Asymptotic analysis of random partitions

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Dedicated to Zhengyan Lin on his sixty fifth birthday

Abstract

In this paper we aim to review some works on the asymptotic behaviors of random partitions. The focus is upon two basic probability models—uniform partitions and Plancherel partitions. One of fundamental results is the existence of limit shapes, which corresponds to the classic law of large numbers. The fluctuations around the limit shape are also considered; the well-known Gumbel distribution, Tracy-Widom distribution and normal distribution will be used to describe the asymptotic fluctuations at the edge and in the bulk of the spectrum of random partitions. The technique used in asymptotic analysis is the so-called poissonization and depoissonization method, a peculiar conditioning argument. We are content with describing some basic facts and remarkable results; no complete proof is given.

1 Introduction

The theory of partitions is one of the very few branches of mathematics that can be appreciated by anyone who is endowed with little more than a lively interest in the subject. Its applications are found wherever discrete objects are to be counted or classified, whether in the molecular and the atomic studies of matter, in the theory of numbers, or in combinatorial problems from all sources.

The theory of partitions has an interesting history. Certain special problems in partitions certainly date back to the Middle Ages; however, the first discovery of any depth was made in the eighteenth century when Euler proved many beautiful and significant partition theorems. Euler indeed laid the foundations of the theory of partitions. Many of

*The author is partly supported by NSF of China (Grant No. 10371109, No.10671176) and the Royal Society K.C.Wong Education Foundation International Incoming Fellowship (2006).
the other great mathematician- Cayley, Gauss, Hardy, Jacobi, Lagrange, Legendre, Littlewood, Rademacher, Ramanujan, Schur, and Sylvester- have contributed to the development of the theory. Andrews [2] is a first thorough survey of this field, which contains many wonderful results and informative historic notes.

Random partitions occur in mathematics and physics in a wide variety of contexts. A partition can record a state of some random growth process. More often it happens that a certain quantity of interest is expressed, explicitly or implicitly, as a sum over partitions. A simple but fruitful perspective is to treat sums over partitions probabilistically, that is, treat them as expectations of some functions of a random partition. See Okounkov [31], Vershik [46] for more.

In the past decades there have been an intensive activity around the theory of distribution and applications of random partitions. The wealth of applications and connections to many areas of different mathematics is such that it is utterly impossible to paint a whole picture in one short paper. Instead, I will give a few illustrative examples, selected according to my own limited expertise and taste. The focus is basically on the probability limit theorems for uniform and Plancherel random partitions, say, limit shapes and fluctuations. This is actually a variant of the classic laws of large numbers and central limit theorems in probability texts. The paper is organized as follows.

The rest of the Introduction introduces some necessary notations, for instance, partition, Young diagram, standard Young tableau. We also give the hook formula and the RSK correspondence.

Uniform and Plancherel measures are two of the most natural and important probability measures on the space of partitions. Their treatment involve apparently different techniques, but is in essence based on a common idea: conditioning device or poissonization and depoissonization method. The limiting distributions involved are Gumbel distribution, Tracy-Widom distribution and Gauss law, three of the most commonly used in random phenomena. The limit shapes of random Young diagrams under uniform and Plancherel measures are described by two distinct curve equations, one of which has infinite length while the other is finite. These two measures and their properties shall be discussed in Section 2 and 3 respectively. For clarity and simplicity, many complicated and technical analysis are not given in the contexts. The interested reader is referred to the original papers in References for more details.

Besides uniform and Plancherel measures, there are many other types of measures on partitions, which are either well studied or worthwhile further work. For sake of survey, we mention in Section 4 three interesting measures: multiplicative measures, Schur measure and 3-dimensional partitions. They are more general, possibly more difficult, than uniform
and Plancherel measures. In Section 4 we also remark that it is very interesting and challenging to use the Stein-Chen method to understand the Tracy-Widom distribution.

Acknowledgement: This paper is dedicated to my teacher, Professor Zhengyan Lin, for his teaching me probability limit theory. I am indebted to him for encouragement and support over the years. This paper was completed when I was visiting the University of Leeds with the KC Wong Education Fellowship. Many thanks also go to Dr. Bogachev for numerous conversation during the visit.

1.1 Partitions

A partition of a positive integer \( n \) is a finite non-increasing sequence of positive integers \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > 0 \) such that \( \sum_{i=1}^{l} \lambda_i = n \). The \( \lambda_i \) are called the parts of the partition, the number \( l \) of parts is the length of \( \lambda \).

Many times the partition \( (\lambda_1, \lambda_2, \cdots, \lambda_l) \) will be denoted by \( \lambda \), and we shall write \( \lambda \vdash n \) to denote "\( \lambda \) is a partition of \( n \)". Sometimes it is convenient to use a notation which indicates the number of times each integer occurs as a part: \( \lambda = (1^{r_1}, 2^{r_2}, \cdots, i^{r_i}, \cdots) \) where exactly \( r_i \) of the parts of \( \lambda \) are equal to \( i \). The number \( r_i = r_i(\lambda) = |\{j : \lambda_j = i\}| \) is called the multiplicity of \( i \) in \( \lambda \). Note \( \sum_{i \geq 1} ir_i = n \) and \( \sum_{i \geq 1} r_i = l \).

The set of all partitions of \( n \) is denoted by \( \mathcal{P}_n \), and the set of all partitions by \( \mathcal{P} \), i.e., \( \mathcal{P} = \cup_{n=0}^{\infty} \mathcal{P}_n \). Here by convention the empty sequence forms the only partition of zero. Among the most important and fundamental is the question of enumerating various set of partitions. Let \( p(n) \) be the partition function, i.e., the number of partitions of \( n \). Trivially, \( p(0) = 0 \); \( p(n) \) increases quite rapidly with \( n \). For example, \( p(10) = 42 \), \( p(20) = 627 \), \( p(50) = 204226 \), \( p(100) = 190569292 \), \( p(200) = 397299029388 \).

One of the most elemental tools for treating partitions is the infinite product generating functions. Euler started the analytic theory of partitions by providing the explicit generating functions

\[
F(q) := \sum_{n=0}^{\infty} p(n)q^n = \prod_{k=1}^{\infty} \frac{1}{1-q^k}
\]

and a good deal more (see Ch.1 of Andrews [2]). We remark that on the one hand, for many problems it suffices to consider \( F(q) \) as a formal power series in \( q \); on the other hand, much asymptotic work requires that the generating functions be analytic functions of the complex variable \( q \). In actual fact, both approaches have their special merits.

The asymptotic theory starts 150 years after Euler, with the first letters of Ramanujan to Hardy in 1913. In a celebrated series of memoirs
published in 1917 and 1918, Hardy and Ramanujan found (and was perfected by Rademacher) very precise estimates for the partition number \( p(n) \). In particular,

\[
p(n) = \frac{1}{4n\sqrt{3}} e^{\pi \sqrt{2/3} n} (1 + O\left(\frac{1}{\sqrt{n}}\right)).
\]

(1.2)

See Ch.5 of Andrews [2] for more historic notes.

1.2 Young diagrams

Another effective elementary device for studying partitions is the graphical representation. To each partition \( \lambda \) is associated its Young diagram (shape), which can be formally defined as the set of points \((i, j) \in \mathbb{Z}^2\) such that \(1 \leq j \leq \lambda_i\). In drawing such diagrams, by convention, the first coordinate \(i\) (the row index) increases as one goes downwards, and the second coordinate \(j\) (the column index) increases as one goes from the left to the right and these points are left justified. More often it is convenient to replace the nodes by unit squares (see Figure 1.1–left).

![Young diagrams](image)

Figure 1.1: Young diagrams

Such a representation is extremely useful when we consider applications of partitions to plane partitions or Young tableaux. Some authors (particularly in French) prefer the representation to be upside down in consistency with Descartes coordinate geometry (see Figure 1.1–right; see also MacDonald [29]); while other authors (for instance, the Russian school) like to flip the diagram and rotate it 135° for effective parameterizations (see Figure 1.2, see also Okounkov [31]).

We remark that these competing traditions of drawing a diagram are in essence equivalent, but have their own advantages in the study of distinct aspects of partitions. The diagram of a partition \( \lambda \) is still denoted by the same symbol \( \lambda \).
The conjugate of a partition $\lambda$ is the partition $\lambda'$ whose diagram is the transpose of the diagram $\lambda$, i.e., the diagram obtained by reflection in the main diagonal. Hence the $\lambda_i'$ is the number of squares in the $i$th column of $\lambda$, or equivalently,

$$\lambda_i' = \sum_{k=i}^{\infty} r_k.$$  \hspace{1cm} (1.3)

In particular, $\lambda'_1 = l(\lambda)$ and $\lambda_1 = l(\lambda')$. Obviously, $\lambda'' = \lambda$.

The purpose of writing a Young diagram instead of just the partition, of course, is to put something in the squares. A standard Young tableau $T$ with the shape $\lambda \vdash n$ is a one-to-one assignment of the numbers $1, 2, \cdots, n$ to the squares of $\lambda$ in such a way that the numbers increase along the rows and down the columns. Let $d_\lambda$ denote the total number of standard Young tableaux associated with a given shape $\lambda$. There are some remarkable closed formulas for $d_\lambda$ in terms of "hook length". For a Young diagram $\lambda$, each square determines a hook, which consists of that square and all squares in its row to the right of the square or in its column below the square. The hook length of a square is the number of squares in its hook; denote by $h(i, j)$ the hook length of the square in the $i$-th row and $j$-th column. Then $h(i, j) = \lambda_i - j + \lambda'_j - i + 1$. Frame, Robinson and Thrall [15] discovered a so-called hook formula:

$$d_\lambda = \frac{n!}{H_\lambda},$$ \hspace{1cm} (1.4)

where $H_\lambda$ is the product of hook lengths of $\lambda$, i.e., $H_\lambda = \prod_{(i,j) \in \lambda} h(i, j)$.

