THE SEQ, LINEAR REGULARITY, AND THE STRONG CHIP FOR AN INFINITE SYSTEM OF CLOSED CONVEX SETS IN NORMED LINEAR SPACES

CHONG LI*, K. F. NG‡, AND T. K. PONG‡

Abstract. We consider a (finite or infinite) family of closed convex sets with nonempty intersection in a normed space. A property relating their epigraphs with their intersection's epigraph is studied, and its relations to other constraint qualifications (such as the linear regularity, the strong CHIP, and Jameson's (G)-property) are established. With suitable continuity assumption we show how this property can be ensured from the corresponding property of some of its finite subfamilies.

Key words. system of closed convex sets, interior-point condition, strong conical hull intersection property

AMS subject classifications. Primary, 90C34, 90C25; Secondary, 52A05, 41A29

DOI. 10.1137/060652087

1. Introduction. In dealing with a lower semicontinuous extended real-valued function \( \phi \) defined on a Banach space (or more generally, a normed linear space) \( X \), it is not only natural but also useful to study its relation with the epigraph \( \text{epi } \phi := \{(x, r) \in X \times \mathbb{R} : \phi(x) \leq r\} \), which is clearly a closed convex subset of the product \( X \times \mathbb{R} \). Conversely, given a nonempty closed convex set \( C \) in \( X \), let \( \sigma_C \) denote the support function of \( C \), which is defined by

\[ \sigma_C(x^*) = \sup \{ \langle x^*, x \rangle : x \in C \}, \quad x^* \in X^*, \]

where \( X^* \) denotes the dual space of \( X \) and \( \langle x^*, x \rangle = x^*(x) \), the value of the functional \( x^* \) at \( x \). Thus \( \sigma_C \) is a \( w^* \)-lower semicontinuous convex function and \( \text{epi } \sigma_C \) is a \( w^* \)-closed convex subset of \( X^* \times \mathbb{R} \). In this paper, we shall apply this simple duality between \( C \) and \( \text{epi } \sigma_C \) to study several important aspects (including the regularity, the strong conical hull intersection property (CHIP), Jameson’s property (G), and other constraint qualifications) for a CCS-system \( \{C_i : i \in I\} \), by which we mean a family of closed convex sets in \( X \) with nonempty intersection \( \cap_{i \in I} C_i \), where \( I \) is an index set.

For the case when \( I \) is finite, the concept of regularity and its quantitative versions were introduced in [4, 5, 6] by Bauschke, Borwein, and Li and were utilized to establish norm or linear convergence results. The concept of the strong CHIP was introduced by Deutsch, Li, and Ward in [12], and was utilized in [13], as well as in [9, 24, 25], to reformulate certain optimization problems with constraints. All the works cited above were in the Hilbert space or Euclidean space setting. The concept of property (G) was introduced by Jameson [17] for a pair of cones, and was utilized to give a duality

*Received by the editors February 13, 2006; accepted for publication (in revised form) January 9, 2007; published electronically August 8, 2007.
http://www.siam.org/journals/siopt/18-2/65208.html
†Department of Mathematics, Zhejiang University, Hangzhou 310027, People’s Republic of China (cli@zju.edu.cn). This author was supported in part by the National Natural Science Foundation of China (grant 10671175) and Program for New Century Excellent Talents in University.
‡Department of Mathematics, Chinese University of Hong Kong, Hong Kong, People’s Republic of China (kfgn@math.cuhk.edu.hk, tkpong@gmail.com). K. F. Ng was supported by a direct grant (CUHK) and an Earmarked Grant from the Research Grant Council of Hong Kong.

643
characterization of the linear regularity. In improving the partial results obtained by Lewis and Pang (see [23, 31]) and by Bauschke, Borwein, and Li [6], Jameson’s result was extended by Ng and Yang [30] to the general case (without the additional assumption that each $C_i$ is a cone), but still only for finite $I$. For the case when $X$ is a Hilbert space, the same result was also independently obtained by Bakan, Deutsch, and Li in [3].

