for analyzing the structure of the more complicated combinatorial graphs—the (space) truncation method. They wrote $L_n - EL_n$ as a sum $\sum_{k=1}^{n} \Delta_k$ of martingale differences and they approximated $\Delta_k$ by a truncated random variable $\Delta_k(L)$. Their main work was to show that this approximation is good enough for the study of the length $L_n$ of the minimal spanning trees. In this way, they were able to apply Lévy martingale central limit theorem and proved the central limit theorem for the length $L_n$ of the minimal spanning tree. The reader may look at Alexander (1996) and Lee (1997, 1999) for the related results.

The truncation method of Kesten & Lee (1996) which is called the stabilization method has been successfully adopted for many other problems; see for example Penrose & Yukich (2001), Baryshnikov & Yukich (2005). However, still this method is not good enough to prove the BHH conjecture. In this paper we report a new method of constructing yet another truncated random variable $\Delta_k(L)$ in the hope that this method may provide a new insight for the BHH conjecture.

Let $\{x_1, \ldots, x_n\}$ be a finite subset of $\mathbb{R}^d$, $d \geq 2$. A minimal spanning tree (MST) on $\{x_1, \ldots, x_n\}$ is a spanning tree $T(\{x_1, \ldots, x_n\})$ on $\{x_1, \ldots, x_n\}$ such that

$$\sum_{e \in T(\{x_1, \ldots, x_n\})} |e| = \min \left\{ \sum_{e \in T} |e| : T \text{ a spanning tree on } \{x_1, \ldots, x_n\} \right\},$$

where $|e| = |x_i - x_j|$ is the Euclidean length of the edge $e = (x_i, x_j)$.

There are several interesting central limit theorems in connection with an MST; the length of an MST (see Alexander (1996), Kesten & Lee (1996) and Lee (1997)), the number of vertices of a given degree $\alpha$ in an MST (see Lee (1997, 1999)). In this paper we add one more story, the central limit theorem for the independence number of an MST. This problem is moti-
vated by the section 2 of David Aldous's "My Favorite 6 Open Problems" which is available at his homepage. See also Aldous & Steele (2003).

An independent set in a graph $G$ is a set of vertices, no two of which are linked by an edge and the independence number $N(G)$ of a graph $G$ is the maximal size of an independent set in the graph. As one of the most basic parameters for random graphs, the independence number has attracted attentions of a number of authors in the literatures. Meir & Moon (1973) were the first to study the independence number for uniform random trees and obtained a weak law of large numbers. Later on, Pittel (1999) used the recurrence relation for the moment generating functions of uniform random trees and established the central limit theorem. In the case of $G(n, d/n)$, Frieze (1990) used the method of bounded differences and proved that for any $\varepsilon > 0$, and for sufficiently large $d \geq d_0(\varepsilon)$ and $n \geq n_0(\varepsilon)$,

$$P \left( \left| N \left( G \left( n, \frac{d}{n} \right) \right) - \frac{2n}{d} (\log d - \log \log d - \log 2 + 1) \right| \leq \frac{\varepsilon n}{d} \right) \rightarrow 1.$$ 

In this direction, Boucheron, Lugosi & Massart (2000) used a powerful tool of log-Sobolev inequality and got a sharp concentration inequality which is independent of $d$. In the case of random geometric graphs, Penrose (2003), Penrose & Yukich (2005) verified the exponential stabilization property for some random geometric graphs and obtained a law of large numbers and the central limit theorem.

In this paper we prove the central limit theorem for the independence number of an MST when the random points are from the Poisson point process. Our results are as follows:

**Theorem 1.1:** Suppose that $P_1$ is a Poisson point process of intensity 1 in $\mathbb{R}^2$. Let $\hat{T}_n$ be the MST on $P_1 \cap [-n^{1/2}/2, n^{1/2}/2]^2$ and denote by $N(\hat{T}_n)$ the corresponding independence number. Then there are two positive finite constants $c_1$ and $c_2$ such that

$$c_1 n \leq \text{Var} N(\hat{T}_n) \leq c_2 n$$

and $N(\hat{T}_n)$ is asymptotically normally distributed, that is, as $n \to \infty$

$$\frac{N(\hat{T}_n) - EN(\hat{T}_n)}{(\text{Var} N(\hat{T}_n))^{1/2}} \to N(0, 1)$$

in distribution.