Greene, Nijenhuis and Wilf have given a short and elegant probabilistic interpretation of (1.4). An equivalent form of the hook formula is due to Frobenius (see p.10, MacDonald [29])

$$d_\lambda = n! \prod_{1 \leq i < j \leq l} (\lambda_i - \lambda_j + j - i) / \prod_{i=1}^{l} (\lambda_i + l - i)!.$$ \hspace{1cm} (1.5)
There is an intimate connection between $P_n$ and the symmetric group $S_n$ of permutations of $1, 2, \cdots, n$. Most direct is a bijection between the partitions of $n$ and the set of conjugacy classes of $S_n$. Considerably deeper is the observation that the set of irreducible representations of $S_n$ can be parameterized by $\lambda \in P_n$ and $d_\lambda$ is the degree (dimension) of the irreducible representation indexed by $\lambda$.Still, Robinson, Schensted and Knuth (see Sagan [35]) developed an algorithm for obtaining Young tableaux with the help of permutations. Let $T_n$ be the set of standard Young tableaux with $n$ squares. According to this algorithm, for any $n$ there is a bijection, the so-called RSK correspondence, between $S_n$ and pairs $T, T' \in T_n$ with the same shape:

$$S_n \ni \pi \xrightarrow{RSK} (T(\pi), T'(\pi)) \in T_n \times T_n. \quad (1.6)$$

The RSK correspondence is very intricate and has no obvious algebraic meaning at all, but it is very deep and allows us to understand many things. In particular, it gives an explicit proof of the Burnside identity:

$$\sum_{\lambda \vdash n} d_\lambda^2 = n!. \quad (1.7)$$

Moreover, the RSK correspondence provides us with a viable algorithm for computing the length of the longest increasing subsequence. For a given $\pi = (\pi_1, \pi_2, \cdots, \pi_n) \in S_n$ and $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, we say $\pi_{i_1}, \pi_{i_2}, \cdots, \pi_{i_k}$ is an increasing subsequence of $\pi$ if $\pi_{i_1} < \pi_{i_2} < \cdots < \pi_{i_k}$. Let $l_\pi(n)$ be the length of the longest increasing subsequence of $\pi$. The celebrated Erdős and Szekeres theorem states every $\pi$ of $S_n$ contains an increasing and/or decreasing subsequence of length at least $\sqrt{n}$ (see Steele [38, 39]). This can be proved by the pigeon-hole principle, but also follows from the RSK correspondence using the observation that a Young diagram of size $n$ must have either width or height at least $\sqrt{n}$. More interestingly, $l_\pi(n)$ is exactly the number of squares in the first row of $T(\pi)$ or $T'(\pi)$, i.e., $l_\pi(n) = \lambda_1(T(\pi))$.

2 Random uniform partitions

We have defined the set $P_n$ of partitions of $n$ and known how to count its size $p(n)$. Now equip a probability measure on this set. As we will see below, this set bears many various natural measures. Certainly, the first natural measure is uniform, i.e., choose at random a partition with equal probability. Let $P_n$ be the uniform measure on $P_n$ defined by

$$P_n(\lambda) = \frac{1}{p(n)}, \quad \lambda \in P_n. \quad (2.1)$$
It seems interesting to study the behaviors of the typical partitions (Young diagrams). For instance, what are the asymptotics of the maximal summand? Erdős’ school first treated these questions and obtained very deep and interesting asymptotics. The most remarkable Erdős-Lehner theorem (see [13]) is as follows. For all $x \in \mathbb{R}$

$$
\lim_{n \to \infty} P_n(\lambda \in \mathcal{P}_n : \frac{c}{\sqrt{n}} \lambda_1 - \log \frac{\sqrt{n}}{c} \leq x) = e^{-e^{-x}}, \quad (2.2)
$$

where $c = \frac{\pi}{\sqrt{6}}$. Note that the limit is the very famous Gumbel distribution.

To prove (2.2), Erdős and Lehner used the Hardy-Ramanujan asymptotic formula for $p(n)$ (see (1.2)) and some finer combinatorial estimates for numbers of partitions with certain properties to find the limiting distribution of $l(\lambda)$ and thus — by the duality relation $l(\lambda) = \lambda_1’$ — the distribution of $\lambda_1$.

As known to us, the probabilistic argument of many combinatorial problems dates back to Erdős and his colleagues, but they usually investigate a single functional. Instead of a single functional, Vershik posed the problem of the limit shape in the early 1970s. Roughly speaking, they are concerned with the following problem: suppose we have some probability measures on the space $\mathcal{P}$ of partitions for all $n$, say $\mu_n$, how do we scale the space $\mathcal{P}$ in order to obtain the true nontrivial limit distribution of the measures $\mu_n$? Obviously, one can obtain rich information about asymptotics from the properties of the limit shape. See Vershik [44, 45, 46] for history and recent development.

### 2.1 Grand canonical ensembles

Let $q \in (0, 1)$. Define a probability measure $P_q$ on $\mathcal{P}$ as follows:

$$
P_q(\lambda) = \frac{q^{\vert \lambda \vert}}{\mathcal{F}(q)}, \quad \lambda \in \mathcal{P}, \quad (2.3)
$$

where $\vert \lambda \vert$ is the number being partitioned by $\lambda$.

This $P_q$ is particularly useful for the study of $P_n$ because it is easily understood and many limit theorems as $q \to 1$ can be converted with some work to the corresponding statements for $P_n$ as $n \to \infty$. We call the family $(\mathcal{P}, P_n)$ small canonical ensemble of partitions and the $(\mathcal{P}, P_q)$ grand canonical ensemble of partitions in view of similarities with statistical physics (see Vershik and Yakubovich [48]). The following lemma, due to Fristedt [16], summarizes some primary properties concerning the probabilities $P_n$ and $P_q$.

**Lemma 1.** For any $q \in (0, 1)$ and $n \geq 0$, we have
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(1) \[ P_q|p_n = P_n, \quad (2.4) \]
i.e., \( P_n \) is the conditional probability measure induced on \( \mathcal{P}_n \) by \( P_q \);

(2) \[ P_q = \frac{1}{\mathcal{F}(q)} \sum_{n=0}^{\infty} p(n)q^n P_n, \quad (2.5) \]
i.e., \( P_q \) is a convex combination of measures \( P_n \);

(3) Under \( (\mathcal{P}, P_q) \), the random variables \( r_k \) are independent geometrically distributed with mean \( q_k/(1-q_k) \).

In this subsection we restrict ourselves to the grand canonical ensemble and obtain a first insight on the asymptotics of random Young diagrams. Let us start with the extremal distribution.

Since \( \lambda_1 < m \) if and only if \( r_j = 0 \) for all \( j \geq m \), then, by virtue of (3) of Lemma 1, direct calculations can be made to show the limiting distribution of \( \lambda_1 \) is the Gumbel distribution (see 2.2)). More generally, analogous extremal distributions hold for the first \( d \) largest summands (see Fristedt [16], Vershik and Yakubovich [48]).

**Theorem 2.1.** Let \( q = e^{-h} \). Define for \( i \geq 1 \)

\[ W_i = h \lambda_i - |\log h|, \quad (2.6) \]

Then we have as \( h \to 0 \)

\[ (W_1, W_2, \cdots) \overset{d}{\to} (Y_1, Y_2, \cdots), \quad (2.7) \]

where \( Y_1, Y_2, \cdots, \) is a Markov chain such that the density of \( Y_1 \) is \( \exp(-x-e^{-x}) \) and the density of \( Y_i \) conditioned on \( Y_{i-1} = x \) is \( \exp(-y-e^{-y}+e^{-x}), \ y \leq x \).

Note \( \overset{d}{\to} \) in (2.7) means that \( (W_1, W_2, \cdots, W_m) \) converges in distribution to \( (Y_1, Y_2, \cdots, Y_m) \) for \( m \geq 1 \). Next let us look at the asymptotics in the deep bulk of the spectrum of Young diagrams.

To a partition \( \lambda \in \mathcal{P} \), we assign a function \( \varphi_\lambda \) on \((0, \infty)\) by the following rule:

\[ \varphi_\lambda(t) = \sum_{k=\lfloor t \rfloor}^{\infty} r_k, \quad 0 < t < \infty, \quad (2.8) \]

where \( \lfloor t \rfloor \) denotes the integer part of \( t \). Clearly, by definition, \( \varphi_\lambda(\cdot) \) is a monotone decreasing, piecewise constant function of \( t \), and that
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\[ \int_{0}^{\infty} \varphi_{\lambda}(t) \, dt = |\lambda| \]. This function describes actually the boundary of the Young diagram \( \lambda \) used by French mathematicians.

Again, using (3) of Lemma 1, one easily has for each \( t > 0 \)

\[
E_q(\varphi_{\lambda}(t)) = \sum_{k=[t]}^{\infty} \frac{q^k}{1-q^k}, \quad \text{Var}_q(\varphi_{\lambda}(t)) = \sum_{k=[t]}^{\infty} \frac{q^k}{(1-q^k)^2}.
\] (2.9)

Obviously, the Markov inequality can be used to show that \( \varphi_{\lambda}(t) \) concentrates around its mean. In fact, one has a stronger result: the limit shape exists (see Vershik [45]).

**Theorem 2.2.** Let \( q = e^{-h} \). Consider the scaled function

\[
\tilde{\varphi}_h(t) = h\varphi_{\lambda}(\frac{t}{h}), \quad t > 0.
\] (2.10)

Then, as \( h \to 0 \), we have \( \tilde{\varphi}_h \to \Psi \) in the sense of uniform convergence on compact sets, where \( \Psi \) is the function defined by

\[
\Psi(t) = \int_{t}^{\infty} \frac{e^{-u}}{1-e^{-u}} \, du, \quad t > 0.
\] (2.11)

More precisely, for any \( \varepsilon > 0 \) and \( 0 < a < b < \infty \), there exists an \( h_0 > 0 \) such that for \( 0 < h < h_0 \) we have

\[
P_q(\lambda \in \mathcal{P} : \sup_{a \leq t \leq b} |\tilde{\varphi}_h(t) - \Psi(t)| > \varepsilon) < \varepsilon.
\] (2.12)

In other words, the image measure of \( P_q \) under \( \tilde{\varphi}_h : \mathcal{P} \to \mathcal{D}(0, \infty) \) (the space of right continuous function with left limits) has a limit which is singular measure concentrated on some continuous curve \( \Psi \) (see Figure 2.1).