In this paper, we extend the above mentioned results to cover the case when $I$ is infinite. From both theoretical and application points of view, the extension from the finite case to the infinite one is of importance. Regarding the strong CHIP, such an extension has already been done rather successfully with many interesting applications (see, for example, [27, 28]). Our investigation is made through the consideration of epigraphs, and in particular by virtue of that, of a new constraint qualification, defined below. Our works in this connection are inspired by the recent works of Jeyakumar and his collaborates (see [7, 18, 19, 20, 22], for example), who made use of epigraphs to provide sufficient conditions to ensure the strong CHIP (for a finite collection of closed convex sets), and study systems of convex inequalities. We say that a CCS-system $\{C_i : i \in I\}$ satisfies the SECQ (sum of epigraphs constraint qualification) if

$$\text{epi}\sigma_{\cap_{i \in I} C_i} = \sum_{i \in I} \text{epi}\sigma_{C_i}.$$ 

In section 4, we study the interrelationship between this property and other constraint qualifications, especially the linear regularity. Also, since this property is stronger than the strong CHIP (and the converse holds in some important cases; see Theorem 3.1), it is both natural and useful to inquire whether or not the sufficient conditions originally provided to ensure the strong CHIP can in fact ensure the SECQ. In this connection, let us recall the following results proved in [27] (see, in particular, Theorems 4.1 and 5.1 therein). For the remainder of this section, we assume that $I$ is a compact metric space (needless to say, if $I$ is finite, then it is compact under the discrete metric) and see the next section for definitions of the undefined terms.

**Theorem 1.1.** Consider the CCS-system $\{D, C_i : i \in I\}$. Suppose that

(a) $D$ is of finite dimension;

(b) the set-valued map $i \mapsto \text{aff } D \cap C_i$ is lower semicontinuous on $I$;

(c) there exist $x_0 \in D \cap (\cap_{i \in I} C_i)$ and $r > 0$ such that

$$\text{aff } D \cap B(x_0, r) \subseteq C_i \quad \text{for each } i \in I;$$

(d) the pair $\{\text{aff } D, C_i\}$ has the strong CHIP for each $i \in I$.

Then $\{D, C_i : i \in I\}$ has the strong CHIP.

**Theorem 1.2.** Consider the CCS-system $\{D, C_i : i \in I\}$. Suppose that

(a) $D$ is of finite dimension $l$;

(b) the set-valued map $i \mapsto \text{aff } D \cap C_i$ is lower and upper semicontinuous on $I$;

(c) for any finite subset $J$ of $I$ with number of elements $|J| \leq l$, there exist $x_0 \in D$ and $r > 0$ such that

$$\text{aff } D \cap B(x_0, r) \subseteq C_i \quad \text{for each } i \in J;$$

(d) for any finite subset $J$ of $I$, the subsystem $\{D, C_j : j \in J\}$ has the strong CHIP.
Let \( f \) at only for subsystems can be dropped and that (d) can be weakened to require the strong CHIP to hold for general locally convex spaces. We use \( A \) polar cone which coincides with the polar \( A \) and its complement.

x with center \( x \) and radius \( \epsilon \). For a set \( A \) in \( X \) (or in \( \mathbb{R}^n \)), the interior (resp., relative interior, closure, convex hull, convex cone hull, linear hull, affine hull, boundary) of \( A \) is denoted by \( \text{int} A \) (resp., \( \text{ri} A \), \( \text{co} A \), \( \text{cone} A \), \( \text{span} A \), \( \text{aff} A \), \( \text{bd} A \)), and the negative polar cone \( A^\ominus \) is the set defined by

\[
A^\ominus = \{ x^* \in X^* : \langle x^*, z \rangle \leq 0 \text{ for all } z \in A \},
\]

which coincides with the polar \( A^\circ \) of \( A \) when \( A \) is a cone. The normal cone of \( A \) at \( z_0 \) is denoted by \( N_A(z_0) \) and defined by \( N_A(z_0) = (A - z_0)^\ominus \). Let \( Z \) be a closed convex nonempty subset of \( X \). The interior and the boundary of \( A \) relative to \( Z \) are, respectively, denoted by \( \text{rint}_Z A \) and \( \text{bd}_Z A \); they are defined to be, respectively, the interior and the boundary of the set \( \text{aff} Z \cap A \) in the metric space \( \text{aff} Z \). Thus, a point \( z \in \text{rint}_Z A \) if and only if there exists \( \epsilon > 0 \) such that

\[
z \in (\text{aff} Z) \cap B(z, \epsilon) \subseteq A,
\]

while \( z \in \text{bd}_Z A \) if and only if \( z \in \text{aff} Z \) and, for any \( \epsilon > 0 \), \((\text{aff} Z) \cap B(z, \epsilon) \) intersects \( A \) and its complement. For a closed subset \( A \) of \( X \), the indicator function \( \delta_A \) and the support function \( \sigma_A \) of set \( A \) are, respectively, defined by

\[
\delta_A(x) := \begin{cases} 0, & x \in A, \\ \infty, & \text{otherwise} \end{cases}
\]

and

\[
\sigma_A(x^*) := \sup_{x \in A} \langle x^*, x \rangle \text{ for each } x^* \in X^*.
\]

