Central limit theorems for random processes with sample paths in exponential Orlicz spaces

Zhonggen Su\textsuperscript{a,b}

\textsuperscript{a} Department of Mathematics, Hangzhou University, Zhejiang, 310028, People’s Republic of China
\textsuperscript{b} Institute of Mathematics, Fudan University, Shanghai, 200433, People’s Republic of China

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Abstract

This paper is devoted to the study of central limit theorems and the domain of normal attraction for some random processes with sample paths in exponential Orlicz spaces under metric entropy conditions. In particular, the local times of strongly symmetric standard Markov processes and random Fourier series are discussed.

Keywords: Central limit theorem; Orlicz space; Local times; Random Fourier series

1. Introduction

Let us first recall that a mean zero random vector $X$ with values in a separable Banach space $B$ is said to satisfy the central limit theorem if there exists a Gaussian vector $G$ in $B$ such that the normalized sums $\sum_{i=1}^{n}X_i/\sqrt{n}$ converge weakly to $G$, where $X_n$, $n \geq 1$, is a sequence of independent random vectors with the same distribution as $X$. We say that $X$ belongs to the domain of normal attraction of $Y$ if the normalized sums $\sum_{i=1}^{n}X_i/n^{1/p}$ converge weakly to a $p$-stable random vector $Y$ for some $1 < p < 2$. The purpose of this paper is to discuss the central limit theorem and the domain of normal attraction of some random processes.

Suppose that $X = (X(t), t \in T)$ is a real or complex valued random process indexed by $T$ and its sample paths are almost surely in a separable Banach space, then we can consider $X$ as a random vector in $B$. Many common function spaces, however, such as the set $C(T)$ of all bounded continuous functions on $T$ and the Orlicz space $L_\phi$ generated by a general Young function $\phi$, are not of finite Rademacher type, Rademacher cotype or stable type. Thus the general theory of central limit theorems in Banach spaces cannot be directly applied, and hence the study of weak convergence for the stochastic process $X = (X(t), t \in T)$ relies heavily on the sample paths themselves. If $(T,d)$ is a compact metric or pseudometric space, let $N(T,d,\varepsilon)$ denote the minimal number of open balls of radius $\varepsilon$, with centers in $T$, that cover $T$. $\log N(T,d,\varepsilon)$ is called the metric entropy of $(T,d)$.

Jain and Marcus (1975) first investigated the central limit theorem for $C(T)$ valued random variables satisfying a Lipschitz condition under the finiteness of the metric entropy.
entropy integral. Marcus and Pisier (1984b, Corollary 1.4) extended their results to obtain the domain of normal attraction of a $p$-stable continuous stochastic process. These authors make efficient use of the characterization of a compact subset of $C(T)$ given by the Arzela–Ascoli theorem. Random Fourier series, although not satisfying the Lipschitz condition, can induce a translation invariant distance on $T$. Marcus and Pisier (1981, 1984b) use this metric to make a complete study of random Fourier series.

In this paper we will be interested in studying the central limit theorem for certain stochastic processes whose sample paths are in exponential Orlicz spaces. We first prove a general central limit theorem for random processes with sample paths in an exponential Orlicz space by characterizing compact subsets. Our basic assumption is still the finiteness of the metric entropy integral. In Section 3 we investigate the local time of certain strongly symmetric standard Markov processes whose state space is in a locally compact metric space by using the Dynkin isomorphism theorem (see Lemma 3.1 in Section 3). Finally, Theorem 1.1 of Chapter IV in Marcus and Pisier (1981) is extended to Orlicz spaces.

2. Random processes with sample paths in exponential Orlicz spaces

Let $\phi_q(x) = \exp|x|^q - 1$, $1 \leq q \leq \infty$. Suppose that $(T, \mathcal{F}, \mu)$ is a probability measure space, we denote by $L_{\phi_q}(d\mu)$ the so-called Orlicz space formed by all measurable functions $x : T \to C$ for which there is a positive constant $c > 0$ such that $\int_T \phi_q(|x(t)|/c)d\mu(t) < \infty$, and define the norm

$$
\|x\|_{\phi_q(d\mu)} = \inf \left\{ c > 0 : \int_T \phi_q(|x(t)|/c)d\mu(t) \leq 1 \right\}.
$$

In the case $q = \infty$, by definition $\|x\|_{\phi_{\infty}} = \sup_{t \in T} |x(t)|$. It is well known that $(L_{\phi_q}(d\mu), \|\cdot\|_{\phi_q})$ is a Banach space and the closure $L^0_{\phi_q}(d\mu)$ of the subset of all simple functions in $L_{\phi_q}(d\mu)$ is separable.

For any finite family $\pi = \{E_1, E_2, \ldots, E_m\} \subseteq \mathcal{F}$ of disjoint sets with positive finite measure, $U_\pi$ denotes the map from $L_{\phi_q}(d\mu)$ to $L^0_{\phi_q}(d\mu)$ given by

$$
U_\pi(x) = \sum_{i=1}^m \frac{1}{\mu(E_i)} \int_{E_i} x(s)d\mu(s)I_{E_i}.
$$

Let $\{\pi\}$ be directed by defining $\pi \leq \pi_1$ to mean that every set in $\pi$ is, except for a set of measure zero, a union of sets in $\pi_1$. The following basic lemma gives a characterization of compact sets in $L^0_{\phi_q}(d\mu)$; it plays the same role in $L^0_{\phi_q}(d\mu)$ as the Arzela–Ascoli theorem in $C(T)$.

**Lemma 2.1.** Suppose that $(T, \mathcal{F}, \mu)$ is a probability measure space with a countably generated $\sigma$-field, then a subset $K$ in $L^0_{\phi_q}(d\mu)$ is relatively compact if and only if

(i) $K$ is bounded,

(ii) there exists a sequence of $\{\pi_j\}$ directed by $\pi_j \leq \pi_{j+1}$ such that $U_{\pi_j}(x)$ converges uniformly in $K$. 


Proof. Sufficiency: Assume that the conditions (i) and (ii) hold. For each $\pi = (E_1, \ldots, E_m)$, we have

$$\int_T \exp \left( \frac{|U_\pi(x)(t)|^q}{\|x\|_{\phi_q}^q} \right) d\mu(t) \leq \sum_{i=1}^m \int_{E_i} \exp \left( \frac{|1/\mu(E_i)| \int_{E_i} x(s)d\mu(s)|^q}{\|x\|_{\phi_q}^q} \right) d\mu(t)$$

$$\leq \sum_{i=1}^m \mu(E_i) \int_{E_i} \exp \left( \frac{|x(s)|^q}{\|x\|_{\phi_q}^q} \right) \frac{d\mu(s)}{\mu(E_i)}$$

$$\leq \int_T \exp \left( \frac{|x(s)|^q}{\|x\|_{\phi_q}^q} \right) d\mu(s) \leq 2, \quad (2.3)$$

from which we deduce that $\|U_\pi(x)\|_{\phi_q} \leq \|x\|_{\phi_q}$. Thus $\{U_\pi\}$ is a sequence of uniformly bounded linear operators.

Since $U_\pi$ has a finite dimensional range it maps any bounded set into a relatively compact set. Now according to the second part of Lemma IV 5.4 in Dunford and Schwartz (1988), $K$ is relatively compact as desired.

Necessity: Assume $K$ is relatively compact. Since $K$ is obviously bounded, we turn to the proof of (ii). Since $\mathcal{F}$ is a countably generated $\sigma$-field, there exists a countable subfamily $\{B_n, n \geq 1\}$ of $\mathcal{F}$ such that $\sigma(\{B_n, n \geq 1\}) = \mathcal{F}$.