Following the framework of Kesten & Lee (1996), we write the quantity $N(\hat{T}_n) - EN(\hat{T}_n)$ as a sum of martingale differences, and then apply the
Lévy martingale central limit theorem. In this way, the proof of the central limit theorem for \( N(\hat{T}_n) \) is reduced to a kind of weak law of large numbers estimate for certain conditional variances. To get a weak law we need some independence and this required independence is usually obtained by the truncation method using the stabilizing property (see Lee (1997), Penrose & Yukich (2001), Baryshnikov & Yukich (2005), Penrose & Yukich (2005), for the stabilizing property and it applications). However, for our random variable \( N(\hat{T}_n) \) this stabilizing property does not hold or at least we don’t know how to prove it. So, we need a new idea; instead of studying the original random variable \( N(\hat{T}_n) \) we artificially create an approximating random variable, say \( N'_n \). By its construction this approximating random variable \( N'_n \) has the stabilizing property so we can follow the steps of Kesten & Lee (1996) and get the central limit theorem for \( N'_n \). Furthermore, the approximation error turns out to be small enough to dig out the central limit theorem for \( N(\hat{T}_n) \) from the central limit theorem for \( N'_n \). One may hope to prove that Theorem 1.1 holds for the non-Poisson case. Unfortunately, since we don’t have the stabilization property for the independence number \( N(\hat{T}_n) \), we cannot follow the de-Poissonization step of Kesten & Lee (1996) and even worse we cannot prove the existence of \( \text{Var}N(\hat{T}_n) \).

Our main tool is the percolation technique regarding the existence of the open and closed circuit at criticality. So, our approximation method works only for \( d = 2 \) and hence the problem is still open for high dimensions.

In Section 2, we review MSTs and continuum percolation for further use. Since there are a large number of literatures on MSTs and continuum percolation, we will omit the details. In Section 3, we prove Theorem 1.1.

In this paper, there are lots of strictly positive but finite constants whose specific values are not of interest to us. We denote them by \( c_i, C(q) \).

2. Minimal spanning trees and continuum percolation

In this section, we recall several facts regarding minimal spanning trees and continuum percolation for further use.

**Lemma 2.1:** Let \( G = (V, E, w) \) be a connected weighted graph satisfying \( |V| < \infty, |E| < \infty \). If there exists a path \( \pi = (v_1, v_2, \ldots, v_n) \) in \( G \) with \( (v_j, v_{j+1}) \in E, 1 \leq j \leq n - 1 \), from \( v_1 \in V \) to \( v_n \in V \), such that also \( |v_j - v_{j+1}| \leq \lambda, 1 \leq j \leq n - 1 \), then for any MST \( T \) on \( G \) there exists a path \( \pi' = (v'_1, v'_2, \ldots, v'_m) \) in \( G \) with \( (v'_j, v'_{j+1}) \in T, 1 \leq j \leq m - 1 \), from \( v_1 = v'_1 \) to \( v_n = v'_m \) such that \( |v'_j - v'_{j+1}| \leq \lambda, 1 \leq j \leq m - 1 \).

Lemma 2.2: There exists a finite constant $D_d$, which depends only on the dimension $d$, such that, for any MST $T(A)$ on any finite subset $A$ of $\mathbb{R}^d$, $T(A)$ has maximum vertex degree bounded by $D_d$. In particular, $D_2 = 6$.


Now, we review continuum percolation. For a set $W$ of points of $\mathbb{R}^2$ and for $r \geq 0$, $x \in \mathbb{R}^2$, denote by $\{x \xrightarrow{r} \infty \text{ in } W\}$ the event that there exist an infinite path $x = x_0, x_1, x_2, \ldots, x_n, \ldots$, and $|x_n| \to \infty$ in $W$ such that $|x_i - x_{i-1}| \leq r, i = 1, 2, \ldots$. The fundamental theorem of continuum percolation says that there exists a constant $0 < r_c = r_c(2) < \infty$ such that

$$P(0 \xrightarrow{r} \infty \text{ in } \mathcal{P}_1) = \begin{cases} > 0 & \text{if } r > r_c, \\ 0 & \text{if } r < r_c. \end{cases}$$