Let \( f \) be a proper lower semicontinuous extended real-valued function on \( X \). The domain of \( f \) is denoted by \( \text{dom} f := \{ x \in X : f(x) < +\infty \} \). Then the subdifferential of \( f \) at \( x \in \text{dom} f \), denoted by \( \partial f(x) \), is defined by

\[
\partial f(x) := \{ z^* \in X^* : f(x) + \langle z^*, y - x \rangle \leq f(y) \text{ for all } y \in X \}.
\]

Let \( f, g \) be proper functions, respectively, defined on \( X \) and \( X^* \). Let \( f^*, g^* \) denote their conjugate functions, that is,

\[
f^*(x^*) := \sup \{ \langle x^*, x \rangle - f(x) : x \in X \} \text{ for each } x^* \in X^*,
\]

\[
g^*(x) := \sup \{ \langle x^*, x \rangle - g(x^*) : x^* \in X^* \} \text{ for each } x \in X.
\]
The epigraph of a function $f$ on $X$ is denoted by $\text{epi} f$ and defined by

$$\text{epi} f := \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}.$$ 

Then, for proper lower semicontinuous extended real-valued convex functions $f_1$ and $f_2$ on $X$, we have

$$f_1 \leq f_2 \iff f_1^* \geq f_2^* \iff \text{epi} f_1^* \subseteq \text{epi} f_2^*,$$

where the forward direction of the first arrow and the second equivalence are easy to verify, while the backward direction of the first arrow is standard (cf. [35, Theorem 2.3.3]).

For closed convex sets $A, B$, the following assertions are well known and easy to verify:

$$\sigma A = \delta_A^*,$$

$$N_A(x) = \partial \delta_A(x) \text{ for each } x \in A,$$

$$\sigma_A(x^*) = \langle x^*, x \rangle \iff x^* \in N_A(x) \iff (x^*, \langle x^*, x \rangle) \in \text{epi} \sigma_A \text{ for each } x \in A,$$

and

$$\text{epi} \sigma_A \subseteq \text{epi} \sigma_B \text{ if } A \supseteq B.$$

Let $\{A_i : i \in J\}$ be a family of subsets of $X$ containing the origin. The set $\sum_{i \in J} A_i$ is defined by

$$\sum_{i \in J} A_i = \left\{ \begin{array}{ll}
\{ \sum_{i \in J_0} a_i : & a_i \in A_i, \quad \emptyset \neq J_0 \subseteq J \text{ being finite} \} & \text{if } J \neq \emptyset, \\
\{0\} & \text{if } J = \emptyset.
\end{array} \right.$$ 

Let $I$ be an arbitrary index set. The following concept of the strong CHIP plays an important role in optimization theory (see [3, 6, 9, 10, 11, 33]) and is due to [12, 13] in the case when $I$ is finite and to [26, 27] in the case when $I$ is infinite.

**Definition 2.1.** Let $\{C_i : i \in I\}$ be a collection of convex subsets of $X$. The collection is said to have

(a) the strong CHIP at $x \in \cap_{i \in I} C_i$, if $N_{\cap_{i \in I} C_i}(x) = \sum_{i \in I} N_{C_i}(x)$, that is,

$$\left( \cap_{i \in I} C_i - x \right)^\ominus = \sum_{i \in I} (C_i - x)^\ominus;$$

(b) the strong CHIP if it has the strong CHIP at each point of $\cap_{i \in I} C_i$;

(c) the SECQ if $\text{epi} \sigma_{\cap_{i \in I} C_i} = \sum_{i \in I} \text{epi} \sigma_{C_i}$.