For each $j \geq 1$, there is a finite partition $\{A_{i,j}, 1 \leq i \leq n(j)\}$ such that

$$\sigma(\{A_{i,j}, 1 \leq i \leq n(j)\}) = \sigma(\{B_n, 1 \leq n \leq j\}). \quad (2.4)$$

Let $\pi_j = (A_{1,j}, \ldots, A_{n(j),j})$. (2.3) implies $\{U_{\pi_j}\}$ is a sequence of uniformly bounded linear operators. On the other hand, for any indicator function $I_E$ and $p \geq 1$,

$$\|U_{\pi}(I_E) - I_E\|_p^p = \left\| \sum_{i=1}^{n(j)} \frac{1}{\mu(A_i)} \int_{A_i} I_E d\mu I_{A_i} - I_E \right\|_p^p$$

$$= \left\| \sum_{i=1}^{n(j)} \left( \frac{\mu(A_i \cap E)}{\mu(A_i)} I_{A_i \cap E} - \frac{\mu(A_i \cap E^c)}{\mu(A_i)} I_{A_i \cap E^c} \right) I_{(\cup A_i \cap E)} \right\|_p^p$$

$$\leq \|U_{\pi_j}(I_E) - I_E\|_1 - E_\mu|E_\mu|E_\mu|\sigma((A_{i,j})_{i=1}^{n(j)})| - I_E|, \quad (2.5)$$

where $E_\mu$ denotes the expectation operation with respect to $\mu$. By the martingale convergence theorem the last term of (2.5) tends to zero as $j \to \infty$. Thus, for any $\varepsilon > 0$ there is an integer $j$ large enough to insure that

$$E_\mu \exp \left( \frac{|U_{\pi_j}(I_E) - I_E|^q}{\varepsilon^q} \right) = \sum_{n=1}^{\infty} E_\mu |U_{\pi_j}(I_E) - I_E|^n \varepsilon^n \leq \sum_{n=1}^{\infty} \|U_{\pi_j}(I_E) - I_E\|_1^n \varepsilon^n \leq 2 \cdot (2.6)$$

This shows that $\|U_{\pi_j}(I_E) - I_E\|_{\phi_q}$ tends to zero as $j \to \infty$. If $x = \sum_{i=1}^k a_i I_{E_i}$, then

$$\lim_{j \to \infty} \|U_{\pi_j}(x) - x\|_{\phi_q} = 0. \quad This \ gives \ \lim_{j \to \infty} \|U_{\pi_j}(x) - x\|_{\phi_q} = 0 \ for \ any \ x \ in$$
$L^0_{\Phi_x}(d\mu)$ because the set of all simple functions is dense in $L^0_{\Phi_x}(d\mu)$. Now the desired (ii) follows directly from the first part of Lemma IV 5.4 in Dunford and Schwartz (1988).

**Remark 2.1.** Suppose that $(T,d)$ is a compact metric or pseudometric space, then $(T,d)$ is separable and the Borel $\sigma$-field $\mathcal{F}$ can be countably generated. Moreover, in the above lemma we can also choose $\pi_j = (A_{1,j}, \ldots, A_{n(j),j})$ such that $\max_{1 \leq i \leq n(j)} d(A_{i,j})$ tends to zero as $j \to \infty$, where $d(A_{i,j})$ is the diameter of $A_{i,j}$.

Prokhorov’s theorem and Lemma 2.1 easily gives

**Lemma 2.2.** Suppose that $(T, \mathcal{F}, \mu)$ is a probability measure space with a countably generated $\sigma$-field. A sequence of bounded measures $\{\nu_n, n \geq 1\}$ in $L^0_{\Phi_q}(d\mu)$ is relatively compact if and only if

(i) $\lim_{M \to \infty} \sup_n \nu_n(\|x\|_{\Phi_q} \geq M) = 0$,

(ii) $\lim_{j \to \infty} \sup_n \nu_n(\|x - U_n(x)\|_{\Phi_q} \geq \varepsilon) = 0$ for any $\varepsilon > 0$.

**Proof.** Since $L^0_{\Phi_q}(d\mu)$ is a separable Banach space, Prokhorov’s theorem indicates that $\nu_n, n \geq 1$, is relatively compact if and only if $\nu_n, n \geq 1$, is uniformly tight, that is, for each $\varepsilon > 0$ there exists a compact subset $K$ in $L^0_{\Phi_q}(d\mu)$ such that

$$\sup_{n \geq 1} \nu_n(x \in K^c) \leq \varepsilon. \quad (2.7)$$

Therefore, it is enough for us to show that the hypotheses (i) and (ii) are equivalent to (2.7). This can be done in a completely similar way to the proof of Theorem 8.2 in Billingsley (1968) by using Lemma 2.1 instead of Arzela–Ascoli theorem.

We now state the main result of this section.

**Theorem 2.1.** Suppose that $(T,d)$ is a compact metric or pseudometric space with Borel $\sigma$-field. Let $(X(t), t \in T)$ be a real valued random process with zero mean, defined on a probability space $(\Omega, \mathcal{F}, P)$ such that

$$\|X(t) - X(s)\|_{\Phi_q(dP)} \leq d(s,t)$$

for some $1 \leq q \leq 2$ and any $s,t \in T$.

If $\int_0^\infty (\log N(T,d,\varepsilon))^{(1/q) - (1/q')} \, d\varepsilon < \infty$ for some $q < q' < \infty$, then $X$ satisfies the central limit theorem as a random vector in $L^0_{\Phi_q}(d\mu)$.

**Proof.** According to Theorem 3.1 in Marcus and Pisier (1985), it is known that under the hypotheses stated there exists a version of $X = (X(t), t \in T)$, still denoted by $X$, such that $X$ has its sample paths in the separable Banach space $L^0_{\Phi_q}(d\mu)$ almost surely, and further

$$E\|X\|_{\Phi_q(d\mu)} \leq C_{q,q'} \left( E\|X(t_0)\| + \hat{d} + \int_0^\infty (\log N(T,d,\varepsilon))^{(1/q) - (1/q')} \, d\varepsilon \right), \quad (2.9)$$

where $t_0 \in T$, $\hat{d} = \sup_{s,t \in T} d(s,t)$ and $C_{q,q'}, C_q$ (defined later) are numerical constants depending only on $q, q'$. 

Since a random vector $X$ with $EX = 0$ satisfies the central limit theorem if and only if $\varepsilon X$ does, where $\varepsilon$ is a Rademacher random variable independent of $X$ (see Ledoux and Talagrand, 1991, p. 279), without loss of generality we can assume that $X$ is a symmetric process. Let $(X, X_n, n \geq 1)$ be an independent and identically distributed sequence, and write $S_n = \sum_{i=1}^{n} X_i$. The following result due to Morrow (1984) will be used. We include a proof since Morrow's paper is unpublished.

**Lemma 2.3.** Let $0 < q \leq 2$. If $X, X_1, \ldots, X_n$ are independent and identically distributed real random variables with $EX = 0$ and $\|X\|_{\phi_q(dP)} < \infty$, then there exists a positive constant $C_q$ such that

$$\left\| \frac{X_1 + \cdots + X_n}{\sqrt{n}} \right\|_{\phi_q(dP)} \leq C_q \|X\|_{\phi_q(dP)}.$$  \hfill (2.10)

**Proof.** Without loss of generality we can assume that $X$ is symmetric. Recall that there exists positive constants $A_q, B_q$ depending only on $q$ so that

$$A_q \sup_{m \geq 1} \|X\|_m^{1/q} \leq \|X\|_{\phi_q(dP)} \leq B_q \sup_{m \geq 1} \|X\|_m^{1/q}.$$ \hfill (2.11)

Indeed, this is easily obtained via the expansion of exponential function (see Ledoux and Talagrand, 1991, Lemma 3.7).