3. Proof of Theorem 1.1

Let us start with the proof of the statement that the variance $\text{Var}N(\hat{T}_n)$ is asymptotically the same order as $n$. By a standard block argument like that of Kesten & Lee (1996) Theorem 2, pp.525-527, we easily see that

$$\text{Var}N(\hat{T}_n) \geq c_1n. \quad (3.1)$$

Suppose that $\{X_i : i \geq 1\}$ are i.i.d. uniform points on $[-n^{1/2}/2, n^{1/2}/2]^2$. Let $\alpha_n$ be a Poisson random variable with mean $n$, then $X_1, X_2, \ldots, X_{\alpha_n}$ is identical in distribution to $\mathcal{P}_1 \cap [-n^{1/2}/2, n^{1/2}/2]^2$. Now let $T_n$ be the MST on $\{X_i : 1 \leq i \leq n\}$ and denote by $N(T_n)$ the corresponding independence number. To stress the dependency to the points, we denote by $N(S)$ the independence number of the MST through $S$. By Steele, Eddy & Shepp (1987) Lemma 2, it follows for any $m \geq 1$ and $x_1, x_2, \ldots, x_m$

$$|N(x_1, x_2, \ldots, x_m) - N(x_1, x_2, \ldots, x_{m-1})| \leq c_3.$$

Thus by Efron-Stein inequality,

$$\text{Var}N(T_n) \leq \sum_{k=1}^{n} E[N(X_1, \ldots, X_{k-1}, X_{k+1}, \ldots, X_n) - N(X_1, X_2, \ldots, X_n)]^2$$

$$\leq c_4n. \quad (3.2)$$
Also, note that

$$E(N(\tilde{T}_n) - N(T_n))^2 \leq c_5 E(\alpha_n - n)^2 = c_5 n.$$ 

This, together with \(\text{Var} N(T_n) \leq c_4 n\), implies

$$\text{Var} N(\tilde{T}_n) \leq c_2 n. \quad (3.3)$$

Next, let us turn to prove the asymptotic normality. Fix \(1/4 < \alpha < \beta < 1/3\). Let \(A_n(x)\) be the moat with center \(x = (x_1, x_2)\), inner radius \(n^\alpha\), outer radius \(n^\beta\), that is

$$A^0_n(x) := \prod_{k=1}^2 [x_k - n^\beta, x_k + n^\beta],$$

$$A^i_n(x) := \prod_{k=1}^2 [x_k - n^\alpha, x_k + n^\alpha],$$

$$A_n(x) := A^0_n(x) \setminus A^i_n(x).$$

We say a finite point set \(A\) is nice if all inter point distances in \(A\) are distinct. For any nice set of points \(A\) in \(\mathbb{R}^2\), let \(\tilde{T}_n(A)\) be the MST through \(A \cap [-n^{1/2}/2, n^{1/2}/2]^2\), \(N_n(A)\) the corresponding independence number. For the critical value \(r_c\) of Poisson continuum percolation, call an edge \(e = (x, y)\) open in \(A\) if \(|e| \leq r_c\). Now we would like to have the following properties for the point configuration \(A\) on the moat \(A_n(x)\);

(P1) there exists a closed circuit \(C_1\) in \(A_n(x)\) at criticality whose interior contains \(A^i_n(x)\), i.e., there exists a continuous path \(\pi_1 : [0, 1] \to A_n(x)\) with \(\pi_1(0) = \pi_1(1)\) such that

$$\cup_{0 \leq t \leq 1} \{x \in \mathbb{R}^2 : |x - \pi_1(t)| \leq r_c/2\} \cap A = \emptyset,$$

$$\bar{\pi}_1 := \cup_{0 \leq t \leq 1} \{x \in \mathbb{R}^2 : |x - \pi_1(t)| \leq r_c/2\} \subset A_n(x),$$

the bounded component of \(\mathbb{R}^2 \setminus \bar{\pi}_1\) contains \(A^i_n(x)\),

(P2) there exists an open circuit \(O_2\) in \(A_n(x)\) at criticality whose interior contains \(C_1\), i.e., there exists a continuous path \(\pi_2 : [0, 1] \to A_n(x)\) with \(\pi_2(0) = \pi_2(1)\) such that

there is an \(m\) with \(\pi_2(j/m) \in A\),

and \(|\pi_2((j+1)/m) - \pi_2(j/m)| \leq r_c\) for \(0 \leq j < m\),
$$\bar{\pi}_2 := \cup_{0 \leq t \leq 1} \{x \in \mathbb{R}^2 : |x - \pi_2(t)| \leq r_c/2 \} \subset A_n(x) ,$$

the bounded component of $\mathbb{R}^2 \setminus \bar{\pi}_2$ contains $\bar{\pi}_1$.