Note that $N_{\cap_{i \in I} C_i}(x) \supseteq \sum_{i \in I} N_{C_i}(x)$ holds automatically for $x \in \cap_{i \in I} C_i$. Hence $\{C_i : i \in I\}$ has the strong CHIP at $x$ if and only if

$$N_{\cap_{i \in I} C_i}(x) \subseteq \sum_{i \in I} N_{C_i}(x).$$

To establish a similar property regarding the SECQ, we first need to extend [16, part X, Theorem 2.4.4] to the setting of normed linear spaces. We recall that for an arbitrary function $f$ defined on $X^*$, we define $\overline{\text{co} f}^\circ$ by (cf. [35, page 63])

$$\text{epi}(\overline{\text{co} f}^\circ) := \overline{\text{co}(\text{epi} f)^\circ}. $$
LEMMA 2.2. Let \( \{g_i : i \in I\} \) be a family of proper convex lower semicontinuous functions on \( X \) with \( \sup_{i \in I} g_i(x_0) < +\infty \) for some \( x_0 \in X \). Then for all \( y^* \in X^* \),
\[
(\sup_{i \in I} g_i)^*(y^*) = \co(\inf_{i \in I} g_i^*)^{\omega^*}(y^*).
\]

Proof. It is well known (and immediate from the definition of the conjugate) that for a family of proper convex lower semicontinuous functions \( (f_i)_{i \in I} \) on \( X \),
\[
(\inf_{i \in I} f_i)^* = \sup_{i \in I} f_i^*.
\]
Now, since \( g_i \) is proper convex lower semicontinuous for each \( i \in I \), \( g_i^{**} = g_i \) and \( g_i^* \) is proper (cf. [35, Theorem 2.3.3]). Applying (2.5) to functions on \( X\) for the sake of completeness (note that the condition that "sup \( g_i \) is proper" is needed).

\[
(\inf_{i \in I} g_i)^* = \sup_{i \in I} g_i^{**} = \sup_{i \in I} g_i.
\]
From this and the properness assumption on \( \sup_{i \in I} g_i \), we obtain from [35, Theorem 2.3.4] that \( (\inf_{i \in I} g_i)^{**} = \co(\inf_{i \in I} g_i^*)^{\omega^*} \). The result follows. \( \Box \)

The following lemma was stated without proof in [21, page 902]. We give a proof here for the sake of completeness (note that the condition that "sup \( g_i \) is proper" is needed).

LEMMA 2.3. Let \( \{g_i : i \in I\} \) be a system of proper convex lower semicontinuous functions on \( X \) with \( \sup_{i \in I} g_i(x_0) < +\infty \) for some \( x_0 \in X \). Then
\[
(\sup_{i \in I} g_i)^* = \co(\bigcup_{i \in I} \text{epi} g_i^{\omega^*}).
\]
Proof. For the family \( \{g_i^* : i \in I\} \) of proper convex lower semicontinuous functions on \( X \) we have \( \bigcup_{i \in I} \text{epi} g_i^* \subset \text{epi}(\inf_{i \in I} g_i^*) \subset \co(\bigcup_{i \in I} \text{epi} g_i^{\omega^*}) \). This implies that \( \text{epi}(\inf_{i \in I} g_i^*)^{\omega^*} = \co(\bigcup_{i \in I} \text{epi} g_i^{\omega^*}) \). The conclusion follows on invoking Lemma 2.2. \( \Box \)

PROPOSITION 2.4. Let \( \{C_i : i \in I\} \) be a collection of closed convex sets in \( X \) with \( C := \cap_{i \in I} C_i \neq \emptyset \). Then
\[
\text{epi} \sigma_C = \sum_{i \in I} \text{epi} \sigma_{C_i}^{\omega^*}.
\]
Proof. Note that \( \sup_{i \in I} \delta_{C_i} = \delta_C \) and that \( \sigma_C = \delta_C^* \) by (2.2). It follows that \( \text{epi} \sigma_C = \text{epi}(\sup_{i \in I} \delta_{C_i})^* \). Consequently, by (2.6) and (2.2), one has that
\[
\text{epi} \sigma_C = \co(\bigcup_{i \in I} \text{epi} \delta_{C_i}^{\omega^*}) = \co(\bigcup_{i \in I} \text{epi} \sigma_{C_i}^{\omega^*}) = \sum_{i \in I} \text{epi} \sigma_{C_i}^{\omega^*},
\]
where the last equality holds because \( \text{epi} \sigma_{C_i} \) is clearly a cone for each \( i \in I \). \( \Box \)

COROLLARY 2.5. Let \( \{C_i : i \in I\} \) be a collection of closed convex sets in \( X \) with \( C := \bigcap_{i \in I} C_i \neq \emptyset \). Then the following equivalences are true:
\[
\{C_i : i \in I\} \text{ satisfies the SEQC} \iff \sum_{i \in I} \text{epi} \sigma_{C_i} \text{ is } w^*-closed
\]
\[
\iff \text{epi} \sigma_C \subseteq \sum_{i \in I} \text{epi} \sigma_{C_i}.
\]

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.
The following simple proposition states that the SECQ is invariant under translation.