Thus, in order to prove the lemma we only need to show

$$\left\| \frac{X_1 + \cdots + X_n}{\sqrt{n}} \right\|_{\phi_q(dP)} \leq C_q m^{1/q} \|X\|_{\phi_q(dP)} \quad \text{for all } m \geq 1.$$ \hfill (2.12)

For this we notice that

$$E\left( \sum_{i=1}^{n} X_i \right)^{2m} \leq \sum_{r=1}^{m} \binom{n}{r} \sum_{m_{1}, \ldots, m_{r} = m} \left( \begin{array}{c} 2m \\ 2m_{1}, \ldots, 2m_{r} \end{array} \right) \times \prod_{i=1}^{r} (2m_i)^{2m_i} \left(A_q \|X\|_{\phi_q(dP)}\right)^{2m}.$$ \hfill (2.13)

Let now $c_0$ and $c_1$ be universal constants such that

$$(c_0 m)^m \leq m! \leq (c_1 m)^m, \quad m \geq 1.$$ \hfill (2.14)

Then by expanding the multinomial coefficient, the right-hand side of (2.13) does not exceed the following expression

$$f\left( \frac{1}{q} \right) := \left( \frac{c_1}{c_0} \right)^{2m} \sum_{r=1}^{m} \binom{n}{r} \sum_{m_{1}, \ldots, m_{r} = m} \exp \left( 2m \log 2m + \left( \frac{1}{q} - 1 \right) \sum_{i=1}^{r} 2m_i \log 2m_i \right) \times \left(A_q \|X\|_{\phi_q(dP)}\right)^{2m}.$$ \hfill (2.14)
Now some simple calculations yield that

\[ f\left(\frac{1}{2}\right) \leq (c')^{2m} \sum_{r=1}^{m} \sum_{m_1 + \ldots + m_r = m} \left( \sum_{i=1}^{2m} \right)^{m_i} (A_q \|X\|_{\phi_q(dP)})^{2m} \]

with \( c' = 2(c_1/c_0)^2 \).

Now differentiate \( f(1/q) \) as defined in (2.14). Obtain

\[ \frac{d}{d(1/q)} f\left(\frac{1}{q}\right) \leq (2m \log 2m) f\left(\frac{1}{q}\right) \]

or, after integration,

\[ \log f(1/q) = \int f(1/2) (2m \log 2m) \frac{d(1/q)}{f(1/2)} \leq (2m \log 2m) \left( \frac{1}{q} - \frac{1}{2} \right). \]

Thus

\[ f\left(\frac{1}{q}\right) \leq (2m)^{2m(1/q) - (1/2)} m^m (c' A_q \|X\|_{\phi_q(dP)})^{2m}. \]

Combining these inequalities we obtain (2.12), and hence conclude the proof of Lemma 2.3.

We continue our proof of Theorem 2.1 by using Lemma 2.3. In fact, we have

\[ \left\| \frac{S_n(s) - S_n(t)}{\sqrt{n}} \right\|_{\phi_q(d\mu)} \leq C_q \|X(s) - X(t)\|_{\phi_q(dP)} \leq C_q d(s, t) \]

for any \( s, t \in T \), from which we deduce, using (2.9), that for some \( t_0 \in T \)

\[ E \left\| \frac{S_n}{\sqrt{n}} \right\|_{\phi_q(d\mu)} \leq C_{q, q'} \left( E \left\| \frac{S_n(t_0)}{\sqrt{n}} \right\| \right. + \hat{d} + \int_0^1 \left( \log N(T, d, \varepsilon) \right)^{(1/q) - (1/q')} d\varepsilon \]

\[ \leq C_{q, q'} \left( E \|X(t_0)\|^2 \right)^{1/2} + \hat{d} + \int_0^1 \left( \log N(T, d, \varepsilon) \right)^{(1/q) - (1/q')} d\varepsilon \]

\[ < \infty, \quad (2.15) \]

where \( \hat{d} = \sup_{s, t \in T} d(s, t) \).

Thus, \( \lim_{M \to \infty} \sup_{n} P(\|S_n/\sqrt{n}\|_{\phi_q(d\mu)} \geq M) = 0 \).

Lemma 2.2 shows that a sequence of probability measures induced by the normalized sums \( (S_n/\sqrt{n}, n \geq 1) \) is relatively compact if there exists \( \{\pi_j, j \geq 1\} \) where \( \pi_j = (A_1, \ldots, A_{n(j)} \}) \) such that

\[ \lim_{j \to \infty} \sup_{n} P \left( \left\| \frac{S_n - U_{\pi_j}(S_n)}{\sqrt{n}} \right\|_{\phi_q(d\mu)} \geq \delta \right) = 0 \quad (2.16) \]

holds for any \( \delta > 0 \). The next step is to establish (2.16).
Using Lemma 2.3 again gives

\[
\left\| \frac{S_n - U_{n_j}(S_n)}{\sqrt{n}}(s) - \frac{S_n - U_{n_j}(S_n)}{\sqrt{n}}(t) \right\|_{\phi_q(d\mu)} \leq C_q \| (I - U_{n_j})(X(s) - X(t)) \|_{\phi_q(d\mu)}.
\] (2.17)

Define \( d_j(s, t) = C_q \| (I - U_{n_j})(X(s) - X(t)) \|_{\phi_q(d\mu)} \). Then \( d_j \) is a metric in \( T \) and can be estimated as follows: if \( s, t \in A_{i,j} \) for some \( 1 \leq i \leq n(j) \), then \( d_j(s, t) \leq d(s, t) \); if \( s \in A_{i,j} \), \( t \in A_{k,j} \) for some \( 1 \leq i, k \leq n(j) \), then

\[
d_j(s, t) = \left\| X(s) - X(t) - \left( \frac{1}{\mu(A_{i,j})} \right) \int_{A_{i,j}} X(u) \, d\mu(u) - \frac{1}{\mu(A_{k,j})} \int_{A_{k,j}} X(u) \, d\mu(u) \right\|_{\phi_q(d\mu)} \leq 2 \max \left\{ \sup_{s,t \in A_{i,j}} d(s, t), \sup_{s,t \in A_{k,j}} d(s, t) \right\}.
\] (2.18)

By the previous Remark 2.1 it is possible to choose \( \{n_j\} \) such that \( \max_{1 \leq i \leq n(j)} \sup_{s,t \in A_{i,j}} d(s, t) \) tends to zero as \( j \to \infty \).

By (2.9),

\[
E \left\| \frac{S_n - U_{n_j}(S_n)}{\sqrt{n}} \right\|_{\phi_q'(d\mu)} \leq C_{q,q'} \left( E \left\| \frac{S_n - U_{n_j}(S_n)}{\sqrt{n}}(t_0) \right\|_{\phi_q'(d\mu)} + \hat{d}_j + \int_0^1 (\log N(T,d_j,\varepsilon))^{(1/q)-(1/q')} \, d\varepsilon \right),
\] (2.19)

where \( \hat{d}_j = \sup_{s,t \in T} d_j(s, t) \).

On the other hand, \( E \| S_n - U_{n_j}(S_n)(t_0) \|^{2} = E \| (I - U_{n_j})X(t_0) \|^{2} \) tends to zero as \( j \to \infty \) and so does the integral \( \int_0^1 (\log N(T,d_j,\varepsilon))^{(1/q)-(1/q')} \, d\varepsilon \) since \( d_j \) decreases to zero. This completes the proof of (2.16).

In order to complete the proof of Theorem 2.1 it is now enough to verify that each real valued random variable \( f(X) \) satisfies the corresponding central limit theorem for every \( f \) in the dual space of \( L^\infty(\phi_q'(d\mu)) \). For properties of this dual space we refer readers to Marcus and Pisier (1981, p. 107). The arguments in the proof of Theorem 3.1 of Marcus and Pisier (1985) yield \( E\| X \|^2_{\phi_q'(d\mu)} < \infty \) which in turn establishes the required result.

3. Local times of strongly symmetric standard Markov processes

We first establish some notation (see Marcus and Rosen, 1992) and summarize some known results.

Let \( S \) be a locally compact metric space with a countable base and let \( X = (\Omega, \mathcal{F}_t, X_t, \mathbb{P}_t), t \in \mathbb{R}^+ \), be a strongly symmetric standard Markov process with state space \( S \). Such a process \( X \) has a symmetric transition density function \( p_t(x, y) \) and \( \alpha \)-potential density \( u^\alpha(x, y) = \int_0^\infty e^{-\alpha t} p_t(x, y) \, dt \). A continuous additive functional
At, \ t \geq 0 of the Markov process \ X \ is \ called \ a \ local \ time \ at \ y \ if \ for \ any \ x \neq y
\begin{align*}
P^x(R_A = 0) &= 1, \\
P^y(R_A = 0) &= 0,
\end{align*}
(3.1)
where \( R_A = \inf \{t \mid A_t > 0 \} \).