![Figure 2.1: Temperley-Vershik curve](image-url)
Note that $\Psi$ in (2.11) is computable: let $s = \Psi(t)$, then
\[
e^{-s} + e^{-t} = 1. \quad (2.13)
\]
This curve was firstly obtained via a heuristic argument by the physicist Temperley [42]; now it is called Temperley-Vershik’s equation. On the other hand, we remark that 0 is a point of singularity of integrand in (2.11). In fact, the scaling constants at the edge and in the bulk are completely different.

For a probabilist, the law of large numbers is only the beginning and the questions about central limit theorem correction to the limit shape and local statistics of the Young diagram in various regions of the limit shape follow immediately. Conjecturally, the limit shape controls the answers to all these questions.

**Theorem 2.3.** Let $q = e^{-h}$. Define
\[
V_h(t) = \frac{1}{h^{1/2}} (\tilde{\varphi}_h(t) - \Psi(t)), \quad t > 0. \quad (2.14)
\]

Then under $(P, P_q)$, we have as $h \to 0$

(1) For each $t > 0$,
\[
V_h(t) \xrightarrow{d} N(0, \sigma^2(t)), \quad (2.15)
\]

where
\[
\sigma^2(t) = \int_t^\infty \frac{e^{-u}}{(1 - e^{-u})^2} du. \quad (2.16)
\]

(2) In $D(0, \infty)$,
\[
V_h \Rightarrow G \quad (2.17)
\]

where $G$ is a continuous Gaussian process with independent increments.

Note the classic theory of weak convergence is applicable to the normalized partial sums $V_h(t)$ of independent geometrically distributed random variables $r_k$ (see Billingsley [5]). Indeed, (2.15) can be proved by a direct computation of characteristic functions, and finite dimensional distributions hold similarly; while the uniform tightness for (2.17) can be checked by using the fourth moment.

As the reader might notice, $|\lambda|$ is itself random under $(P, P_q)$ and is actually an infinite sum of the $kr_k$’s, so the mean is
\[
E_q |\lambda| = \sum_{k=1}^\infty \frac{ke^{-kh}}{1 - e^{-kh}} \sim \frac{1}{h^2} \int_0^\infty ue^{-u} du = \frac{\pi^2}{6h^2}, \quad (2.18)
\]
and $|\lambda|$ fluctuates around this mean like a Gaussian random variable (see Bloch and Okounkov [6], Pittel [33]).

### 2.2 Small canonical ensembles

In this subsection we turn to the second issue: how do we convert the results in the independent setting to the original partitions? By Lemma 1, for each $n$ and $q$, the probability measure $P_n$ is equal to the conditional probability measure of $P_q$ obtained by conditioning $P_q$ on the event $\{\lambda : |\lambda| = n\}$ (just $P_n$). That is, for any set of partitions,

$$P_n(A) = P_q(A | |\lambda| = n) = \frac{P_q(A, |\lambda| = n)}{P_q(|\lambda| = n)} \leq \frac{P_q(A)}{P_q(|\lambda| = n)}.$$  \hspace{1cm} (2.19)

For this bound to be most effective, one needs to determine $q$ for which $P_q(|\lambda| = n)$ is asymptotically the largest possible. It turns out that $q = e^{-h_n} = e^{-\frac{c}{\sqrt{n}}}$ where $c = \frac{\pi}{\sqrt{6}}$ (see (2.2)) meets this requirement and that for this $q$,

$$P_q(|\lambda| = n) = (1 + o(1)) \frac{1}{(96n^3)^{1/4}}.$$ \hspace{1cm} (2.20)

This indeed follows from the local limit theorem; on the other hand, the choice of $h_n = \frac{c}{\sqrt{n}}$ is most natural if one notices that the equation $E_q |\lambda| = n$ has a unique solution, which is asymptotically equivalent to $e^{-h_n}$ by (2.18).

We have the following weak equivalence of ensembles (see Vershik [45]).

**Lemma 2.** (1) If the probability measures $P_n$ have weak limit in $\mathcal{D}(0, \infty)$ as $n \to \infty$, then the measures $P_q$ have the same limit as $q \to 1$;

(2) Conversely, if the measures $P_q$ have weak limit as $q \to 1$ and the limit measure is singular, then the measures $P_n$ have the same limit as $n \to \infty$.

Statement (1) is obvious since $P_q$ is a convex combination of the measures $P_n$. The proof of (2) is substantially based on the fact that the measures $P_q$ are multiplicative; in fact, it is a Tauberian-type theorem. The problem of Tauberian-type theorem in general setting (for non-singular case and for general multiplicative measures) is of great interest and, seemingly, has not been studied yet. Now Lemma 2 together with Theorem 2.2 implies
Theorem 2.4. Let $h_n$ be as above. Consider the scaled function

$$\tilde{\varphi}_n(t) = h_n \varphi(\frac{t}{h_n}), \quad t > 0. \quad (2.21)$$

Then for any $\varepsilon > 0$ and $0 < a < b < \infty$, there exist $n_0$ such that for $n > n_0$

$$P_n(\lambda \in \mathcal{P}_n : \sup_{a \leq t \leq b} |\tilde{\varphi}_n(t) - \Psi(t)| > \varepsilon) < \varepsilon. \quad (2.22)$$

We say that the grand and small canonical ensembles are strongly equivalent for some functional $f$ on partitions if the distributions of the functional $f$ are asymptotically the same with respect to $P_n$ and $P_q$ with $q = e^{-h_n}$. According to Vershik and Yakubovich [48], the ensembles are equivalent for the functional of scaled largest parts in partition. Thus Theorem 2.1 yields the following results, which particularly recovers the Erdős and Lehner extremal distribution (see 2.2).

Theorem 2.5. Let $h_n$ be as above. Define for $i \geq 1$

$$\hat{W}_i = h_n \lambda_i - |\log h_n|. \quad (2.23)$$

Then as $n \to \infty$,

$$(\hat{W}_1, \hat{W}_2, \cdots) \xrightarrow{d} (Y_1, Y_2, \cdots), \quad (2.24)$$

where $Y_i$’s are given by Theorem 2.1.

Note that there is no guarantee about the equivalence of distributions for a certain functional (see Vershik [44]). To extend the central limit theorem like Theorem 2.3 to the case of $(\mathcal{P}_n, \mathcal{P}_n)$, we need some additional work.

By Lemma 1, the probability generating function of $\lambda'_k$ under $P_q$ is

$$E_q e^{x \lambda'_k} = \frac{1}{\mathcal{F}(q)} \sum_{n=0}^{\infty} p(n) q^n E_n e^{x \lambda'_k}. \quad (2.25)$$

On the other hand, $E_q e^{x \lambda'_k}$ is easily written as

$$E_q e^{x \lambda'_k} = \prod_{j=k}^{\infty} E_q e^{x r_j} = \prod_{j=k}^{\infty} \frac{1 - q^j}{1 - e^x q^j}. \quad (2.26)$$

Since this function is analytic in $q$ in the disk $\{q \in \mathbb{C} : |q| < 1\}$, by the Cauchy integral formula, we have for any $\gamma > 0$

$$E_n e^{x \lambda'_k} = \frac{1}{2\pi i p(n) \gamma^n} \int_{-\pi}^{\pi} e^{-i n \theta} \mathcal{F}(\gamma e^{i \theta}) \prod_{j=k}^{\infty} \frac{1 - (\gamma e^{i \theta})^j}{1 - e^x (\gamma e^{i \theta})^j} d\theta. \quad (2.27)$$
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It remains to choose a proper \( \gamma > 0 \) and to give a precise estimate for the contour integral. Note (1.2) and the following simple formula:
\[
F(e^{-z}) = \exp\left(\frac{\pi^2}{6z} + \frac{1}{2} \log \frac{z}{2\pi} + O(|z|)\right)
\]
uniformly for \( z \to 0 \) within a corner \( \{z : \text{Im} z \leq \varepsilon \text{Re} z, \text{Re} z > 0\}, \varepsilon > 0 \) being fixed. One can now prove for any real \( u \) and some \( \tau^2(t) \)
\[
E_n \exp\left(\frac{u\tilde{\varphi}_n(t)}{h_n^{1/2}}\right) = \exp\left(\frac{u\Psi(t)}{h_n^{1/2}} + \frac{u^2}{2} \tau^2(t) + o(1)\right),
\]
which obviously implies that \( \tilde{\varphi}_n(t) \) after suitably scaled is approximately normal with variance \( \tau^2(t) \) for each \( t > 0 \). Analogous results hold for multidimensional vectors. These are summarized as follows (see Pittel [33]).