(P3) there exists a closed circuit $C_3$ in $A_n(x)$ at criticality whose interior contains $O_2$.

(P4) there exists an open circuit $O_4$ in $A_n(x)$ at criticality whose interior contains $C_3$.

If (P1), (P2), (P3) and (P4) hold simultaneously, we say that the property $G_n(x)$ (or $G_n(x; A)$) to denote the dependency of the property to the point configuration $A$) holds. Note that the property $G_n(x)$ is completely characterized by $A \cap A_n(x)$. If for $x$ with $A_n(x) \subset [-n^{1/2}/2, n^{1/2}/2]^2$ the property $G_n(x)$ holds, then without loss of generality we may and do assume that $O_2$ and $O_4$ are the most inner and most outer open circuits respectively. By Lemma 2.1, we know that in the MST $\hat{T}_n(A)$ there exists a unique self-avoiding path $\pi = (y_1, y_2, \ldots, y_m)$ from $O_2$ to $O_4$ with $y_1 \in O_2$, $y_m \in O_4$, $y_j \notin O_2 \cup O_4$ for $2 \leq j \leq m - 1$. Suppose there is another such path $\pi^* = (z_1, z_2, \ldots, z_l)$. In addition, without loss of generality suppose

$$\max_{1 \leq j \leq l-1} |z_j - z_{j+1}| > \max_{1 \leq j \leq m-1} |y_j - y_{j+1}| ,$$

$$|z_{j_0} - z_{j_0+1}| = \max_{1 \leq j \leq l-1} |z_j - z_{j+1}| .$$

Note that since the path $\pi$ should cross the closed circuit $C_3$, $\max_{1 \leq j \leq m-1} |y_j - y_{j+1}| > r_c$.

Now, we travel from $z_{j_0}$ to $O_2$ using $(z_{j_0}, z_{j_0-1}, \ldots, z_1)$, from $z_1$ to $y_1$ using only the points in $O_2$, from $y_1$ to $y_m$ using $\pi$, from $y_m$ to $z_l$ using only the points in $O_4$, from $z_l$ to $z_{j_0+1}$ using $(z_l, z_{l-1}, \ldots, z_{j_0+1})$. In this way we find a path $\pi^{**} = (w_1, w_2, \ldots, w_k)$ from $w_1 = z_{j_0}$ to $w_k = z_{j_0+1}$ with $\max_{1 \leq j \leq k-1} |w_j - w_{j+1}| < |z_{j_0} - z_{j_0+1}|$. By Lemma 2.1, the edge $(z_{j_0}, z_{j_0+1})$ cannot be in the MST $\hat{T}_n(A)$ and this leads to a contradiction. Therefore, there is indeed a unique path $\pi = (y_1, \ldots, y_m)$ from $O_2$ to $O_4$ in the MST $\hat{T}_n(A)$. In this case, we find the first $y_{i_0}$ in $\pi$ with jump size greater than the critical value, that is

$$|y_i - y_{i+1}| < r_c, \quad j = 1, \ldots, i_0 - 1,$$

$$|y_{i_0} - y_{i_0+1}| > r_c.$$
Now, we call such points \( y_{i_0} \) big jump points and we denote the set of the big jump points in \([-n^{1/2}/2, n^{1/2}/2]^2\) by \( B_n \).