In fact the function \( t \to A_t \) is the distribution function of a measure supported on the set \( \{t \mid X_t = y\} \). It is well-known that a necessary and sufficient condition for \( X \) to have a local time process is that
\begin{align*}
u^\alpha(x, y) < \infty, \quad \forall x, y \in \mathcal{S}
\tag{3.2}
\end{align*}
for some (hence for all) \( \alpha > 0 \). Since local times are defined up to a multiplicative constant, we can choose a version of the local time at \( y \), which we will denote by \( L_t^y \), by requiring that
\begin{align*}
E^x \int_0^\infty e^{-t} \, dL_t^y = u^1(x, y), \quad \forall x \in \mathcal{S}.
\tag{3.3}
\end{align*}

On the other hand, the function \( u^\alpha(x, y) \) is positive definite on \( \mathcal{S} \times \mathcal{S} \) for each \( \alpha > 0 \). Therefore for each \( \alpha > 0 \), we can define a mean zero Gaussian process \( (G_\alpha(x), x \in \mathcal{S}) \) with covariance
\begin{align*}
E G_\alpha(x) G_\alpha(y) &= u^\alpha(x, y), \quad \forall x, y \in \mathcal{S}.
\end{align*}
The process \( X \) and \( (G_\alpha(x), x \in \mathcal{S}) \), assumed to be independent, are related through the \( \alpha \)-potential density \( u^\alpha(x, y) \). To simplify the following statements, we will only consider \( \hat{X} \) and the associated Gaussian process corresponding to \( \alpha = 1 \) and denote this Gaussian process by \( G = (G(x), x \in \mathcal{S}) \). The metric induced by \( G \) is as follows:
\begin{align*}
d(x, y) &= (E(G(x) - G(y))^2)^{1/2} = (u^1(x, y) + u^1(y, y) - 2u^1(x, y))^{1/2}.
\tag{3.4}
\end{align*}

Let \( \lambda \) be an exponential random variable, defined on a probability space \((\Omega', P')\), with mean 1 independent of \( X \). We can now obtain a stochastic process \( (L_{\lambda}^x, x \in \mathcal{S}) \) by replacing \( t \) in \( L_t^x \) by \( \lambda \). In fact, let \( A \) be the cemetery state for \( \hat{X} \) and define the killed version of \( \hat{X} \) by
\begin{align*}
\hat{X}_t &= \begin{cases} X_t & t < \lambda, \\
A & t \geq \lambda.
\end{cases}
\tag{3.5}
\end{align*}
\end{document}
K of S, then \((G(x), x \in S)\), and of course \((G(x)^2, x \in S)\), has a version with continuous sample paths. Now assume that \(\sup_{x \in S} EG(x)^2 < \infty\), then

\[
\|G(x)^2 - G(y)^2\|_{\phi(dP)} \leq c d(x, y)
\]

for some constant \(c > 0\) (possibly depending on \(\sup_x EG(x)^2\)). In addition, suppose that \((K, \mu)\) is a compact topological probability measure space, and \(\int_0 \left(\log N(K, d, \varepsilon)\right)^{1-1/q} d\varepsilon < \infty\) for some \(1 < q < \infty\). Then by Theorem 3.1 of Marcus and Pisier (1985), \((G(x)^2, x \in S)\) is a.s. in a separable Banach space \(L^0_{\phi_q}(d\mu)\).

We next state a lemma which is Theorem 4.1 from the paper of Marcus and Rosen (1992). This lemma, which we shall refer to as the Dynkin isomorphism theorem, provides a link between Gaussian processes and the local times of their associated strongly symmetric Markov processes.

**Lemma 3.1** (Dynkin isomorphism theorem). Let \(l = \{l_i\}_{i=1}^\infty\) and \(G = \{G_i\}_{i=1}^\infty\) be \(\mathbb{R}^\infty\)-valued random variables and let \(G_\alpha\) and \(G_\beta\) be real-valued random variables such that \(\{G, G_\alpha, G_\beta\}\) are jointly Gaussian with probability space \((\Omega_G, P_G)\) and expectation operator \(E_G\). Let \((\Omega, Q)\) be the underlying space for \(l\) and define \((G_i, G_j) = E_G G_i G_j\). Assume that for any \(i_1, \ldots, i_n\), not necessarily distinct, we have

\[
Q \left( \prod_{j=1}^n l_{i_j} \right) = \sum_{\pi} (G_{i_1}, G_{i_1(1)}) (G_{i_1(1)}, G_{i_1(2)}) \cdots (G_{i_1(n)}, G_{i_1(n)}),
\]

where \(\pi\) runs over all permutations of \(\{1, 2, \ldots, n\}\). Then for all \(\mathcal{C}\) measurable non-negative functions \(F\) on \(\mathbb{R}^\infty\), we have

\[
Q E_G \left( F \left( l + \frac{1}{2} G^2 \right) \right) = E_G \left( F \left( \frac{1}{2} G^2 \right) G_\alpha G_\beta \right),
\]

where \(\mathcal{C}\) denotes the \(\sigma\)-algebra generated by the cylinder sets of \(\mathbb{R}^\infty\).

The Dynkin isomorphism theorem may be used to show that the process \((L^x, x \in S)\) is a.e. in \(L^0_{\phi_q}(K, d\mu)\) whenever \((G^2(x), x \in S)\) is.

For each \(x \in S\), let \(h(y) = u^l(x, y), y \in S\). Then \(h\) is an excessive function. For every bounded measurable function \(f\) defined on \(S\), we define

\[
P_t^{(h)} f(x) = \frac{1}{h(x)} P_t (f \cdot h)(x) = \frac{1}{h(x)} E^x f(h(X_t)).
\]

It is easy to see that \(P_t^{(h)}\) is a semigroup. There exists a unique Markov process \((\Omega, \mathcal{F}_t, X_t, P^{x/h})\), called the \(h\)-transform of \(X\), with transition operators \(P_t^{(h)}\), for which

\[
P^{x/h} \left( F(\omega) I_{t < \zeta(\omega)} \right) = \frac{1}{h(x)} P^x( F(\omega) h(X_t) )
\]

where and in the following context \(\zeta\) denotes the natural life time of \(X\). In particular, for \(0 < t_1 < \cdots < t_k\) and Borel measurable sets \(B_1, \ldots, B_k\) we have

\[
E_{x, \lambda} E^{x/h} I_{t_1 \in B_1, \ldots, t_k \in B_k} I_{t_k < \zeta} = E_{x, \lambda} E^{x/h} I_{t_1 \in B_1, \ldots, t_k \in B_k} I_{t_k < \zeta} I_{t_k < \lambda}
\]
$$= e^{-u} \cdot \frac{1}{h(x)} E^{x,h} I_{(X_i \in B_1, \ldots, X_i \in B_k)} h(X_i)$$

$$= P^*(\hat{X}_1 \in B_1, \ldots, \hat{X}_k \in B_k | \hat{X}_0 = x, \hat{X}_k = x).$$

Thus $P^* \times P^{x,h}$ describes a process starting and finishing (after an exponential killing time) at $x$. In the following discussion, the local times ($L^x_i, x \in S$) will be defined on a probability space $(\Omega' \times \Omega, P^i \times P^{x,h})$.

**Lemma 3.2.** Under the above hypotheses and notations, for each $x \in S$

$$E_{x} E^{x,h} \prod_{j=1}^{n} L^y_{x_j} = \sum_{\pi} \frac{1}{\langle G(x), G(x) \rangle} \langle G(x), G(y_{\pi(1)}) \rangle \cdots \langle G(y_{\pi(n)}), G(x) \rangle$$

$$= \sum_{\pi} \frac{1}{u^i(x,x)} \langle G(x), G(y_{\pi(1)}) \rangle \cdots \langle G(y_{\pi(n)}), G(x) \rangle,$$

where $\pi$ is the set of all the permutation $(\pi(1), \ldots, \pi(n))$ of $(1, \ldots, n)$. Moreover, we have

$$E_{x} E^{x,h} E_{G^F} \left( L^y_{x} + \frac{G(y)^2}{2} \right) = E_{G^F} \left( \frac{G(y)^2}{2} \right) \frac{G(x)^2}{u^i(x,x)},$$

where $F$ is any positive functional on the space of functions from $S$ to $R^1$.