**Theorem 2.6.** Let \( h_n \) be as above. Define for \( t > 0 \)
\[
V_n(t) = \frac{1}{h_n^{1/2}}(\tilde{\varphi}_n(t) - \Psi(t)).
\]
Then for any \( m \geq 1 \) and \( 0 < t_1 < t_2 < \cdots < t_m \)
\[
(V_n(t_1), V_n(t_2), \ldots, V_n(t_m)) \to^d (V(t_1), V(t_2), \ldots, V(t_m)),
\]
where \( V(t), t > 0 \) is a Gaussian process with covariance structure: for \( s \leq t \)
\[
\text{Cov}(V(s), V(t)) = \frac{\tau^2}{(1 - e^{-s})(1 - e^{-t})} \left( \frac{e^{-t} - l(s)l(t)}{2e^2} \right),
\]
where
\[
l(s) = se^{-s} + (1 - e^{-s}) \log(1 - e^{-s}).
\]
It is worth mentioning that the weak convergence of the process \( V_n \) is not proven yet. We don’t know whether the uniform tightness is true for \( V_n \). However, a slightly weaker condition can be proved, i.e., \( V_n \) is stochastically equi-continuous, uniformly for \( t > 0 \):
\[
\lim_{\delta \to 0} \lim_{n \to \infty} \sup_{t,s:|t-s| \leq \delta} P_n(|V_n(t) - V_n(s)| \geq \varepsilon) = 0.
\]
And, fortunately, this stochastic equi-continuity can be used to prove convergence in distribution of \( f(V_n) \) for a broad class \( \mathcal{G} \) of the integral functionals \( f \) of a form
\[
f(x) = \int_0^\infty g(t, x(t))dt,
\]
i.e., for \( f \in \mathcal{G} \)
\[
f(V_n) \xrightarrow{d} f(V). \tag{2.36}
\]

Since \( \lambda \) is uniformly distributed on \( \mathcal{P}_n \), then so \( \lambda' \). Equivalently,
\[
(\lambda_1, \lambda_2, \cdots, \lambda_l) \xrightarrow{d} (\lambda'_1, \lambda'_2, \cdots, \lambda'_l). \tag{2.37}
\]

Therefore, we can reformulate Theorems 2.4 and 2.6 in terms of \( \lambda \); while Theorem 2.5 can be restated in terms of \( \lambda' \). In fact, this symmetry can be directly seen from the limit curve equation (see (2.13)).

So far, we have seen some important limit theorems concerning the limit shape and the fluctuations of random uniform partitions. Frankly speaking, the probabilistic structure of a uniform partition is quite complicated. According to Theorem 2.5, for a fixed \( k \geq 1 \), the \( \lambda_k \) suitably scaled has exponential-type limit distribution. A natural question is what we can say about the \( k \)-th largest part \( \lambda_k \) if \( k = k(n) \) varies as \( n \), say, \( k(n) \to \infty \) sufficiently slowly that \( k(n) = o(n^{1/4}) \). Likewise, Theorem 2.6 only describes the asymptotic distributional result for the intermediate parts of size \( O(n^{1/2}) \) (equivalently in the middle bulk); one naturally asks for the limiting distribution for the first \( o(n^{1/2}) \) parts and/or parts near to the other end. The reader is referred to Fristedt [16] and Pittel [33] for related answers.

\subsection*{2.3 Normal approximation of \( d_{\lambda} \)}

In conclusion to this section, we review some work on the \( d_{\lambda} \) — the total number of standard Young tableaux associated with the partition \( \lambda \). As we have seen, the \( d_{\lambda} \) is of great interest in its own right. In a seminal sequence of three papers, Szalay and Turán [41] undertook an asymptotic study of the random uniform partitions. A highlight of their work is a pioneering analysis of the limiting distribution of \( d_{\lambda} \). Using the classic formula (1.5), Szalay and Turán showed that

\[
\log d_{\lambda} = \frac{1}{2} \log n! - An = O_p(n^{7/8} \log^4 n), \tag{2.38}
\]

where

\[
A = \frac{1}{2} \left( 1 - \log \frac{\pi^2}{6} \right) + \frac{6}{\pi^2} \int_0^\infty \frac{t \log t}{e^t - 1} dt. \tag{2.39}
\]

Pittel [33] used the distribution identity (2.37) and refined estimates for the \( \lambda'_k \) to improve the error term to \( O_p(n^{3/4} \log^{3/2} n) \). The power \( 3/4 \) is indeed optimal. The following result implies that \( \log d_{\lambda}/\sqrt{n!} \) is asymptotically normal, with mean \(-An\), and standard deviation of order \( n^{3/4} \) (see Pittel [34]).
**Theorem 2.7.** Let \( K(s, t) \) be a symmetric function on \([0, 1] \times [0, 1]\) defined by

\[
K(s, t) = \frac{1}{c} \left(s(1-t) - \frac{1}{2} L(s)L(t)\right), \quad 0 \leq s \leq t \leq 1
\]

where \( L(t) = \frac{1}{c}(t \log t - (1-t) \log(1-t)) \) (remind \( c = \frac{\pi}{\sqrt{6}} \)). Introduce

\[
\sigma^2 = \frac{1}{c^2} \int_0^1 \int_0^1 K(s, t)\phi(s)\phi(t)dsdt
\]

where

\[
\phi(t) = \int_0^\infty \frac{\log |\log u|}{(1 - t + tu)^2} du
\]

Then under \( P_n \)

\[
\frac{1}{n^{3/4}} \left( \log \frac{d\lambda}{\sqrt{n!}} + A n \right) \xrightarrow{d} N(0, \sigma^2).
\]

The proof of (2.40) consists of two steps: (1) show that, for almost all diagrams \( \lambda \), \( \log d\lambda \) is well approximated by a linear function of the dual diagram \( \lambda' \), namely, for every \( \varepsilon > 0 \),

\[
\log \frac{d\lambda}{\sqrt{n!}} = -A n + \sum_{k=1}^{\lambda_1} \nu(k)(\lambda'_k - E(k)) + O_p(n^{2/3+\varepsilon})
\]

for some deterministic functions \( \nu(\cdot), E(\cdot) \); (2) apply the functional central limit theorem (2.36) to the linear sums in (2.41).

### 3 Random Plancherel partitions

In this section we consider the Plancherel measure \( P_{p,n} \) on the set \( \mathcal{P}_n \) of partitions of \( n \) defined by

\[
P_{p,n}(\lambda) = \frac{d^2}{n!} , \quad \lambda \in \mathcal{P}_n,
\]

where \( p \) in the subscript stands for Plancherel. This measure arises naturally in representation-theoretic, combinatorial, and probabilistic problems. It is called Plancherel because the Fourier transform

\[
L^2(S_n, \mu_s) \xrightarrow{\text{Fourier}} L^2(\hat{S}_n, P_{p,n})
\]

is an isometry just like in the classical Plancherel theorem, where \( \mu_s \) is the uniform measure on \( S_n \) and \( \hat{S}_n \) is the set of irreducible representations of \( S_n \).
Recall $l_n(\pi)$ denotes the length of the longest increasing subsequence of $\pi$. Then by the RSK correspondence
\[
\mu_s(\pi \in S_n : l_n(\pi) = k) = \frac{|\{\pi \in S_n : l_n(\pi) = k\}|}{n!} = \sum_{\lambda : \lambda_1 = k} \frac{d_\lambda^2}{n!} = P_{p,n}(\lambda \in \mathcal{P}_n : \lambda_1 = k). \tag{3.3}
\]
In words, the Plancherel measure $P_{p,n}$ on $\mathcal{P}_n$ is the push-forward of the uniform measure $\mu_s$ on $S_n$. Thus the analysis of $l_n$ is equivalent to a statistical problem in the "geometry" of the Young diagram.

On the other hand, we can understand the Plancherel measure in the following way. Consider the following chain of partitions:
\[
\emptyset = \lambda^{(-n)} \nearrow \cdots \nearrow \lambda^{(-1)} \nearrow \lambda^{(0)} \searrow \lambda^{(1)} \searrow \cdots \searrow \lambda^{(n)} = \emptyset, \tag{3.4}
\]
where $\mu \nearrow \nu$ means that $\nu$ is obtained from $\mu$ by adding a square, while $\mu \searrow \nu$ means that $\nu$ is obtained from $\mu$ by removing a square.

It is clear that the number of increasing chains is identical to that of decreasing chains. Also, the increasing chain may be viewed as a path in the Young graph which produces a standard tableau. Thus the Plancherel measure is the distribution of the central slice $\lambda^{(0)}$ of this chain induced by the uniform measure on all chains of the form (3.4).

### 3.1 Limit shapes

In this subsection we will give the limit shape of Young diagrams under the Plancherel measure in the spirit of Theorem 2.4. But let us first consider a basic problem on increasing subsequences, that is to determine the asymptotics of $l_n$ as $n \to \infty$. Again, the study of $l_n$ goes back to Erdős. It follows from the Erdős and Szekeres theorem that the expected value $E_s l_n$ satisfies
\[
E_s l_n \geq \frac{1}{2} \sqrt{n}. \tag{3.5}
\]
In 1960s Ulam, who had a long and enduring friendship with Erdős, computed $E_s l_n$ for $n$ lying in the range $1 \leq n \leq 10$ using Monte Carlo techniques and conjectured that
\[
\lim_{n \to \infty} \frac{E_s l_n}{\sqrt{n}} = \kappa \tag{3.6}
\]
exists (see Deift [11], Steele [38]). What became known as Ulam’s problem was to prove that the limit in (3.6) indeed exists and compute
the constant $\kappa$. The first analytic result was due to Hammersley, who introduced a certain Poisson point process in the first quadrant with the express purpose of studying $l_n$. Then he proved the existence of the limit $\kappa$ and

$$\frac{\ln n}{\sqrt{n}} \xrightarrow{p} \kappa \quad (3.7)$$

as an application of subadditive ergodic theory (H. Kesten strengthened (3.7) to almost sure convergence). There has been significant interest in the determination of the exact value $\kappa$. see Aldous and Diaconis [1], Deift[11] for various associated results and some of the history.