**Proposition 2.6.** Let \( \{C_i : i \in I\} \) be a family of closed convex sets in \( X \). Suppose that \( C := \bigcap_{i \in I} C_i \neq \emptyset \). Then \( \{C_i : i \in I\} \) satisfies the SECQ if and only if the system \( \{C_i - x : i \in I\} \) does for each \( x \in X \).

**Proof.** Let \( x \in X \). Note that
\[
(y^*, \alpha) \in \text{epi} \sigma_{C-x} \iff (y^*, \alpha + \langle y^*, x \rangle) \in \text{epi} \sigma_C
\]
and
\[
(y^*, \alpha) \in \sum_{i \in I} \text{epi} \sigma_{C_i-x} \iff (y^*, \alpha + \langle y^*, x \rangle) \in \sum_{i \in I} \text{epi} \sigma_{C_i}.
\]

Hence the conclusion follows from Corollary 2.5. \( \square \)

We will need the following notion of semicontinuity of set-valued maps in sections 4 and 5. Readers may refer to standard texts such as [1, 32].

**Definition 2.7.** Let \( Q \) be a metric space. Let \( X \) be a normed linear space and let \( t_0 \in Q \). A set-valued function \( F : Q \to 2^X \) is said to be
(i) lower semicontinuous at \( t_0 \) if, for any \( y_0 \in F(t_0) \) and any \( \epsilon > 0 \), there exists a neighborhood \( U(t_0) \) of \( t_0 \) such that \( B(y_0, \epsilon) \cap F(t) \neq \emptyset \) for each \( t \in U(t_0) \);
(ii) lower semicontinuous on \( Q \) if it is lower semicontinuous at each \( t \in Q \).

The following characterization regarding the lower semicontinuity is a reformulation of the equivalence of (i) and (ii) in [27, Proposition 3.1]. For a closed convex set \( S \) in a normed linear space \( X \), let \( d_S(\cdot) \) denote the distance function of \( S \) defined by \( d_S(x) = \inf \{\|x - y\| : y \in S\} \) for each \( x \in X \). Furthermore, let \( \liminf_{t \to t_0} F(t) \) denote the lower limit of the set-valued function \( F \) at \( t_0 \in Q \) which is defined by
\[
\liminf_{t \to t_0} F(t) := \{z \in X : \exists \{z_t\}_{t \in Q} \text{ with } z_t \in F(t) \text{ such that } z_t \to z \text{ as } t \to t_0\}.
\]

**Proposition 2.8.** Let \( Q \) be a metric space. Let \( F : Q \to 2^X \) be a set-valued function and let \( t_0 \in Q \). Then the following statements are equivalent:
(i) \( F \) is lower semicontinuous at \( t_0 \);
(ii) For any \( y_0 \in F(t_0) \), \( \liminf_{t \to t_0} d_{F(t)}(y_0) = 0 \);
(iii) \( F(t_0) \subseteq \liminf_{t \to t_0} F(t) \).

We collect some properties of the lower limit of the set-valued function \( F \) at \( t_0 \in Q \) in the following proposition. The first property is direct from the definition and the second property is a direct consequence of [32, Proposition 4.15].

**Proposition 2.9.** Let \( Q \) be a metric space and \( X \) a normed linear space. Let \( F : Q \to 2^X \) be a set-valued function such that \( F(t) \) is convex for each \( t \in Q \). Let \( t_0 \in Q \). Then \( \liminf_{t \to t_0} F(t) \) is convex.

Moreover, if \( X \) is finite dimensional and \( B \) is a compact subset contained in \( \text{int}(\liminf_{t \to t_0} F(t)) \) (e.g., \( F \) is lower semicontinuous and \( B \) is a compact set contained in \( \text{int}(F(t_0)) \)), then there exists a neighborhood \( U(t_0) \) of \( t_0 \) such that \( B \subseteq \text{int}(F(t)) \) for each \( t \in U(t_0) \).