The proof of Lemma 3.2 is omitted since it is Theorem 4.1, Example 2 in Marcus and Rosen (1992).

Now we state our main results on the central limit theorem for the local time process $(L^x_i, x \in S)$. Here since $L^x_i$ does not have mean zero, we shall restrict ourselves to the study of its centered process $L^i - EL^i$.

**Theorem 3.1.** Let $X = (X_t, t \in R^1)$ be a strongly symmetric standard Markov process with state space $S$, which is a locally compact metric space with countable basis; and suppose that its 1-potential density $u^i(x,y)$ of $X$ is continuous and $\sup_{x,y \in S} u^i(x,y) < \infty$. Let $\lambda$ be an exponential random variable with mean 1 independent of $X$. Define

$$d(s,t) = (u^i(x,x) - 2u^i(x,y) + u^i(y,y))^{1/2}.$$  

Suppose that $(S, d, \mu)$ is a topological probability space and there is a positive constant $1 < q < \infty$ such that $\int_{K} (\log N(K, d, x))^q \, d\mu < \infty$ for each compact set $K \subset S$. Then $L^i - EL^i$ satisfies the central limit theorem as a random vector in $L^0_{\mu}(d\mu)$. 

**Remark 3.1.** Since $(S, d)$ is locally compact, the statement that $L^i - EL^i$ satisfies the central limit theorem in $L^0_{\mu}(d\mu)$ means that $L^i - EL^i$, when restricted to $K$, satisfies the central limit theorem in $L^0_{\mu}(d\mu/\mu(K))$ for each compact set $K$ with $\mu(K) > 0$. Similar notation has been used by other authors Adler et al. (1990). Therefore, without loss of generality, we can assume that $(S,d)$ itself is compact.
Proof. Assume \((S,d)\) is compact. We first show that \(L^x_h\) is a.s. in \(L^0_\Phi(d\mu)\). In fact, for each \(x \in S\) we have by using Lemma 3.2 for \(F = \| \cdot \|_{\Phi(d\mu)}\).

\[
E_h E^{x/h} E_G \left\| L^x_h + \frac{G(y)^2}{2} \right\|_{\Phi(d\mu)} \leq E_G \left\| \frac{G(y)^2}{2} \right\|_{\Phi(d\mu)} + \frac{G(x)^2}{2} u^1(x,x),
\]

from which we deduce

\[
E_h E^{x/h} \left\| L^x_h \right\|_{\Phi(d\mu)} \leq E_G \left\| \frac{G(y)^2}{2} \right\|_{\Phi(d\mu)} + \frac{G(x)^2}{2} u^1(x,x) + E_G \left\| \frac{G(y)^2}{2} \right\|_{\Phi(d\mu)}
\]

\[
\leq \left( E_G \left\| \frac{G(y)^2}{2} \right\|_{\Phi(d\mu)}^2 \right)^{1/2} \left( E \left( \frac{G(x)^2}{u^1(x,x)} \right)^2 \right)^{1/2}
\]

\[
+ E_G \left\| \frac{G(y)^2}{2} \right\|_{\Phi(d\mu)}. \tag{3.12}
\]

On the other hand, \(G(x)\) is a normal random variable with variance \(u^1(x,x)\), so that \(E^h G(x) = 3u^1(x,x)^2\). Moreover, when \(\int_0^1 (\log N(K,d,\varepsilon))^{1-1/q} d\varepsilon < \infty\), by Theorem 3.1 and the proof in Marcus and Pisier (1985) we have \((G^2(y), y \in S)\) is a.s. in \(L^0_\Phi(d\mu)\) and \(E\|G^2(y)\|_{\Phi(d\mu)} < \infty\). Thus (3.12) is finite.

Similarly, for each \(\tau_j\) defined as in the proof of Theorem 2.1 we have

\[
E_h E^{x/h} \left\| (I - U_{\tau_j}) L^x_h \right\|_{\Phi(d\mu)} \leq \left( E_G \left\| (I - U_{\tau_j}) \frac{G(y)^2}{2} \right\|_{\Phi(d\mu)} \right)^{1/2} \left( E \left[ \frac{G(x)^2}{u^1(x,x)} \right] \right)^{1/2}
\]

\[
+ E_G \left\| (I - U_{\tau_j}) \frac{G(y)^2}{2} \right\|_{\Phi(d\mu)}. \tag{3.13}
\]

Since the conditions of Theorem 2.1 holds for the process \((G(y)^2, y \in S)\), then

\[
\lim_{j \to \infty} E_G \left\| (I - U_{\tau_j}) \frac{G(y)^2}{2} \right\|_{\Phi(d\mu)} = 0.
\]

This together with (3.13) implies that

\[
\lim_{j \to \infty} E_h E^{x/h} \left\| (I - U_{\tau_j}) L^x_h \right\|_{\Phi(d\mu)} = 0. \tag{3.14}
\]

Thus \(L^x_h\) is also a.s. in the separable Banach space \(L^0_\Phi(d\mu)\).

Suppose that \((L^x_{\lambda}, L^x_{\lambda,n}, \ n \geq 1)\) is a sequence of independent and identically distributed random processes on \(S\) and \((\varepsilon, \varepsilon_n, \ n \geq 1)\) an independent Rademacher sequence. As in (1.5a) of Adler et al. (1990), let

\[
Z_n = \sum_{i=1}^n \varepsilon_i L^x_{\lambda,i} / \sqrt{n}. \tag{3.15}
\]

In order to prove that \(L^x_h - EL^x_h\) satisfies the central limit theorem we first verify that \(Z_n\) is relatively compact. For this we show as in the proof of Theorem 2.1 that

\[
\sup_n E_h E^{x/h} \left\| Z_n \right\|_{\Phi(d\mu)} < \infty, \tag{3.16}
\]
for each $x \in S$. These two statements can be proved by an argument similar to the proof of Theorem 1.2 of Adler et al. (1990).

In fact, it is easily seen by using repeatedly Lemma 3.2 that

\[
E_t E^{x/h} \| Z_n \|_{\phi_t(d \mu)} \leq E_t E^{x/h} E_t E_G \left\| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \varepsilon_i \left( L_{x,i}^r + \frac{G_i^2(y)}{2} \right) \right\|_{\phi_t(d \mu)} + E_t E_G \left\| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \varepsilon_i \frac{G_i^2(y)}{2} \right\|_{\phi_t(d \mu)}
\]

\[
= E_t E_G \left\| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \varepsilon_i \frac{G_i^2(y)}{2} \right\|_{\phi_t(d \mu)} \prod_{i=1}^{n} \frac{G_i^2(x)}{u^l(x,x)} + E_G \left\| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \varepsilon_i \frac{G_i^2(y)}{2} \right\|_{\phi_t(d \mu)},
\]

and

\[
E_t E^{x/h} \left\| \frac{1}{\sqrt{n}} (I - U_{x,j}) \left( \sum_{i=1}^{n} \varepsilon_i L_{x,i}^r \right) \right\|_{\phi_t(d \mu)} \leq E_t E_G \left\| \frac{1}{\sqrt{n}} (I - U_{x,j}) \left( \sum_{i=1}^{n} \varepsilon_i \frac{G_i^2(y)}{2} \right) \right\|_{\phi_t(d \mu)} \prod_{i=1}^{n} \frac{G_i^2(x)}{u^l(x,x)} + E_t E_G \left\| \frac{1}{\sqrt{n}} (I - U_{x,j}) \left( \sum_{i=1}^{n} \varepsilon_i \frac{G_i^2(y)}{2} \right) \right\|_{\phi_t(d \mu)}.
\]