Next we derive the value of $\kappa$ making use of the distribution identity (3.3). To emphasize the dependence of $\lambda$ upon the size $n$, we use $\lambda^{(n)}$ in place of $\lambda$ in the following argument. Clearly, we can and do assume $\lambda^{(n)}$ is generated by adding a square to $\lambda^{(n-1)}$. Note $\lambda^{(n)} = \lambda^{(n-1)} + 1$ if and only if $\lambda^{(n)}$ is generated from $\lambda^{(n-1)}$ by adding an extra square at the first row of an extra column. Therefore,

$$E_{p,n}\lambda^{(n)} = \sum_{k=1}^{n} E_{p,k}(\lambda^{(k)} - \lambda^{(k-1)})$$

$$= \sum_{k=1}^{n} P_{p,k}(\lambda^{(k)} - \lambda^{(k-1)} = 1). \quad (3.8)$$

In turn,

$$P_{p,k}(\lambda^{(k)} - \lambda^{(k-1)} = 1) = \frac{d_{\lambda^{(k-1)}}d_{\lambda^{(k)}}}{k!}, \quad (3.9)$$

where $\lambda^{(k)*}$ is the Young diagram with an extra square in the first row of $\lambda^{(k-1)}$. Now it easily follows

$$E_{p,n}\lambda^{(n)} \leq 2\sqrt{n}. \quad (3.10)$$

For the lower bound of $\kappa$, define the graph function $\lambda(x)$ and $\bar{\lambda}(x)$ associated with $\lambda$ by

$$\lambda(0) = \lambda(1), \quad \lambda(x) = \lambda_{\lfloor x \rfloor}, \quad x > 0 \quad (3.11)$$

and

$$\bar{\lambda}(x) = \frac{1}{\sqrt{n}} \lambda(\sqrt{n}x), \quad x \geq 0. \quad (3.12)$$

Obviously, $\int_{0}^{\infty} \bar{\lambda}(x)dx = 1$. Let $\mathcal{H}$ be the class of nonnegative nonincreasing functions on $[0, \infty)$ of integral unity. For $f \in \mathcal{H}$, define

$$I(f) = \int_{0}^{\infty} \int_{0}^{f(x)} \log(f(x) - y + f^{-1}(y) - x)dxdy. \quad (3.13)$$
Then by the hook formula (1.3)

\[ P_{p,n}(\lambda) = \frac{n!}{H^2} \]

\[ = \exp(-(1 + o(1))2n(I(\tilde{\lambda}) + \frac{1}{2})), \] (3.14)

where \( o(1) \) is uniform over all \( \lambda \). It immediately follows for any \( \varepsilon > 0 \)

\[ P_{p,n}(\lambda : \lambda_1 \leq (2 - \varepsilon)\sqrt{n}) \]
\[ = \sum_{\lambda_1 \leq (2 - \varepsilon)\sqrt{n}} P_{p,n}(\lambda) \]
\[ \leq p(n) \exp(-(1 + o(1))2n(\inf_{f \in \mathcal{H}_0(\varepsilon)} I(f) + \frac{1}{2})), \] (3.15)

where \( \mathcal{H}_0(\varepsilon) = \{ f \in \mathcal{H} : f(0) = 2 - \varepsilon \} \).

Note \( p(n) \) is at most \( e^{\sqrt{6n}/3} \). It suffices to prove the infimum is strictly larger than \(-1/2\) for \( \varepsilon > 0 \). This analysis leads to a calculus of variations minimization problem. The problem of minimizing the functional \( I(f) \) over \( \mathcal{H} \) was posed by Stanley, and Logan and Shepp [28] showed that the minimum is unique and has the value \(-1/2\). Moreover, the minimizing function \( y = \omega(x) \) such that \( I(\omega) = -1/2 \) is given parametrically by

\[ x = \frac{2}{\pi}(\sin \theta - \theta \cos \theta), \quad y = x + 2 \cos \theta, \quad 0 \leq \theta \leq \pi. \] (3.16)

They also found the minimum of \( I(f) \) subject to the constraints \( f(0) \leq a \) and \( f^{-1}(0) \leq b \); in particular

\[ \inf_{f \in \mathcal{H}_0(\varepsilon)} I(f) = -1 + \frac{(2 - \varepsilon)^2}{8} + \log \frac{2 - \varepsilon}{2} \]
\[ - (1 + \frac{(2 - \varepsilon)^2}{4}) \log \frac{2(2 - \varepsilon)^2}{4 + (2 - \varepsilon)^2}; \] (3.17)

which is a strictly convex, monotone decreasing function with minimum \(-\frac{1}{2}\) at \( \varepsilon = 0 \). Thus, it follows for any \( \varepsilon > 0 \)

\[ P_{p,n}(\lambda : \lambda_1 \leq (2 - \varepsilon)\sqrt{n}) \to 0. \] (3.18)

Combining (3.10) and (3.18), we have proved

**Theorem 3.1.**

\[ \kappa = 2. \] (3.19)
It is natural to ask how the $\lambda_2, \lambda_3, \cdots$ behave asymptotically. Instead of a specific $\lambda_k$, Vershik and Kerov [47] investigated the limit shape of the Young diagram under the Plancherel measure. It is convenient to use the rotated coordinate system

$$u = y - x, \quad v = x + y.$$  \hspace{1cm} (3.20)

In the $(u, v)$ plane, the function $\lambda$ in (3.11) ($\check{\lambda}$ in (resp. 3.12)) transforms into $\Lambda$ (resp. $\check{\Lambda}$). Note that $\Lambda'(u) = \pm 1$, $\Lambda(u) \geq |u|$ and $\check{\Lambda}(u) = |u|$ for sufficiently large values of $u$.

Let $\hat{H}$ be the class of piece smooth functions $\hat{f}$ with $\hat{f}(u) \geq u$. Set for each piece smooth function $\hat{f} \in \hat{H}$

$$\hat{I}(\hat{f}) = \frac{1}{4} \int \int_{v < u} [\log(v - u)](1 - \hat{f}'(v))(1 + \hat{f}(u))dudv. \hspace{1cm} (3.21)$$

$\hat{I}(\hat{f})$ can be expressed in the form

$$1 + 2\hat{I}(\hat{f}) = \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \frac{\hat{f}(u) - \hat{f}(v)}{u - v} \right)^2dudv + 2 \int_{|u| > 2} \hat{f}(u)\text{arch}\frac{u}{2}|du$$

$$= \frac{1}{2}||\hat{f}||^2_\theta + 2 \int_{|u| > 2} \hat{f}(u)\text{arch}\frac{|u|}{2}|du. \hspace{1cm} (3.22)$$

Note $||\hat{f}||_\theta$ is the Sobolev norm. Define the function $\Omega(u)$ by

$$\Omega(u) = \begin{cases} \frac{2}{\pi} (u \arcsin \frac{u}{2} + \sqrt{4 - u^2}), & |u| \leq 2, \\ |u|, & |u| > 2. \end{cases} \hspace{1cm} (3.23)$$

The following lemma, due to Vershik and Kerov [47], plays an important role in finding the limit shape (see Figure 3.1 and Theorem 3.2 below).

Figure 3.1: The limit shape $\Omega(u)$ under Plancherel measure
Lemma 3. (1) \( \Omega \) is the unique solution of the variational problem minimizing the functional \( \hat{I} \) over \( \hat{H} \):

\[
\hat{I}(\Omega) = -\frac{1}{2}.
\]

It is easy to see

\[
\Omega'(u) = \frac{2}{\pi} \arcsin \frac{u}{2}, \quad \Omega''(u) = \frac{2}{\pi \sqrt{4 - u^2}}.
\]

So, \( \Omega(u) \) and \( \Omega'(u) \) are continuous on the whole real line, while \( \Omega''(u) \) is not.

Theorem 3.2. Under \((\mathcal{P}_n, P_{p,n})\) we have as \( n \to \infty \)

\[
\sup_{-\infty < u < \infty} |\hat{\Lambda}(u) - \Omega(u)| \xrightarrow{P} 0. \tag{3.24}
\]

To prove (3.24), observe an important estimate: there exists a constant \( C \) such that for any 1-Lipschitz function \( \hat{f} \) with compact support

\[
\sup_{-\infty < u < \infty} |\hat{f}(u)| \leq C ||\hat{f}||_\theta^{2/3}. \tag{3.25}
\]

Thus it suffices to show

\[
||\hat{\Lambda} - \Omega||_\theta \xrightarrow{P} 0. \tag{3.26}
\]

Since \( \hat{\Lambda}(u) - \Omega(u) \xrightarrow{P} 0 \) for each \( |u| > 2 \), then

\[
\int_{|u| > 2} |\hat{\Lambda}(u) - \Omega(u)| \arcsin \frac{u}{2} \, du
\]

is negligible. On the other hand,

\[
1 + 2\hat{I}(\hat{\Lambda}) = \frac{1}{2} ||\hat{\Lambda} - \Omega||_\theta^2 + 2 \int_{|u| > 2} |\hat{\Lambda}(u) - \Omega(u)| \arcsin \frac{u}{2} \, du. \tag{3.27}
\]

Thus for (3.26), it is sufficient to prove

\[
\hat{I}(\hat{\Lambda}) \xrightarrow{P} -\frac{1}{2}. \tag{3.28}
\]

Observe \( I(\lambda) = \hat{I}(\hat{\Lambda}) \) for all Young diagrams \( \lambda \in \mathcal{P}_n \). Then a similar argument to (3.14) and (3.15) concludes the proof. An alternative proof can be found in Ivanov and Olshanski [21].

As a consequence of Theorem 3.2, we have for each \( k \geq 1 \)

\[
\frac{\lambda_k}{\sqrt{n}} \xrightarrow{P} 2 \tag{3.29}
\]

and

\[
\lambda_k - \lambda_{k+1} \xrightarrow{P} \infty. \tag{3.30}
\]
3.2 Tracy-Widom distribution at the edge

In Logan and Shepp [28] the question about the fluctuations of $\bar{\lambda}$ around the curve $\omega$ and/or $\bar{\Lambda}$ around the curve $\Omega$ was posed (see p. 211, [28] for their original guess).

As in the uniform partitions in Section 2, one needs to consider separately two different cases: at the edge and in the bulk. The present subsection is devoted to the study of fluctuation at the edge. The appearance of Tracy-Widom distribution is definitely a miracle.