**3. The strong CHIP and the SECQ.** Recall that \( I \) is an arbitrary index set and \( \{C_i : i \in I\} \) is a collection of nonempty closed convex subsets of \( X \). We denote \( \bigcap_{i \in I} C_i \) by \( C \) and assume that \( 0 \in C \) throughout the whole paper. The following theorem describes a relationship between the strong CHIP and the SECQ for the system \( \{C_i : i \in I\} \). Results in this section are folklore; we include their proofs here for the sake of completeness.
THEOREM 3.1. If \( \{ C_i : i \in I \} \) satisfies the SECQ, then it has the strong CHIP; the converse conclusion holds if \( \text{dom} \sigma_C \subseteq \text{Im} \partial \delta_C \), that is, if

\[
(3.1) \quad \text{dom} \sigma_C \subseteq \bigcup_{x \in C} N_C(x).
\]

Proof. Suppose that \( \{ C_i : i \in I \} \) satisfies the SECQ. Let \( x \in C \) and \( y^* \in N_C(x) \). Then \( (y^*, \langle y^*, x \rangle) \in \text{epi} \sigma_C \) by (2.4). Hence, if \( \{ C_i : i \in I \} \) satisfies the SECQ, one can apply (2.7) to express \( (y^*, \langle y^*, x \rangle) \) as

\[
(y^*, \langle y^*, x \rangle) = \sum_{j \in J} (y_j^*, u_j)
\]

for some finite set \( J \subseteq I \) and \( (y_j^*, u_j) \in \text{epi} \sigma_{C_j}(x) \) for each \( j \in J \). Then \( \langle y_j^*, x \rangle \leq \sigma_{C_j}(y_j^*) \leq u_j \) for all \( j \in J \) and \( \sum_{j \in J} \langle y_j^*, x \rangle = \sum_{j \in J} u_j \). It follows that \( \langle y_j^*, x \rangle = u_j \) for each \( j \in J \) and hence that \( y_j^* \in N_{C_j}(x) \) by (2.4). Therefore \( y^* \in \sum_{i \in I} N_{C_i}(x) \).

Thus the strong CHIP for \( \{ C_i : i \in I \} \) is proved.

Conversely, assume that \( \text{dom} \sigma_C \subseteq \text{Im} \partial \delta_C \) and that the strong CHIP for \( \{ C_i : i \in I \} \) is satisfied. We have to show that

\[
(3.2) \quad \text{epi} \sigma_C \subseteq \bigcup_{i \in I} \text{epi} \sigma_{C_i}.
\]

To do this, let \( (y^*, \alpha) \in \text{epi} \sigma_C \), that is, \( \alpha \geq \sigma_C(y^*) \). Hence \( y^* \in \text{dom} \sigma_C \). Then, by the assumption and (2.4), there exists \( x \in C \) such that \( y^* \in N_C(x) \). By the strong CHIP assumption, it follows that there exist a finite index set \( J \subseteq I \) and \( y_j^* \in N_{C_j}(x) \) for each \( j \in J \) such that

\[
(3.3) \quad y^* = \sum_{j \in J} y_j^*.
\]

Note that, for each \( j \in J \), \( \sigma_{C_j}(y_j^*) \leq \langle y_j^*, x \rangle \) because \( y_j^* \in N_{C_j}(x) \). Since \( \alpha \geq \langle y^*, x \rangle = \sum_{j \in J} \langle y_j^*, x \rangle \), there exists a set \( \{ \alpha_j : j \in J \} \) of real numbers such that

\[
\alpha = \sum_{j \in J} \alpha_j \quad \text{and} \quad \sigma_{C_j}(y_j^*) \leq \langle y_j^*, x \rangle \leq \alpha_j \quad \text{for each} \ j \in J.
\]

This implies that \( (y_j^*, \alpha_j) \in \text{epi} \sigma_{C_j} \) for each \( j \) and \( (y^*, \alpha) \in \sum_{i \in I} \text{epi} \sigma_{C_i} \), thanks to (3.3). Hence (3.2) is proved.

Given a closed convex set \( C \) and a finite dimensional linear subspace \( Y \) containing \( C \), recall from [2, section 2.4] and [14] that \( C \subseteq Y \) is said to be continuous if \( y^* \mapsto \sup \{ \langle y^*, y \rangle : y \in C \} \) is continuous on \( Y^* \setminus \{0\} \). Here, the continuity at \( y_0^* \) with \( \sup \{ \langle y_0^*, y \rangle : y \in C \} = +\infty \) means that for each \( \alpha \in \mathbb{R} \) there exists a neighborhood \( V_0 \) of \( y_0^* \) such that \( \sup \{ \langle v^*, y \rangle : y \in C \} > \alpha \) for all \( v^* \in V_0 \).

For convenience, we use \( x|Z \) to denote the restriction to \( Z \) of the functional \( x^* \in X^* \), where \( Z \) is a linear subspace of \( X \).

PROPOSITION 3.2. Let \( C \) be a nonempty closed convex set in \( X \). Then condition (3.1) holds in each of the following cases:

(i) There exists a weakly compact convex set \( D \) and a closed convex cone \( K \) such that \( C = D + K \).