Since by the proof of Theorem 2.1 we have

\[
\sup_n E_t E_G \left\| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \varepsilon_i \frac{G_i^2(y)}{2} \right\|_{\phi_t(d \mu)} < \infty \tag{3.18}
\]

and

\[
\lim_{j \to \infty} E_t E_G \left\| \frac{1}{\sqrt{n}} (I - U_{x,j}) \left( \sum_{i=1}^{n} \varepsilon_i \frac{G_i^2(y)}{2} \right) \right\|_{\phi_t(d \mu)} = 0, \tag{3.19}
\]

we need only show that

\[
\sup_n E_t E_G \left\| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \varepsilon_i \frac{G_i^2(y)}{2} \right\|_{\phi_t(d \mu)} \prod_{i=1}^{n} \frac{G_i^2(x)}{u^l(x,x)} < \infty \tag{3.20}
\]
Let \( \tilde{G}(y) = G(y) - \gamma(y)G(x) \) where \( \gamma(y) = EG(x)G(y)/u^1(x,x) \). Thus \( \tilde{G}(y) \) is a Gaussian process independent of \( G(x) \). Next we estimate \( \| \tilde{G}(y)^2 - \tilde{G}(z)^2 \|_{\phi_1(dP)} \). Recall that for a mean zero Gaussian variable \( \xi \), \( \| \xi \|_m \leq cm^{1/2} \| \xi \|_2 \), \( m \geq 1 \), where \( c \) is an absolute constant (see Ledoux and Talagrand, 1991, p. 60). Thus for every integer \( m \geq 1 \),

\[
\| \tilde{G}(y)^2 - \tilde{G}(z)^2 \|_m = (E|\tilde{G}(y) - \tilde{G}(z)|^m|\tilde{G}(y) + \tilde{G}(z)|^m)^{1/m} \\
\leq (\|G(y) - G(z)\|_{2m} + |\gamma(y) - \gamma(z)|\|G(x)\|_{2m}) \\
\quad \cdot (\|\tilde{G}(y)\|_{2m} + \|\tilde{G}(z)\|_{2m}) \\
\leq cm \, d(y,z),
\]

where \( c \) is a positive constant possibly depending on \( \sup_x u^1(x,x) \). Hence we have

\[
\| \tilde{G}(y)^2 - \tilde{G}(z)^2 \|_{\phi_1(dP)} \leq cd(y,z)
\]

using the expansion formula for the exponential function. From the proof of Theorem 2.1,

\[
\sup_n E_n \tilde{G} \left\| \frac{1}{\sqrt{n}} \sum_{i=1}^n \xi_i \tilde{G}^2(y) \right\|_{\phi_1(dP)} < \infty
\]

and

\[
\lim_{j \to \infty} E_n \tilde{G} \left\| \frac{1}{\sqrt{n}} (I - U_{n_j}) \left( \sum_{i=1}^n \xi_i \tilde{G}^2(y) \right) \right\|_{\phi_1(dP)} = 0.
\]

Hence,

\[
\sup_n E_n \tilde{G} \left\| \frac{1}{\sqrt{n}} \sum_{i=1}^n \xi_i \tilde{G}^2(y) \right\|_{\phi_1(dP)} E \prod_{i=1}^n \frac{G^2(x)}{u^1(x,x)}
\]

\[
= \sup_n E_n \tilde{G} \left\| \frac{1}{\sqrt{n}} \sum_{i=1}^n \xi_i \tilde{G}^2(y) \right\|_{\phi_1(dP)} E \prod_{i=1}^n \frac{G^2(x)}{u^1(x,x)}
\]

\[
< \infty
\]

and

\[
\lim_{j \to \infty} E_n \tilde{G} \left\| \frac{1}{\sqrt{n}} (I - U_{n_j}) \left( \sum_{i=1}^n \xi_i \tilde{G}^2(y) \right) \right\|_{\phi_1(dP)} E \prod_{i=1}^n \frac{G^2(x)}{u^1(x,x)}
\]

\[
= \lim_{j \to \infty} E_n \tilde{G} \left\| \frac{1}{\sqrt{n}} (I - U_{n_j}) \left( \sum_{i=1}^n \xi_i \tilde{G}^2(y) \right) \right\|_{\phi_1(dP)} E \prod_{i=1}^n \frac{G^2(x)}{u^1(x,x)}
\]

\[
= 0.
\]
Similarly,

\[ \|\gamma(y)\tilde{G}(y) - \gamma(z)\tilde{G}(z)\|_{\phi_i(d\mu)} \leq |\gamma(y) - \gamma(z)| \|\tilde{G}(y)\|_{\phi_i(d\mu)} + |\gamma(z)| \|\tilde{G}(y) - \tilde{G}(z)\|_{\phi_i(d\mu)} \]

\[ \leq cd(y, z), \quad (3.23) \]

where \( c \) is a positive constant possibly depending on \( u^l(x, x) \) and \( \sup_x u^l(x, x) \). Again, using the proof of Theorem 2.1,

\[ E_{\tilde{G}}|\gamma(y)\tilde{G}(y)|_{\phi_i(d\mu)} < \infty \quad (3.24) \]

and

\[ \lim_{j \to \infty} E_{\tilde{G}}\|(I - U_{\tilde{Y}})\gamma(y)\tilde{G}(y)\|_{\phi_i(d\mu)} = 0. \quad (3.25) \]

Hence it easily follows that

\[ \sup_n E_c E_G E_{\tilde{G}} \left\| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \varepsilon_i G_i(x)\gamma(y)\tilde{G}(y) \right\|_{\phi_i(d\mu)} \prod_{i=1}^{n} \frac{G_i^2(x)}{u^l(x, x)} \]

\[ = \sup_n E_c E_G E_{\tilde{G}} \left\| \left( \frac{1}{n} \sum_{i=1}^{n} G_i^2(x) \right)^{1/2} \gamma(y)\tilde{G}(y) \right\|_{\phi_i(d\mu)} \prod_{i=1}^{n} \frac{G_i^2(x)}{u^l(x, x)} \]

\[ = \sup_n E_G \left( \frac{1}{n} \sum_{i=1}^{n} G_i^2(x) \right)^{1/2} \prod_{i=1}^{n} \frac{G_i^2(x)}{u^l(x, x)} E_{\tilde{G}}|\gamma(y)\tilde{G}(y)|_{\phi_i(d\mu)} \]

\[ \leq \sup_n \left( E_G \frac{1}{n} \sum_{i=1}^{n} G_i^2(x) \prod_{i=1}^{n} \frac{G_i^2(x)}{u^l(x, x)} \right)^{1/2} E_{\tilde{G}}|\gamma(y)\tilde{G}(y)|_{\phi_i(d\mu)} < \infty \quad (3.26) \]

and

\[ \lim_{j \to \infty} \sup_n F_e F_G F_{\tilde{G}} \left\| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \varepsilon_i G_i(x)(I - U_{\tilde{Y}})\gamma(y)\tilde{G}(y) \right\|_{\phi_i(d\mu)} \prod_{i=1}^{n} \frac{G_i^2(x)}{u^l(x, x)} \]

\[ = \lim_{j \to \infty} \sup_n E_G E_{\tilde{G}} \left\| \left( \frac{1}{n} \sum_{i=1}^{n} G_i^2(x) \right)^{1/2} (I - U_{\tilde{Y}})\gamma(y)\tilde{G}(y) \right\|_{\phi_i(d\mu)} \prod_{i=1}^{n} \frac{G_i^2(x)}{u^l(x, x)} \]

\[ = \lim_{j \to \infty} \sup_n E_G \left( \frac{1}{n} \sum_{i=1}^{n} G_i^2(x) \right)^{1/2} \prod_{i=1}^{n} \frac{G_i^2(x)}{u^l(x, x)} E_{\tilde{G}}|(I - U_{\tilde{Y}})\gamma(y)\tilde{G}(y)|_{\phi_i(d\mu)} \]

\[ \leq \sup_n \left( E_G \frac{1}{n} \sum_{i=1}^{n} G_i^2(x) \prod_{i=1}^{n} \frac{G_i^2(x)}{u^l(x, x)} \right)^{1/2} \lim_{j \to \infty} E_{\tilde{G}}|(I - U_{\tilde{Y}})\gamma(y)\tilde{G}(y)|_{\phi_i(d\mu)} \]

\[ = 0, \quad (3.27) \]
where we need only note that \( \{(I - U_{n_j})\gamma(y)\tilde{G}(y), y \in S\} \) is still a Gaussian process in the first equality.