When we consider the independent set, we would like to exclude those big jump points. In other words, instead of \( N_n(A) \) we now consider \( N'_n(A) \) where \( N'_n(A) \) is given by

\[
N'_n(A) = \max\{|A|; A \text{ is an independent set in } \tilde{T}_n(A), A \cap B_n = \emptyset\}. \tag{3.4}
\]

Since for each big jump point \( y_{i_0} \) there is a corresponding open circuit \( \mathcal{O}_2 \) whose interior \( K(y_{i_0}) \) has its volume at least \((2n^a)^2\), and since by the choice of the big jump points in \([-n^{1/2}/2, n^{1/2}/2]^2\) for two distinct big jump points \( y_{i_0} \) and \( y'_{i_0} \),

\[
K(y_{i_0}) \cap K(y'_{i_0}) = \emptyset,
\]

the number \(|B_n|\) of big jump points is at most

\[
|B_n| \leq \frac{\text{Volume of } [-n^{1/2}/2, n^{1/2}/2]^2}{\text{Volume of } A^1_n(x)} = O(n^{1-2\alpha}) = o(n^{1/2}). \tag{3.5}
\]

Also, by the maximality of \( N_n(A) \) we have \( N_n(A) \geq N'_n(A) \). Now, note that any independent set \( A \) can be written as \( A = (A \cap B_n) \cup (A \cap B_n^c) \). Since a subset \( A \cap B_n^c \) of the independent set \( A \) is still an independent set, by the maximality of \( N'_n(A) \) we have \(|A \cap B_n^c| \leq N'_n(A)\) and also \(|A| = |A \cap B_n| + |A \cap B_n^c| \leq |B_n| + N'_n(A)\). Hence,

\[
|N_n(A) - N'_n(A)| \leq |B_n| = o(n^{1/2}). \tag{3.6}
\]

Next let us focus upon the Poisson point configuration \( \mathcal{P}_1 \). Observe first that by the standard argument of the RSW inequality in continuum percolation (see Alexander (1996) and Grimmett (1999) for the detail) in \( \mathbb{R}^2 \),

\[
P(G_n(x; \mathcal{P}_1)) \to 1 \text{ as } n \to \infty
\]

uniformly in \( x \).

Write for simplicity \( N'_n = N'_n(\mathcal{P}_1) \) and \( \sigma^2_n = \text{Var}N'_n \). By (3.6), it is easy to see that

\[
\lim_{n \to \infty} \frac{\sigma^2_n}{\text{Var}N(\tilde{T}_n)} = 1
\]

and so \( c_0 n \leq \sigma^2_n \leq c_7 n \) and to prove Theorem 1.1, it suffices to prove the central limit theorem for \( N'_n \), that is, as \( n \to \infty \)

\[
\frac{N'_n - EN'_n}{\sigma'_n} \to N(0, 1) \tag{3.7}
\]
in distribution.

Now, we represent $N'_n - EN'_n$ as a sum of martingale differences and we apply Lévy’s martingale central limit theorem to the sum of martingale differences. In this way, the proof of the central limit theorem for $N'_n$ is reduced to a kind of weak law of large numbers estimate for certain conditional variances. Even though a weak law of large numbers is much easier to obtain, in general, than a central limit theorem, it still requires some independence. The required independence is obtained by approximating the conditional variances by quantities which depend only on the local configuration of $P_1$. This approximation method is motivated by the bounded rooted dual of Redmond & Yukich (1994) and works well due to the existence of the open and closed circuits at criticality.

For each $v = (v_1, v_2) \in \mathbb{Z}^2$, we let $Q(v) = \prod_{k=1}^2 [v_k - \frac{1}{2}, v_k + \frac{1}{2}]$. We order the points $v$ of $\mathbb{Z}^2$ with $Q(v) \cap [n^{1/2}/2, n^{1/2}/2]^2 \neq \emptyset$ in some way, say lexicographically, as $v(1), \cdots, v(l)$, and we define $\mathcal{F}_k$ by

$$\mathcal{F}_k = \sigma(\mathcal{P}_1 \cap [\cup_{i \leq k} Q(v(i))])$$

($\mathcal{F}_0$ is the trivial $\sigma$-field). Then, $N'_n - EN'_n$ can be written as a sum of martingale differences, i.e.,

$$N'_n - EN'_n = \sum_{k=1}^l \Delta_k,$$  \hspace{1cm} (3.8)

where

$$\Delta_k = \mathbb{E}(N'_n|\mathcal{F}_k) - \mathbb{E}(N'_n|\mathcal{F}_{k-1})$$

$$= \int P(da_k, \cdots, da_l)$$

$$\left[N'_n([\cup_{i \leq k} A_i] \cup [\cup_{i > k} a_i]) - N'_n([\cup_{i < k} A_i] \cup [\cup_{i \geq k} a_i])\right]$$