Let us start with the notion of the determinantal point process. A random point process $X$ on the real line $\mathbb{R}$ is called determinantal with respect to a reference measure $\mu$ (the Lebesgue measure or the count measure) if the $m$-point correlation function ($m \geq 1$) are given by

$$P(\{x_1, \cdots, x_m\} \subset X) = \det(K(x_i, x_j))_{1 \leq i, j \leq m},$$

(3.31)

where $K(x, y)$ is a kernel of an integral operator $K : L^2(\mathbb{R}, \mu) \to L^2(\mathbb{R}, \mu)$, non-negative and locally trace class.

A Poisson point process on $\mathbb{R}$ with density $\rho(x)$ can be viewed as a somewhat degenerate determinantal process with kernel (see Johansson [24]),

$$K_{\text{ext}}(x, y) = \begin{cases} 0, & x \neq y, \\ \rho(x), & x = y. \end{cases}$$

(3.32)

Other important examples are sine, Bessel and Airy point processes. The Airy kernel is defined by

$$A(x, y) = \frac{A(x)A'(y) - A(y)A'(x)}{x - y},$$

(3.33)

where $d\mu = dx$ and $A(x)$ is the Airy function satisfying the equation $A''(x) = xA(x)$. Equivalently,

$$A(x, y) = \int_{0}^{\infty} A(x + t)A(y + t)dt.$$  

(3.34)

Let $\zeta = (\zeta_1 > \zeta_2 > \cdots)$ be the corresponding random configuration. The following result describes the fluctuations of the Young diagram at the edge.

Theorem 3.3. Under the Plancherel measure we have

$$\frac{\lambda_k - 2\sqrt{n}}{n^{1/6}}, k = 1, 2, \cdots \xrightarrow{d} (\zeta_k, k = 1, 2, \cdots).$$

(3.35)
The distributions of $\zeta_k$ can be explicitly calculated. In fact, it is easy to see for any $t \in \mathbb{R}$

$$F_1(t) = P(\zeta_1 \leq t) = E \prod_{j=1}^{\infty} (1 - I_{(\zeta_j > t)})$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_{[0,t]^k} \det(A(x_i, x_j))_{1 \leq i, j \leq k} dx_1 \cdots dx_k$$

$$= \det(1 - A)_{L^2((t, \infty), dx)},$$

(3.36)

where the last determinant is called Fredholm determinant of the operator $A$ on the space $L^2((t, \infty), dx)$.

It was Tracy and Widom [43] who discovered the Fredholm determinant could be governed by the Painlevé II equation. Let $u$ be the global positive solution of $u''(x) = xu(x) + 2u^3(x)$ with boundary asymptotics

$$u(x) = \begin{cases} 
  -(\frac{x}{2})^{1/2}(1 + O(\frac{1}{x^2})), & x \to -\infty, \\
  -A(x) + O(\frac{e^{-4x^{3/2}/3}}{x^{1/4}}), & x \to \infty.
\end{cases}$$

(3.37)

Then for any $t \in \mathbb{R}$

$$F_1(t) = \exp \left( - \int_t^{\infty} (x-t)u^2(x)dx \right).$$

(3.38)

Note that the right hand side of (3.38) is indeed a distribution function — the so called Tracy-Widom distribution. Interestingly, the Tracy-Widom distribution has occurred in many distinct areas, say, increasing subsequence, random matrices, random growth models, queueing theory.

It is worth mentioning that $F_1$ is asymmetric with mean $\sim -1.7711$ and variance $\sim 0.8132$, and tail behaviors (see Figure 3.2): for some positive constant $C$,

$$C^{-1}e^{-C|t|^3} \leq F_1(t) \leq Ce^{-|t|^3/C}, \quad t \to -\infty$$

and

$$C^{-1}e^{-Ct^{3/2}} \leq 1 - F_1(t) \leq Ce^{-t^{3/2}/C}, \quad t \to \infty.$$

For $k \geq 2$, one can prove the distribution $F_k$ of $\zeta_k$

$$F_k(t) = F_{k-1}(t) + \frac{1}{(k-1)!} \left( -\frac{\partial}{\partial \kappa} \right)^{(k-1)}|_{\kappa=1} F(t, \kappa),$$

(3.39)

where $F(t, \kappa)$ is defined in a similar way to (3.37) with $u(x) \sim -\sqrt{\kappa}A(x)$ when $x \to \infty$.

Now Theorem 3.3 is restated as follows.
Theorem 3.4. Under the Plancherel measure, we have for each \( k \geq 1 \)

\[
\frac{\lambda_k - 2\sqrt{n}}{n^{1/6}} \overset{d}{\to} F_k.
\] (3.40)

The discovery of Tracy-Widom distribution is one of the most spectacular achievements in probability theory and mathematical physics in the 1990s. Baik, Deift and Johansson [3] first proved that \( F_1 \) in (3.38) is the limiting distribution of \( l_n \) (equivalently \( \lambda_1 \)) suitably scaled; then [4] gave a proof of analog for \( \lambda_2 \) and conjectured the analogs are also true for general \( k \geq 3 \). Over the year 1999, almost within six months, three different groups had independently proved Theorem 3.3. The proof in Borodin, Okounkov and Olshanski [8] is representation theoretic, while the proof in Johansson [23] involves ideas from statistical mechanics together with the asymptotics of polynomials orthogonal with respect to a certain distinguished discrete measure. The proof in Okounkov [30], however, provides a picture relating random matrices and random permutations via the equivalence of two points of view on topological surfaces.

For the purpose of this survey, we sketch quickly the proof of [8]. For \( \theta > 0 \), consider the poissonization of the Plancherel measure

\[
P_p^\theta(\lambda) = e^{-\theta} \sum_{n=0}^{\infty} \frac{\theta^n}{n!} P_{p,n}(\lambda).
\] (3.41)

This is a probability measure on \( \mathcal{P} \). The fact of the matter is that one can compute the correlation functions of \( P_p^\theta \).

Set \( D(\lambda) = \{\lambda_i - i\} \subset \mathbb{Z} \) and define the modified Frobenius coordi-
nates $Fr(\lambda)$ of a partition $\lambda$ by
\[
Fr(\lambda) =: (D(\lambda) + \frac{1}{2}) \Delta (Z_{\leq 0} - \frac{1}{2}) = \{ p_1 + \frac{1}{2}, \ldots, p_d + \frac{1}{2}, -q_1 - \frac{1}{2}, \ldots, -q_d - \frac{1}{2} \},
\]
where $\Delta$ stands for the symmetric difference of two sets, $d$ is the number of squares on the diagonal of $\lambda$. We rewrite the hook formula (1.4) to get
\[
d_{\lambda} \frac{\lambda!}{\lambda!} = \det \left( \frac{1}{(p_i + q_j + 1)p_i q_j} \right)_{1 \leq i, j \leq d}.
\]
Thus for a partition $\lambda$
\[
P^\theta_p(\lambda) = e^{-\theta \lambda! \left( \frac{d_{\lambda}}{\lambda!} \right)^2} = e^{-\theta \det(L(x_i, x_j; \theta))}_{1 \leq i, j \leq 2d},
\]
where $Fr(\lambda) = \{ x_1, \ldots, x_{2d} \}$ and
\[
L(x, y; \theta) = \begin{cases} 0, & xy > 0, \\ \frac{1}{x-y} \frac{\theta(|x|+|y|)/2}{\Gamma(|x|+\frac{1}{2})\Gamma(|y|+\frac{1}{2})}, & xy < 0. \end{cases}
\]
From this it follows

**Lemma 4.** For any $X = \{ x_1, x_2, \ldots, x_m \} \subset \mathbb{Z}$ we have
\[
P^\theta_p(\lambda \in \mathcal{P} : \lambda \subset D(\lambda)) = \det(J(x_i, x_j; \theta)))_{1 \leq i, j \leq m}.
\]

Here the kernel $J$ is given by the following formula
\[
J(x, y; \theta) = \sqrt{\theta} J_x J_y \frac{1}{x-y} J_x J_y + 1 - J_x J_y + 1, 
\]
where $J_x = J_x(2\sqrt{\theta})$ is the Bessel function of order $x$ and argument $2\sqrt{\theta}$.

This lemma can be used to show Theorem 3.3 holds under the poissonized Plancherel measure $P^\theta_p$, i.e,
\[
\left( \frac{\lambda_k - 2\sqrt{\theta}}{\theta^{1/6}}, k = 1, 2, \ldots \right) \xrightarrow{d} (\zeta_k, k = 1, 2, \ldots), \quad \theta \to \infty.
\]
To illustrate the basic idea for (3.48), we look only at the case $k = 1$:
\[
P^\theta_p(\lambda \in \mathcal{P} : \frac{\lambda_1 - 2\sqrt{\theta}}{\theta^{1/6}} < t) = P^\theta_p(\lambda \in \mathcal{P} : \lambda_1 - 1 < 2\sqrt{\theta} + \theta^{1/6} t) = \det(1 - J)_{t^2(m, m+1, \ldots)}.
\]
where $m = [2\sqrt{\theta} + \theta^{1/6}t]$. Since for all $t \in R$,
\begin{equation}
\det(1 - \mathbf{J})_{L^2((m,m+1,\cdots) \to \det(1 - \mathbf{A})_{L^2((t,\infty),dx)}, \quad (3.50)
\end{equation}
then
\begin{equation}
\frac{\lambda_1 - 2\sqrt{\theta}}{\theta^{1/6}} \to \zeta_1, \quad \theta \to \infty. \quad (3.51)
\end{equation}

We now convert to the original measures using the depoissonization techniques, which is slightly easier than the conditioning argument in Section 2. Given a sequence $b_n, n \geq 0$, its poissonization is by definition, the function

\begin{equation}
B(\theta) = e^{-\theta} \sum_{k=0}^\infty \frac{\theta^k}{k!} b_k = E b_N, \quad (3.52)
\end{equation}
where $N$ is a Poisson random variable with mean $\theta$. One expects that $B(n) \sim b_n$ as $n \to \infty$ provided the variations of $b_k$ for $|k - n| \leq \text{const} \sqrt{n}$ are small. One possible regularity condition on $b_n$ which implies (3.52) is monotonicity. In the setting of Young diagrams, this monotonicity was established by Johansson (see also [8]): if we define

\begin{equation}
F_n(x_1, \cdots, x_m) = P_{p,n}(\lambda \in \mathcal{P}_n : \lambda_i < x_i, 1 \leq i \leq m), \quad (3.53)
\end{equation}
then for $(x_1, \cdots, x_m) \in R^m$

\begin{equation}
F_{n+1}(x_1, \cdots, x_m) \leq F_n(x_1, \cdots, x_m). \quad (3.54)
\end{equation}
This concludes Theorem 3.3. We remark that convergence of the moments also holds, but it requires pretty delicate tail probability estimates. See Baik, Deift and Johansson [3], Ledoux [26] and Su [40] for large deviations and related results.