Using these results gives

\[
\sup_n E_{\epsilon} E_G \left\| \frac{1}{\sqrt{n}} \sum_{i=1}^n \varepsilon_i \frac{G_i^2(y)}{2} \right\|_{\phi_{d}(d_{\mu})} \left( \prod_{i=1}^n \frac{G_i^2(x)}{u^1(x,x)} \right) \\
\leq \sup_n E_{\epsilon} E_G E_{\tilde{G}} \left\| \frac{1}{\sqrt{n}} \sum_{i=1}^n \varepsilon_i G_i^2(y) \right\|_{\phi_{d}(d_{\mu})} \left( \prod_{i=1}^n \frac{G_i^2(x)}{u^1(x,x)} \right) \\
+ 2 \sup_n E_{\epsilon} E_G E_{\tilde{G}} \left\| \frac{1}{\sqrt{n}} \sum_{i=1}^n \varepsilon_i G_i(x)\gamma(y)\tilde{G}(y) \right\|_{\phi_{d}(d_{\mu})} \left( \prod_{i=1}^n \frac{G_i^2(x)}{u^1(x,x)} \right) \\
+ \sup_n E_{\epsilon} E_G \left\| \frac{1}{\sqrt{n}} \sum_{i=1}^n \varepsilon_i G_i^2(x)\gamma^2(y) \right\|_{\phi_{d}(d_{\mu})} \left( \prod_{i=1}^n \frac{G_i^2(x)}{u^1(x,x)} \right) \\
< \infty \tag{3.28}
\]

since the third term on the right-hand side of the first inequality can be estimated as follows

\[
\sup_n E_{\epsilon} E_G \left\| \frac{1}{\sqrt{n}} \sum_{i=1}^n \varepsilon_i G_i^2(x)\gamma^2(y) \right\|_{\phi_{d}(d_{\mu})} \left( \prod_{i=1}^n \frac{G_i^2(x)}{u^1(x,x)} \right) \\
\leq \|\gamma^2(y)\|_{\phi_{d}(d_{\mu})} \sup_n E_{\epsilon} E_G \left\| \frac{1}{\sqrt{n}} \sum_{i=1}^n \varepsilon_i G_i^2(x) \right\|_{\phi_{d}(d_{\mu})} \left( \prod_{i=1}^n \frac{G_i^2(x)}{u^1(x,x)} \right) \\
\leq \|\gamma^2(y)\|_{\phi_{d}(d_{\mu})} \sup_n E_{\tilde{G}} \left\| \frac{1}{n} \sum_{i=1}^n G_i^4(x) \right\|^{1/2} \left( \prod_{i=1}^n \frac{G_i^2(x)}{u^1(x,x)} \right) \\
\leq \|\gamma^2(y)\|_{\phi_{d}(d_{\mu})} \sup_n \left( E_{\tilde{G}} \left\| \frac{1}{n} \sum_{i=1}^n G_i^4(x) \right\|^{1/2} \left( \prod_{i=1}^n \frac{G_i^2(x)}{u^1(x,x)} \right) \right)^{1/2} \\
< \infty; \tag{3.29}
\]

and

\[
\limsup_{j \to \infty} n E_{\epsilon} E_G \left\| \frac{1}{\sqrt{n}} (I - U_{n_j}) \left( \sum_{i=1}^n \varepsilon_i G_i^2(y) \right) \right\|_{\phi_{d}(d_{\mu})} \left( \prod_{i=1}^n \frac{G_i^2(x)}{u^1(x,x)} \right) \\
\leq \limsup_{j \to \infty} n E_{\epsilon} E_G E_{\tilde{G}} \left\| \frac{1}{\sqrt{n}} (I - U_{n_j}) \left( \sum_{i=1}^n \varepsilon_i G_i^2(y) \right) \right\|_{\phi_{d}(d_{\mu})} \left( \prod_{i=1}^n \frac{G_i^2(x)}{u^1(x,x)} \right) \\
+ 2 \limsup_{j \to \infty} n E_{\epsilon} E_G E_{\tilde{G}} \left\| \frac{1}{\sqrt{n}} (I - U_{n_j}) \left( \sum_{i=1}^n \varepsilon_i G_i(x)\gamma(y)\tilde{G}(y) \right) \right\|_{\phi_{d}(d_{\mu})} \\
\times \left( \prod_{i=1}^n \frac{G_i^2(x)}{u^1(x,x)} \right) + \limsup_{n} \sup E_{\epsilon} E_G \left\| \frac{1}{\sqrt{n}} (I - U_{n_j}) \left( \sum_{i=1}^n \varepsilon_i G_i^2(y)\gamma^2(y) \right) \right\|_{\phi_{d}(d_{\mu})} \\
\times \left( \prod_{i=1}^n \frac{G_i^2(x)}{u^1(x,x)} \right)
\]
since the last term on the right-hand side of this inequality can also be estimated similarly as follows:

\[
\lim_{j \to \infty} \sup_n E_\phi(L_{j}) \left\| \frac{1}{\sqrt{n}} (I - U_{t_j}) \left( \sum_{i=1}^{n} e_{i} G_{i}(x) \gamma_{i}(y) \right) \right\|_{\psi_{\phi}(d\mu)} \left( \prod_{i=1}^{n} \frac{G_{i}(x)}{u(x,x)} \right)^{1/2} = 0.
\]

To complete our proof, it is now enough to show that the real valued random variable \( \psi f(L_{j}) \) satisfies the classical central limit theorem for each \( f \) in the dual space of \( L_{\psi}^{0}(d\mu) \). In fact, for any \( f \) in the dual space of \( L_{\psi}^{0}(d\mu) \) there exists a corresponding \( u \) in \( L_{\psi}^{0}(d\mu) \) such that

\[
f(x) = \int_{S} x(y) u(y) d\mu(y),
\]

where \( \psi_{\phi} \) is the conjugate function of \( \phi_{\psi} \). Thus for any \( x \in S \) we have

\[
F_{\lambda} E^{x/h} f(L_{\lambda})^{2} = F_{\lambda} E^{x/h} \left( \int_{S} L_{\lambda}^{y} u(y) d\mu(y) \right)^{2} = \int_{S} \int_{S} E_{\lambda} E^{x/h} L_{\lambda}^{y} L_{\lambda}^{z} u(y) u(z) d\mu(y) d\mu(z).
\]

Since \( \sup_{y,z \in S} E_{\lambda} E^{x/h} L_{\lambda}^{y} L_{\lambda}^{z} = 2u^{1}(x,y) u^{1}(x,z) u^{1}(y,z) / u^{1}(x,x) < \infty \), the limit distribution of the normalized sums \( 1/\sqrt{n} \sum_{i=1}^{n} \epsilon_{i} f(L_{\lambda}) \) exists and is uniquely determined by

\[
E_{\lambda} E^{x/h} L_{\lambda}^{y} L_{\lambda}^{z} \text{ for any } y,z \in S.
\]

4. Random Fourier series

Let \( G \) be a compact Abel group, \( \{r_{n}, n \geq 1\} \) its character group. Suppose that \( \{a_{n}, n \geq 1\} \) is a sequence of complex numbers, \( \{\xi_{n}, n \geq 1\} \) a sequence of independent real valued symmetric random variables and \( \{\theta_{n}, n \geq 1\} \) standard symmetric \( p \)-stable \( (1 < p \leq 2) \) real valued random variables, i.e., \( E \exp(it\theta) = \exp(-|t|^{p}) \), \( t \in \mathbb{R} \). If \( p = 2 \), \( \theta \) is a normal variable.

Consider the random Fourier series

\[
X(t) = \sum_{n=1}^{\infty} a_{n} \xi_{n} r_{n}(t), \quad Y(t) = \sum_{n=1}^{\infty} a_{n} \theta_{n} r_{n}(t).
\]
If \( \sum_{n=1}^{\infty} |a_n|^p < \infty \), then define a translation invariant metric \( d(s, t) \) as follows:

\[
d(s, t) = \left( \sum_{n=1}^{\infty} |a_n|^p |r_n(s) - r_n(t)|^p \right)^{1/p}
\]

for \( s, t \in G \).

**Theorem 4.1.** Suppose that \( \mu \) is a probability measure on \( G \) and \( \int_{0}^{\log N(G, d, c)} (1/q) - (1/q') \, dc < \infty \), where \( q \) is conjugate of \( p \) and \( q < q' < \infty \).