(ii) \( \dim C < \infty \), \( \text{Im} \partial \delta_C \) is convex, and \( C \) is a continuous set as a subset of \( \text{span} C \).
Proof. (i) Suppose that (i) holds and let $y^* \in \text{dom } \sigma_C$. Then since $K$ is a cone,

$$\sup_{d \in D} \langle y^*, d \rangle = \sup_{d \in D} \langle y^*, d \rangle + \sup_{k \in K} \langle y^*, k \rangle = \sup_{d \in D, k \in K} \langle y^*, d + k \rangle = \sigma_C(y^*) < +\infty.$$  

(3.4)

Since $D$ is weakly compact, there exists $\bar{x} \in D (\subseteq C)$ such that $\langle y^*, \bar{x} \rangle = \sup_{d \in D} \langle y^*, d \rangle$. Thus by (3.4), $\langle y^*, \bar{x} \rangle = \sigma_D(y^*) = \sigma_C(y^*)$. Hence $y^* \in N_C(\bar{x})$ and (3.1) is proved.

(ii) Suppose that (ii) holds. If $C$ is bounded, then $C$ is compact because span $C$ is finite dimensional. Hence (3.1) in this case follows from part (i). If $C$ is the whole space, then (3.1) holds trivially as $\text{dom } \sigma_C = \text{Im } \partial \delta_C = \{0\}$. Thus we may assume that $C$ is a proper and unbounded subset of the finite dimensional space $Z := \text{span } C$. Let $\hat{\delta}_C$ denote the indicator function of the set $C$ as a set in the space $Z$, and let $\hat{\sigma}_C(c^*) := \sup\{\langle c^*, c \rangle : c \in C\}$ for each $c^* \in Z^*$; that is, $\hat{\delta}_C$ and $\hat{\sigma}_C$ are the indicator function and the support function of $C$ as a subset of $Z$, respectively. It is easy to see from definitions that

$$\text{dom } \sigma_C = \{y^* \in X^* : y^*|_Z \in \text{dom } \hat{\sigma}_C\} \quad \text{and} \quad \text{Im } \partial \delta_C = \{y^* \in X^* : y^*|_Z \in \text{Im } \partial \hat{\delta}_C\}.  

(3.5)$$

Now, by the assumption, it follows that $\text{Im } \partial \hat{\delta}_C$ is convex in $Z^*$. We claim that

$$\text{dom } \hat{\sigma}_C \subseteq \text{Im } \partial \hat{\delta}_C. \quad (3.6)$$

Since $C$ is proper, unbounded, and continuous as a subset of $Z$, we know from [2, Proposition 2.4.3] that

$$\text{dom } \hat{\delta}_C \setminus \{0\} = \text{int}(\text{dom } \hat{\sigma}_C) \neq \emptyset. \quad (3.7)$$

On the other hand, since $\text{Im } \partial \hat{\delta}_C$ is a convex set in the finite dimensional Banach space $Z^*$, one has (cf. [35, Proposition 1.2.1.2 and Corollary 1.3.4])

$$\text{int}(\text{Im } \partial \hat{\delta}_C) = \text{int}(\overline{\text{Im } \partial \hat{\delta}_C}). \quad (3.8)$$

Moreover, by [35, Theorem 3.1.2], one has $\text{dom } \hat{\sigma}_C \subseteq \overline{\text{Im } \partial \hat{\delta}_C}$. Consequently, by (3.6)–(3.8), we get that

$$\text{dom } \hat{\sigma}_C \setminus \{0\} = \text{int}(\text{dom } \hat{\sigma}_C) \subseteq \text{int}(\overline{\text{Im } \partial \hat{\delta}_C}) = \text{int}(\text{Im } \partial \hat{\delta}_C) \subseteq \text{Im } \partial \hat{\delta}_C.$$

Therefore claim (3.6) stands because $0 \in \text{Im } \partial \hat{\delta}_C$. Consequently, (3.1) follows from (3.5), (3.6), and the Hahn–Banach theorem. The proof is complete. $\square$

Combining Theorem 3.1 and Proposition 3.2, we immediately have the following corollary.

**Corollary 3.3.** Let $\{C_i : i \in I\}$ be a family of closed convex sets in $X$. Then the strong CHIP and the SECQ are equivalent for $\{C_i : i \in I\}$ in each of the following cases.

(i) There exists a weakly compact convex set $D$ and a closed convex cone $K$ such that $C = D + K$.