### 3.3 CLT in the bulk

In this subsection we investigate the fluctuations in the bulk of the spectrum of the Young diagrams. Just recently, Bogachev and Su [7] proved the classic central limit theorems hold like in the case of random uniform partitions. Define for $0 < x < 2$

\begin{equation}
\Delta_n(x) = \frac{\lambda(\sqrt{n}x) - \sqrt{n}\omega(x)}{\frac{1}{2\pi} \sqrt{\log n}}. \quad (3.55)
\end{equation}

**Theorem 3.5.** Under the Plancherel measure, we have, as $n \to \infty$, for $m \geq 1$ and $0 < x_1 < x_2 < \cdots < x_m < 2$

\begin{equation}
(\Delta_n(x_1), \Delta_n(x_2), \cdots, \Delta_n(x_m)) \overset{d}{\to} (Z_1, Z_2, \cdots, Z_m), \quad (3.56)
\end{equation}
where $Z_1, Z_2, \ldots, Z_m$ are independent random variables and each $Z_i$ is normally distributed with mean zero and variance $\varrho^{-2}(x_i)$ where $\varrho(x) = \frac{1}{\pi} \arccos \frac{x}{2}$.

One can reformulate Theorem 3.5 in coordinates $u, v$. Define for $-2 < u < 2$
\[
\hat{\Delta}_n(u) = \frac{\Lambda(\sqrt{n}u) - \sqrt{n}\Omega(u)}{\frac{1}{\pi} \sqrt{\log n}}.
\]

An elementary geometric argument shows
\[
\hat{\Delta}_n(u) = \frac{1}{2} \varrho(x) \Delta_n(x)(1 + \eta_n),
\]
where $x_n \xrightarrow{P} x$ and $\eta_n \xrightarrow{P} 0$.

**Theorem 3.6.** Under the Plancherel measure, we have as $n \to \infty$ for $m \geq 1$ and $0 < u_1 < u_2 < \cdots < u_m < 2$
\[
(\hat{\Delta}_n(u_1), \hat{\Delta}_n(u_2), \cdots, \hat{\Delta}_n(u_m)) \xrightarrow{d} (\hat{Z}_1, \hat{Z}_2, \cdots, \hat{Z}_m),
\]
where $\hat{Z}_1, \hat{Z}_2, \cdots, \hat{Z}_m$ are independent standard normal random variables.

That the normalization constant is the same for all points is not surprising if one observes a Young diagram looks more symmetric and more balanced in coordinates $u, v$. The proof of Theorem 3.6 is again based on the poissonization and depoissonization technique. We show only the following 1-dimensional case: under the poissonized Plancherel measure, for $0 < x < 2$
\[
\frac{\lambda(\sqrt{\varrho x}) - \sqrt{\varrho \omega(x)}}{\frac{1}{2\pi} \sqrt{\log \varrho}} \xrightarrow{d} N(0, \varrho^{-2}(x)), \quad \varrho \to \infty
\]

Equivalently, for each $y \in \mathbb{R}$,
\[
P^\varrho_\theta(\lambda(\sqrt{\varrho x}) - \sqrt{\varrho \omega(x)} \leq a_\theta(x, y)) \to \Phi_x(y), \quad \varrho \to \infty
\]

where $a_\theta(x, y) = \sqrt{\varrho} (\omega(x) - x) + \frac{\varrho}{2\pi} \sqrt{\log \varrho}$ and $\Phi_x(y)$ denotes the distribution function of $N(0, \varrho^{-2}(x))$.

Let $I_\theta$ be the half infinite interval $[a_\theta(x, y), \infty)$, and let $|I_\theta|$ denote the number of $\lambda_i - i$ it contains. Thus, (3.61) can be rewritten as
\[
P^\varrho_\theta \left( \lambda \in \mathcal{P} : \frac{|I_\theta| - E^\varrho_\theta(|I_\theta|)}{\sqrt{\text{Var}^\varrho_\theta(|I_\theta|)}} \leq \frac{[\sqrt{\varrho x}] - E^\varrho_\theta(|I_\theta|)}{\sqrt{\text{Var}^\varrho_\theta(|I_\theta|)}} \right) \to \Phi_x(y).
\]

At this point, we need the following two lemmas.
Lemma 5. Assume that $\mathcal{X}_t, t \in R_+$ is a family of determinantal point process on the real line with kernel $K_t(x, y)$. Let $I_t, t \in R_+$ be a set of intervals and $|I_t|$ the number of points of $\mathcal{X}_t$ in $I_t$. Assume, in addition,

1. the integral operator $A_t =: K - t \cdot \chi_{I_t}$ is a family of trace class operators;
2. $\text{Var}(|I_t|) = \text{tr}(A_t - A_t^2) \to \infty$ as $t \to \infty$.

Then as $t \to \infty$

$$\frac{|I_t| - E|I_t|}{\sqrt{\text{Var}(|I_t|)}} \xrightarrow{d} N(0, 1).$$

(3.63)

This lemma was first proved by Costin and Lebowitz [10] when $K_t(x, y) \equiv \frac{\sin \pi(x - y)}{\pi(x - y)}$. Soshinikov [36] extended it to a general determinantal random fields and studied at length the Airy, Bessel and sine kernels. Hough et al [20] gave a simple and elegant probabilistic interpretation.

Note that (1) of Lemma 5 is satisfied for many kernels, in particular $J$ in Lemma 4. So it remains to verify (2) and to find out what the expectation $E|I_t|$ and variance $\text{Var}(|I_t|)$ behave asymptotically like.

Lemma 6. Fix $0 < x < 2$ and $y \in R$. We have as $\theta \to \infty$

$$E_p^\theta(|I_\theta|) = \sqrt{\theta} x - y \phi(x) \frac{1}{2\pi} \sqrt{\log \theta} + O(1)$$

and

$$\text{Var}_p^\theta(|I_\theta|) = \frac{1}{4\pi^2} \log \theta(1 + o(1)).$$

(3.64) (3.65)

The proof of Lemma 6 is based on a direct asymptotic analysis of the expressions for the expectation and variance. In so doing, the calculations are quite laborious and heavily use the asymptotics of the Bessel function $J_m(2\sqrt{\theta})$ in various regions of variation of the parameters (see Bogachev and Su [7]).

The multidimensional normal convergence follows similar ideas using Soshinikov’s central limit theorem for linear statistics of the form $\sum \alpha_i|I_{\theta_i}|$ (see Soshinikov [37]). A direct calculation shows that the covariance is 0 between $Z_i$ and $Z_j$ so that $Z_i$ and $Z_j$ are independent of each other. By independence, we have for any $\varepsilon > 0$ and two distinct $u_1, u_2 \in (-2, 2)$

$$\lim_{n \to \infty} P_{p,n}(|\hat{\Delta}_n(u_1) - \hat{\Delta}_n(u_2)| > \varepsilon) = P(|\hat{Z}_1 - \hat{Z}_2| > \varepsilon) > 0. \quad (3.66)$$

Hence, the weak convergence of process is not true at least under the natural choice of the space of continuous functions $C[-2, 2]$ since the necessary condition of tightness breaks down. Analogous remark applies to the process $\Delta_n(u), u \in [0, 2]$, considered in the space $D[0, 2]$ of right continuous function with left limits.
3.4 Global fluctuations

Kerov obtained and announced a global central limit theorem in his short note [25] in 1993. There also contained the scheme of the proof and a number of fruitful ideas. Around 1999 Kerov found a simpler derivation of the theorem and started writing a detailed paper on this subject but had time only to finish the preliminary section. Ivanov and Olshanski [21] reconstructed the proof from Kerov’s unpublished notes.

Consider the modified Chebyshev polynomials

\[ u_k(u) =: U_k\left(\frac{u}{2}\right) = \sum_{j=0}^{[k/2]} (-1)^j \binom{k-j}{j} u^{k-2j}, \quad k = 0, 1, 2, \ldots. \]

Note that

\[ u_k(2\cos \theta) = \frac{\sin(k+1)\theta}{\sin \theta} \]

and

\[ \frac{1}{2\pi} \int_{-2}^{2} u_k(u)u_l(u)\sqrt{4-u^2}du = \begin{cases} 1, & k = l, \\ 0, & k \neq l. \end{cases} \]

**Theorem 3.7.** Let for \( k \geq 1 \)

\[ u_k^{(n)} = \int_{-\infty}^{\infty} u_k(u)(\Lambda(u) - \sqrt{n}\Omega(u))du, \quad (3.67) \]

and let \( \xi_2, \xi_3, \ldots \), stand for a sequence of independent standard normal random variables. Then under the Plancherel measure we have

\[ (u_k^{(n)}, k = 1, 2, \ldots) \xrightarrow{d} \left( \frac{2\xi_{k+1}}{\sqrt{k+1}}, k = 1, 2, \ldots \right). \quad (3.68) \]

The proof used essentially the moment method (see [21]). Consider the random series

\[ \Xi(u) = \sum_{k=1}^{\infty} \xi_{k+1} u_k(u)\sqrt{4-u^2} \pi \sqrt{k+1}. \quad (3.69) \]

This series correctly defines a generalized Gaussian process on the space of distributions with support on \([-2, 2]\). Its trajectories are not ordinary functions but generalized functions, i.e., the functions in the space \((C^\infty(R))'\) of compactly supported distributions on the real line. Informally, Theorem 3.7 claims that for the Plancherel diagram \( \lambda \in \mathcal{P}_n \),

\[ \hat{\Lambda}(u) \sim \Omega(u) + \frac{\Xi(u)}{\sqrt{n}}. \quad (3.70) \]
As we have seen, the fluctuations at the edge are large (of order $n^{1/6}$) and so might present a danger in the integral (3.67). But Kerov’s theorem shows the edge of the spectrum in fact does not give any considerable contribution into the integral fluctuations.