(i) \((1 < p < 2)\) If \( \sup_n P(|\xi_n| > c) < c^{-p} \) for any \( c > 0 \) and each \( \xi_n \) belongs to the normal attraction domain of \( \theta \), then \( X \) itself belongs to the normal attraction domain of \( Y \) in \( L^{q'}_{\phi'}(d\mu) \).

(ii) \((p = 2)\) If \( E\xi_n^2 = 1 \) and \( \inf_n E|\xi_n| > 0 \), then \( X \) satisfies the central limit theorem in \( L^{q'}_{\phi'}(d\mu) \).

The following lemma will be used.

**Lemma 4.1.** Under the hypotheses of Theorem 4.1 there exists a positive constant \( C_{q, q'} \) such that

\[
E\|X\|_{\phi', (d\mu)} \leq C_{q, q'} \left( \left( \sum_{n=1}^{\infty} |a_n|^p \right)^{1/p} + \int_{0}^{\log N(G, d, c)} (1/q) - (1/q') \, dc \right).
\]

**Proof.** Let \( \{\varepsilon_n, n \geq 1\} \) be a Rademacher sequence independent of \( \{\xi_n, n \geq 1\} \). We may assume that each \( \varepsilon_n \) is defined on probability space \( (\Omega_1, \mathcal{F}_1, P_1) \) and \( \xi_n \) on \( (\Omega_2, \mathcal{F}_2, P_2) \). Define

\[
\tilde{X}(t, \omega_1, \omega_2) = \sum_{n=1}^{\infty} a_n \varepsilon_n(\omega_1) \xi_n(\omega_2) r_n(t)
\]

for \( t \in G_1, \omega_1 \in \Omega_1, \omega_2 \in \Omega_2 \). Then \( X \) is equal to \( \tilde{X} \) in distribution. We deduce from Lemma 2.1 and Theorem 3.1 in Marcus and Pisier (1985) that

\[
E_{\omega_1} \|\tilde{X}(t, s, \omega_2)\|_{\phi', (d\mu)} \leq C_{q, q'} (E_{\omega_1} |X(0, s, \omega_2)|) + \sup_{s, t \in G} \delta_{\omega_2}(s, t)
\]

\[
\quad + \int_{0}^{\log N(G, \delta_{\omega_2}, c)} (1/q) - (1/q') \, dc
\]

for almost all \( \omega_2 \in \Omega_2 \), where \( \delta_{\omega_2}(s, t) \) is defined by

\[
\delta_{\omega_2}(s, t) = \|\{a_n \xi_n(\omega_2)(r_n(s) - r_n(t))\|_{p, \infty}
\]

for \( 1 < p < 2 \); and

\[
\delta_{\omega_2}(s, t) = \left( \sum_{n=1}^{\infty} |a_n|^2 |\xi_n(\omega_2)|^2 |r_n(s) - r_n(t)|^2 \right)^{1/2}
\]

for \( p = 2 \).
Since $E_{w_2} \delta_{w_2}(s, t) \leq C_q d(s, t)$ and $E_{w_2} \sup_{s, t \in G} \delta_{w_2}(s, t) \leq C_q \left(\sum_{n=1}^{\infty} |a_n|^p\right)^{1/p}$ for $1 < p \leq 2$, then Lemma 4.1 is completed by using Lemma 3.11 from Marcus and Pisier (1984a).

**Proof of Theorem 4.1.** We only prove (i), since (ii) is similar and easier. Suppose that $(\xi_n, n \geq 1)$, $i = 1, 2, \ldots$, are sequences of independent random variables with the same distribution as $(\xi_n, n \geq 1)$ and independent of $(\xi_n, n \geq 1)$. Let

$$X_i^N(t) = \sum_{n > N} a_n \xi_n r_n(t), \quad Z_m = \frac{1}{m^{1/p}} \sum_{i=1}^{m} X_i^N.$$  \hfill (4.8)

We need the following elementary inequality which has been used by Juknevichiene (1986, p. 285)

$$\sup_{c > 0} c^p \mathbb{P}\left(\left|\frac{1}{m^{1/p}} \sum_{i=1}^{m} \xi_i\right| > c\right) \leq C_p \sum_{i=1}^{m} \sup_{c > 0} c^p \mathbb{P}(|\xi_i| > cm^{1/p}) \leq C_p.$$  \hfill (4.9)

Without loss of generality, we can assume $C_p < 1$, since a change of scale will ensure this. Thus by Lemma 4.1 and (4.9) we have

$$\sup_{m \geq 1} E \left\|Z_m\right\|_{\phi_q'(d\mu)} \leq C_p \left(\sum_{n > N} |a_n|^p\right)^{1/p} + \int_0^\infty \left(\log N(G, d_N, \varepsilon)\right)^{(1/q)-(1/q')} d\varepsilon,$$  \hfill (4.10)

where $d_N(s, t) = \left(\sum_{n > N} |a_n|^p |r_n(s) - r_n(t)|^p\right)^{1/p}$ for any $s, t \in G$.

Since $\sum_{n=1}^{\infty} |a_n|^p < \infty$ and $\int_0^\infty \left(\log N(G, d_N, \varepsilon)\right)^{(1/q)-(1/q')} d\varepsilon < \infty$, it follows that

$$\lim_{N \to \infty} \int_0^\infty \left(\log N(G, d_N, \varepsilon)\right)^{(1/q)-(1/q')} d\varepsilon = 0$$

by the dominated convergence theorem, and hence

$$\lim_{N \to \infty} \sup_{m \geq 1} E \left\|Z_m\right\|_{\phi_q'(d\mu)} = 0.$$

Similarly,

$$\lim_{N \to \infty} E \left\| \sum_{n > N} a_n \theta_n r_n \right\|_{\phi_q'(d\mu)} = 0.$$

Thus, for any $c > 0$ and bounded Lipschitz function $f$ on $L^0_{\phi_q'}(d\mu)$ there exists a positive integer $N \geq 1$ such that

$$\sup_{m \geq 1} E \left\|Z_m\right\|_{\phi_q'(d\mu)} \leq \frac{\varepsilon}{3 \|f\|_{\text{lip}}}, \quad E \left\| \sum_{n > N} a_n \theta_n r_n \right\|_{\phi_q'(d\mu)} \leq \frac{\varepsilon}{3 \|f\|_{\text{lip}}}.$$
In addition, since each $\xi_n$ belongs to the domain of normal attraction of $\theta$, then for fixed $N$ there is a positive integer $m \geq 1$ such that

$$|Ef \left( \frac{1}{m^{1/p}} \sum_{i=1}^{m} \sum_{n=1}^{N} a_n^{\xi_n \xi_i} r_n \right) - Ef \left( \sum_{n=1}^{N} a_n \theta_n r_n \right) | \leq \frac{\varepsilon}{3}.$$  

Hence, for any $\varepsilon > 0$ there exists an $m \geq 1$ such that

$$|Ef \left( \frac{1}{m^{1/p}} \sum_{i=1}^{m} \sum_{n=1}^{\infty} a_n^{\xi_n \xi_i} r_n \right) - Ef \left( \sum_{n=1}^{\infty} a_n \theta_n r_n \right) |$$

$$\leq \left| Ef \left( \frac{1}{m^{1/p}} \sum_{i=1}^{m} \sum_{n=1}^{N} a_n^{\xi_n \xi_i} r_n \right) - Ef \left( \sum_{n=1}^{N} a_n \theta_n r_n \right) \right|$$

$$+ \|f\|_{\text{lip}} \left( E\|Z_m\|_{\phi'}(d\mu) + E \left\| \sum_{n>N} a_n \theta_n r_n \right\|_{\phi'}(d\mu) \right)$$

$$\leq \varepsilon.$$  

(4.11)

The proof is completed.

**Remark 4.1.** Suppose that $1 < p < 2$ and $q' = \infty$. From Theorem C of Marcus and Pisier (1984a) we may conclude that if $X$ belongs to the domain of the normal attraction of a $p$-stable random process $Y$ in $C(G)$, then $\int_0^1 (\log N(G,d,\varepsilon))^{1/q} \, d\varepsilon < \infty$ for $q = p/(p - 1)$. However, the corresponding necessary condition does not hold if $q' < \infty$.

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