$$= \int P(da_k, \cdots, da_l)$$

$$\left[D_n([\cup_{i \leq k} A_i] \cup [\cup_{i > k} a_i], A_k) - D_n([\cup_{i < k} A_i] \cup [\cup_{i \geq k} a_i], a_k)\right]$$

$$=: \int D_{n,k} P(da_k, \cdots, da_l), \hspace{1cm} (3.9)$$

where $A_i = P_1 \cap Q(v(i)), 1 \leq i \leq l$, where $P(da_k, \cdots, da_l)$ is short for $P(A_k \in da_k, \cdots, A_l \in da_l)$, and where

$$D_n(A, B) = N'_n(A) - N'_n(A \setminus B).$$
By virtue of the representation of \( N'_n - EN'_n \) as a sum of martingale differences and by Theorem 67.2 of Lévy (1937), Theorem (2.3) of McLeish (1974), or Theorem 3.2 of Hall & Heyde (1980), to prove (3.7), it suffices to verify the following three relations:

\[
\frac{1}{\sigma_n^2} \sum_{k=1}^{t} \Delta_k^2 \to 1 \text{ in probability as } n \to \infty, \tag{3.10}
\]

\[
\frac{1}{\sigma_n} \max_{1 \leq k \leq l} |\Delta_k| \to 0 \text{ in probability as } n \to \infty, \tag{3.11}
\]

\[
\frac{1}{\sigma_n^2} \mathbb{E}(\max_{1 \leq k \leq l} \Delta_k^2) \text{ is bounded in } n. \tag{3.12}
\]

To this end, we still need more notations. Now, for \( L = n^\theta \) we define \( \tilde{D}_L(x; W) \) by the following. If \( G_n(x) \) happens under the point configuration \( W \), that is, if \( G_n(x; W) \) happens, we let

\[
\tilde{D}_L(x; W) = D_n([\frac{-n^{1/2}}{2}, \frac{n^{1/2}}{2}] \cap W, Q(x) \cap W).
\]

If \( G_n(x; W) \) does not happen, we let

\[
\tilde{D}_L(x; W) = D_n(\prod_{k=1}^{2} [x_k - L, x_k + L] \cap W, Q(x) \cap W).
\]

Note that the event \( G_n(x; W) \) completely depends on the point configuration \( \prod_{k=1}^{2} [x_k - L, x_k + L] \cap W \). Also, if \( G_n(x; W) \) happens, then \( \tilde{D}_L(x; W) \) completely depends on the point configuration \( \prod_{k=1}^{2} [x_k - L, x_k + L] \cap W \).

In this case, even though

\[
\tilde{D}_L(x; W) = D_n([\frac{-n^{1/2}}{2}, \frac{n^{1/2}}{2}] \cap W, Q(x) \cap W)
\]

looks as if it has dependency running over the whole point configuration \( [-n^{1/2}/2, n^{1/2}/2]^2 \cap W \), this is not the case. In fact, when we calculate the modified independence number \( N'_n([-n^{1/2}/2, n^{1/2}/2]^2 \cap W) \) and \( N'_n([-n^{1/2}/2, n^{1/2}/2]^2 \setminus Q(x) \cap W) \), the big jump point does not change, the graph structure outside the big jump point does not change, and the vertices outside the big jump point which are included in the modified independence set also do not change. Therefore, the point configuration outside the big jump point has no influence on \( \tilde{D}_L(x; W) \). Of course, if \( G_n(x; W) \) does not happen, by definition \( \tilde{D}_L(x; W) \) completely depends on the point configuration \( \prod_{k=1}^{2} [x_k - L, x_k + L] \cap W \). Therefore, in either case \( \tilde{D}_L(x; W) \) completely depends on the point configuration.
\[ \prod_{k=1}^{2}(x_k - L, x_k + L) \cap W \] and if \( G_n(x; W) \) happens, then \( \tilde{D}_L(x; W) = D_n([-n^{1/2}/2, n^{1/2}/2]^2 \cap W, Q(x) \cap W) \).