To conclude this section, we remark that there is a striking similarity between random matrices and random partitions. The Tracy-Widom distribution was first discovered in the study of the eigenvalues of random matrices in the Gaussian Unitary Ensemble. A similar result about convergence to a generalized Gaussian process for eigenvalues of random matrices was obtained by Johansson [22]; while Gustavsson [19] established the central limit theorem in the bulk of the spectrum of eigenvalues of random matrices. However, random matrices and random partitions are linked only from the view of point of limiting distributions. It would be interesting to establish a finite identity relation between the eigenvalues of random matrices and the parts of random partitions.

4 Miscellaneous

In the last section we will briefly introduce other natural measures, among which are multiplicative measures and Schur measures. There have been many works on them in the literatures, see Okounkov [31], Vershik [44, 45] and references therein. So far we have only touched the so-called linear partitions; a natural generalization is higher dimensional partitions. We will give the definition of 3-dimensional (plane) partitions in this section. This is a very interesting and new field, which has close link with both multiplicative measures and Schur measures. At the end we will mention applications of the well-known Stein-Chen method to Plancherel measures, a real challenge.

4.1 Multiplicative measures

Consider a sequence of functions $f_k(z), k \geq 1$, analytic in the open disk $D = \{z \in \mathbb{C} : |z| < \gamma \}$, $\gamma = 1$ or $\gamma = \infty$, such that $f_k(0) = 1$ and assume that (i) the Taylor series

\begin{equation}
 f_k(z) = \sum_{j=0}^{\infty} s_k(j) z^j
\end{equation}

have all coefficients $s_k(j) \geq 0$ and (ii) the infinite product

\begin{equation}
 F(z) = \prod_{k=1}^{\infty} f_k(z^k)
\end{equation}
converges in $D$. Then we can define a family of probability measures $P_q$, $q \in (0, \gamma)$, on $\mathcal{P}$ in the following way: put

$$P_q(\lambda \in \mathcal{P} : r_k(\lambda) = j) = \frac{s_k(j)q^k}{f_k(q^k)}, \quad j \geq 0, \quad k \geq 1 \quad (4.3)$$

and assume that different $r_k$ are independent. Thus

$$P_q(\lambda) = \frac{\prod_{k=1}^{\infty} s_k(r_k)}{\mathcal{F}(q)} q^{\lambda\lambda}, \quad \lambda \in \mathcal{P}. \quad (4.4)$$

We call $P_q$ a multiplicative measure with parameter $q$ on $\mathcal{P}$. The generating function $\mathcal{F}(z)$, along with its decomposition $\mathcal{F}(z) = \prod_{k=1}^{\infty} f_k(z^k)$, completely determines such a family. The class of multiplicative measures contain many important examples as discussed in Vershik [45].

Now define the measure $P_n$ on $\mathcal{P}_n$ as follows:

$$P_n(\lambda \in \mathcal{P}_n : r_k(\lambda) = j) = \frac{s_k(j)}{Q_n}, \quad j \geq 0, \quad k \geq 1 \quad (4.5)$$

and

$$P_n(\lambda) = \frac{\prod_{k=1}^{\infty} s_k(r_k)}{Q_n}, \quad \lambda \in \mathcal{P}_n, \quad (4.6)$$

where

$$Q_n = \sum_{\lambda \in \mathcal{P}_n} \prod_{k=1}^{\infty} s_k(r_k) \quad (4.7)$$

Similarly to Lemma 1, we have, for any $q \in (0, \gamma)$ and $n \geq 0$, (1) $P_n$ is the conditional probability measure induced on $\mathcal{P}_n$ by $P_q$; (2) $P_q$ is a convex combination of measures $P_n$.

On the other hand, $P_n$ is no longer uniform and $\lambda$ is not identical in distribution to $\lambda'$. It would be interesting to investigate the limit shape and fluctuations under $(P_n, \mathcal{P}_n)$. See Vershik [45], Vershik and Yakubovich [48] for related works.

### 4.2 Schur measures

A generalization of the Plancherel measure is defined as follows. Introduce the following function of $\lambda$

$$P_{\text{sch}}^x(\lambda) = \frac{1}{Z_i} h_\lambda(x)s_\lambda(\bar{x}), \quad (4.8)$$
where $\bar{x}$ is complex conjugate of $x$, $s_{\lambda}$ are the Schur function in auxiliary variables $x_1, x_2, \cdots$ and $Z$ is the sum in the Cauchy identity for the Schur functions

$$Z = \sum_{\lambda \in \mathcal{P}} s_\lambda(x)s_\lambda(\bar{x}) = \prod_{i,j} \frac{1}{1-x_i \bar{x}_j} < \infty. \quad (4.9)$$

It is clear $P_{x_{sch}}$ is a probability measure on $\mathcal{P}$ which we call the Schur measure. These measure (depending on countably many parameters) are in fact objects of fundamental importance, with profound connections to many central themes of mathematics and physics, including integrable system.

It is convenient to introduce another parameters for the Schur measure

$$t_k = \frac{1}{k} \sum_i x_i^k, \quad \bar{t}_k = \frac{1}{k} \sum_i \bar{x}_i^k, \quad k = 1, 2, \cdots \quad (4.10)$$

In particular,

$$s_\lambda(x) = \sum_\rho \chi_\lambda(\rho) \prod_k \frac{t_k^{r_k(\rho)}}{(r_k(\rho))!}, \quad (4.11)$$

where $\chi_\lambda(\rho)$ is the character of a permutation $\rho$ in the representation labeled by $\lambda$.

If we set $t = \bar{t} = (\sqrt{\theta}, 0, 0, \cdots)$ then $P_{x_{sch}}$ specializes to the poissonized Plancherel measure $P^\theta_{P}$. The fact of the matter is that the correlation function of the Schur measure has still a determinantal structure: for any finite set $X \subset \mathbb{Z}$

$$P_{x_{sch}}(\lambda \in \mathcal{P} : X \subset D(\lambda)) = \det(K(x_i, x_j))_{x_i, x_j \in X} \quad (4.12)$$

for a certain kernel $K$, which has a nice generating function, and thus a contour integral representation, in terms of the parameters. From this one can use the steepest descent method to analyze the asymptotics of the correlation function and limit shapes. See Okounkov [31] for further discussions.

### 4.3 Plane partitions

A plane partition is by definition a 2-dimensional array of nonnegative numbers

$$\pi = (\pi_{ij}), \quad i, j = 1, 2, \cdots \quad (4.13)$$
that are nonincreasing as a function of both $i$ and $j$ and such that its size

$$|\pi| := \sum_{i,j} \pi_{ij} \tag{4.14}$$

is finite. For example,

$$\pi = \begin{pmatrix} 5 & 3 & 2 & 1 \\ 4 & 3 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 \end{pmatrix}$$

is a plane partition of size 29, where the entries that are not shown are zeros. The plot of the function $(x, y) \mapsto \pi_{\lceil x \rceil, \lceil y \rceil}$, $x, y > 0$ is a 3-dimensional Young diagram with the volume $|\pi|$.

Let $P_{3,n}$ be the set of all plane partitions of $n$, and let $P_3$ be the set of all plane partitions, i.e., $P_3 = \bigcup_{n=0}^{\infty} P_{3,n}$. Denote by $p_3(n)$ the number of all plane partitions of $n$, then the generating function is

$$\sum_{n=0}^{\infty} p_3(n) q^n = \prod_{k=1}^{\infty} \frac{1}{(1-q^k)^k}. \tag{4.15}$$

This coincides $\mathcal{F}(q)$ in (4.2) with $f_k(q) = \frac{1}{(1-q^k)^k}$, $k \geq 1$.

Define a probability measure $P_q$ on $P_3$ by

$$P_q(\pi) = \prod_{k=1}^{\infty} \frac{1}{(1-q^k)^k q^{|\pi|}}. \tag{4.16}$$

The restriction $P_{3,n}$ of $P_q$ to $P_{3,n}$ is a uniform measure on $P_{3,n}$. On the other hand, to a plane partition $\pi$, assign a sequence $\{\lambda(t)\}$ of its diagonal slices (linear partitions). In this way one can use the Schur process and correlation functions to study the limit shape and fluctuations of 3-dimensional diagrams. This is a very interesting probability model; only a few of papers devoted to this object. See Cerf and Kenyon [9], Ferrai and Spohn [14], Okounkov and Reshetikhin [32], Vershik and Yakubovich [48].

Up to now, there is no idea what happens in other dimensions. MacMahon conjectured that the generating function for the 4-dimensional partitions has the form

$$\prod_{k=1}^{\infty} \frac{1}{(1-z^k)^{\frac{k}{2}}}.$$ 

However, this is wrong.
4.4 Applications of the Stein-Chen method

One of the most important developments in probability theory in the last decades was the Stein-Chen method. The Stein-Chen method is a highly original technique and has been useful in proving normal and Poisson approximation theorems in probability problems with limited information such as the knowledge of only a few moments of the random variables. Fulman [18] initiated the study of the Plancherel measure by the Stein-Chen method. But as Fulman remarked in [18], it is very interesting and challenging to use the Stein-Chen method to understand the Theorem 3.3, giving explicit bounds on the convergence to the Tracy-Widom type distributions.

References


Asymptotic analysis of random partitions