Now, to proceed, we write \( \Delta_{k,L} \) for the expression arising in (3.9) when \( D_n([\cup_{i \leq k} A_i] \cup [\cup_{i > k} a_i], a_k) \) and \( D_n([\cup_{i < k} A_i] \cup [\cup_{i \geq k} a_i], a_k) \) are replaced by \( \tilde{D}_L(v(k); [\cup_{i \leq k} A_i] \cup [\cup_{i > k} a_i]) \) and \( \tilde{D}_L(v(k); [\cup_{i < k} A_i] \cup [\cup_{i \geq k} a_i]) \), respectively, i.e.,

\[
\Delta_{k,L} = \int P(da_k, \ldots, da_i) \\
\times \left[ \tilde{D}_L(v(k); [\cup_{i \leq k} A_i] \cup [\cup_{i > k} a_i]) - \tilde{D}_L(v(k); [\cup_{i < k} A_i] \cup [\cup_{i \geq k} a_i]) \right] \\
:= \int D_{n,k,L} P(da_k, \ldots, da_i). \tag{3.13}
\]

Note that \( \Delta_k \) depends on \( \mathcal{P}_1 \cap [-n^{1/2}/2, n^{1/2}/2]^2 \) whereas \( \Delta_{k,L} \) depends only on \( \prod([v(k)_i] - L, (v(k))_i + L) \cap \mathcal{P}_1 \).

(3.11) and (3.12) can be easily verified by Chebyshev’s inequality and by the moment estimates in Lemma 3.1 below.

**Lemma 3.1:** For each \( q > 0 \), there exists a constant \( C_q \) such that

\[ E[D_n(\mathcal{P}_1, \mathcal{P}_1 \cap Q(x))]^q \leq C_q \text{ uniformly in } x. \tag{3.14} \]

For \( L = n^\beta \),

\[ E[\tilde{D}_L(x; \mathcal{P}_1)]^q \leq C_q \text{ uniformly in } x. \tag{3.15} \]

**Proof:** To have the moment estimate in (3.15), in the case \( G^c_n(x; P_1) \) first we construct an MST \( T(1) \) on \( \prod([x_i - L, x_i + L] \cap Q(x)) \cap P_1 \). Second we construct an MST \( T(2) \) on \( \prod([x_i - L, x_i + L] \cap P_1) \) by applying the revised add and delete algorithm (see Kesten & Lee (1996) and Lee (1997)) with \( T(2) \setminus T(1) \) to \( T(1) \). When we construct \( T(2) \) from \( T(1) \), the local structure of the graph around some vertices changes. However, since by Lemma 2.2 each \( v \in Q(x) \cap P_1 \) has degree bounded by 6 in \( T(2) \), by the revised add and delete algorithm the number of vertices for which the edges in \( T(2) \) emerging from the particular vertex are different from those in \( T(1) \), is bounded by \( 12|Q(x) \cap P_1| \). Also, by the choice of the big jump point the number of big jump points may change at most \( c_8 \). So, it follows from the revised add and delete algorithm that

\[ |\tilde{D}_L(x; P_1)| \leq c_8|Q(x) \cap P_1| + c_8. \]

We can handle the case \( G_n(x; P_1) \) in a similar way and get the similar bound for \( |\tilde{D}_L(x; P_1)| \). In fact, in this case the number of big jump point
does not change. Indeed, suppose that $A$ and $B$ are two subsets of points and $A \subset B$. If $x, y \in A$ and $e = (x, y)$ is one of edges of an MST of $B$, then $e = (x, y)$ is one of edges of an MST of $A$, too (see Eddy, Shepp & Steele (1987)). Hence the addition or the deletion of the points from $Q(x)$ have no influence upon the existence of a unique path from the most inner open circuit to the most outer open circuit in $P_1 \cap A_n(x)$. So, in this case we have

$$|\tilde{D}_L(x; P_1)| \leq c_8 |Q(x) \cap P_1|.$$  

Therefore, the moment estimate in (3.15) holds. By the same argument, the moment estimate in (3.14) also holds. 

Now (3.11) and (3.12) follow from Chebyshev’s inequality and the moment estimates in Lemma 3.1;

$$P\left(\frac{1}{\sigma_n} \max_{1 \leq k \leq l} |\Delta_k| \geq \varepsilon \right) \leq \frac{\sum_{k=1}^{l} E|\Delta_k|^3}{(\varepsilon \sigma_n)^3} = O(n^{-1/2}),$$

$$\frac{1}{\sigma_n^2} E(\max_{1 \leq k \leq l} \Delta_k^2) \leq \frac{1}{\sigma_n^2} \sum_{k=1}^{l} E\Delta_k^2 \text{ is bounded in } n.$$  

So the heart of the matter lies in the proof of convergence in (3.10). Define $\sigma_{n,L}^2 = \sum_{k=1}^{l} E\Delta_{k,L}^2$. In order to prove (3.10), we shall verify the following two relations:

$$\frac{1}{n} \sum_{k=1}^{l} E|\Delta_k^2 - \Delta_{k,L}^2| \to 0, \quad (3.16)$$

$$\frac{1}{\sigma_{n,L}^2} \sum_{k=1}^{l} \Delta_{k,L}^2 \to 1 \text{ in probability as } n \to \infty. \quad (3.17)$$

Recall that

$$P(G_n(x; \mathcal{P}_1)) \to 1 \text{ as } n \to \infty$$

uniformly in $x$. If $G_n(v(k); (\cup_{i<k} A_i) \cup (\cup_{i>k} A_i))$ happens, then we have $D_{n,k} = D_{n,k,L}$. So, with $\varepsilon_{n,k} = D_{n,k} - D_{n,k,L}$

$$\Delta_k^2 - \Delta_{k,L}^2 = \left( \int D_{n,k} dP \right)^2 - \left( \int D_{n,k,L} dP \right)^2$$

$$= \left( \int D_{n,k} dP \right)^2 - \left( \int D_{n,k} dP - \int \varepsilon_{n,k} dP \right)^2$$

$$= 2 \left( \int D_{n,k} dP \right) \left( \int \varepsilon_{n,k} dP \right) - \left( \int \varepsilon_{n,k} dP \right)^2.$$
Now, since \( \varepsilon_{n,k} = \varepsilon_{n,k} G_n^{\varepsilon}(v(k);(U_i < k A_i) \cup (U_i > k A_i)) \), we have uniformly in \( k \),
\[
E\left( \int \varepsilon_{n,k} dP \right)^2 \leq E \int \varepsilon_{n,k}^2 dP \leq \left( \int \varepsilon_{n,k}^4 dP \right)^{1/2} \left( P(G_n^{\varepsilon}(0)) \right)^{1/2} \rightarrow 0. \tag{3.18}
\]

Thus we have
\[
\frac{1}{n} \sum_{k=1}^{l} E|\Delta_k^2 - \Delta_{k,L}^2| \leq \frac{1}{n} \sum_{k:A_n^0(v(k)) \cap \mathbb{R}^2 \setminus [-n^{1/2}/2,n^{1/2}/2]^2 \neq \emptyset} E|\Delta_k^2 - \Delta_{k,L}^2| \\
+ \frac{1}{n} \sum_{k:A_n^0(v(k)) \subseteq [-n^{1/2}/2,n^{1/2}/2]^2} E|\Delta_k^2 - \Delta_{k,L}^2| \rightarrow 0. \tag{3.19}
\]

Thus, since \( c_6 n \leq \sigma_{n}^2 \leq c_{7} n \), it follows that
\[
\lim_{n \to \infty} \frac{\sigma_{n,L}^2}{\sigma_{n}^2} = 1
\]
and \( c_9 n \leq \sigma_{n,L}^2 \leq c_{10} n \).

Now (3.17) follows from Chebyshev's inequality using the fourth moment. Since the events \( \Delta_{k,L} \) and \( \Delta_{k',L} \) are independent as long as \( A_n^0(v(k)) \cap A_n^0(v(k')) = \emptyset \), and since for each \( k \) the number of \( k' \) with \( A_n^0(v(k)) \cap A_n^0(v(k')) \neq \emptyset \) is of order \( n^{2\beta} \),
\[
E\left( \sum_{k=1}^{l} (\Delta_{k,L}^2 - E\Delta_{k,L}^2)^4 \right) \leq c_{11} \left[ n + n^{1+2\beta} + n^{1+4\beta} + n^{1+2\beta} \right] = O(n^{1+6\beta})
\]
and hence
\[
P\left( \frac{1}{\sigma_{n,L}^2} \sum_{k=1}^{l} \Delta_{k,L}^2 - E\Delta_{k,L}^2 > \varepsilon \right) \leq \frac{c_{12} n^{1+6\beta}}{\varepsilon n^4} \rightarrow 0.
\]

Now, Theorem 1.1 follows.

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