(ii) $\dim C < \infty$, $\text{Im } \partial \delta_C$ is convex, and $C$ is a continuous set as a subset of span $C$.

**Remark 3.1.** Part (i) was known in some special cases; see [7, Proposition 4.2] for the case when $I$ is a two point set and $D = \{0\}$, and see [20] for the case when $I$ is a finite set and $D = \{0\}$.

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.
4. Linear regularity and the SECQ. Let $I$ be an arbitrary index set and let \( \{C_i : i \in I\} \) be a CCS-system with $0 \in C$, where $C = \bigcap_{i \in I} C_i$ as before. Throughout this section, we shall use $\Sigma^*$ to denote the set $B^* \times \mathbb{R}^+$, where $B^*$ is the closed unit ball of $X^*$, while $\mathbb{R}^+$ consists of all nonnegative real numbers. This section is devoted to a study of the relationship between the linear regularity and the SECQ. We begin with the notion of the linear regularity for the system $\{C_i : i \in I\}$ and two simple lemmas (the first one is easy to verify). Recall that, for a closed convex set $S$ in a normed linear space $X$, $d_S(\cdot)$ denotes the distance function of $S$.

**Definition 4.1.** The system $\{C_i : i \in I\}$ is said to be

(i) linearly regular if there exists a constant $\gamma > 0$ such that

\[
d_C(x) \leq \gamma \sup_{i \in I} d_{C_i}(x) \quad \text{for all } x \in X;
\]

(ii) boundedly linearly regular if, for each $r > 0$, there exists a constant $\gamma_r > 0$ such that

\[
d_C(x) \leq \gamma_r \sup_{i \in I} d_{C_i}(x) \quad \text{for all } x \in rB.
\]

**Lemma 4.2.** Let $\gamma > 0$. Then

\[
\co \bigcup_{i \in I}(\epi \sigma_{C_i} \cap \gamma \Sigma^*)^\circ + \{0\} \times \mathbb{R}^+ = \co \bigcup_{i \in I}(\epi \sigma_{C_i} \cap \gamma \Sigma^*)^\circ.
\]

**Lemma 4.3.** Let $\gamma > 0$ and let $f_\gamma := \gamma d_S$. If $0 \in S$, then

\[
\epi f_\gamma^* = \epi \sigma_S \cap (\gamma B^* \times \mathbb{R}^+).
\]

**Proof.** By conjugation computation rules (cf. [35, Theorem 2.3.1(v) and Proposition 3.8.3(i)]), we have for any $x^* \in X^*$,

\[
f_\gamma^*(x^*) = \gamma(\sigma_S + \delta_{B^*}) \left( \frac{x^*}{\gamma} \right) = \sigma_S(x^*) + \delta_{B^*}(x^*).
\]

Then (4.1) follows immediately.

In the next two theorems, we shall use the graph $\gph f$ of a function $f$ which is defined by

\[
\gph f := \{(x, f(x)) \in X \times \mathbb{R} : x \in \dom f\}.
\]

Clearly, $\gph f \subseteq \epi f$ for a function $f$ on $X$.

**Theorem 4.4.** Let $\gamma > 0$. Then the following conditions are equivalent:

(i) For all $x \in X$, $d_C(x) \leq \gamma \sup_{i \in I} d_{C_i}(x)$.

(ii) $\epi \sigma_C \cap \Sigma^* \subseteq \co \bigcup_{i \in I}(\epi \sigma_{C_i} \cap \gamma \Sigma^*)^\circ$.

(iii) $\gph \sigma_C \cap \Sigma^* \subseteq \co \bigcup_{i \in I}(\epi \sigma_{C_i} \cap \gamma \Sigma^*)^\circ$.

**Proof.** By Lemmas 2.3 and 4.3, one has that

\[
\epi \left( \sup_{i \in I} d_{C_i} \right)^* = \co \bigcup_{i \in I} \epi d_{C_i}^* = \co \bigcup_{i \in I}(\epi \sigma_{C_i} \cap \Sigma^*)^\circ.
\]

Noting that $\epi \sigma_S$ is a cone, we see that the equivalence of (i) and (ii) follows from (2.1) and Lemma 4.3.
By Lemma 4.2, (iii) implies that
\[ \text{epi}\sigma_C \cap \Sigma^* \subseteq \text{gph}\sigma_C \cap \Sigma^* + \{0\} \times \mathbb{R}^+ \]
\[ \subseteq \text{co}\bigcup_{i \in I} (\text{epi}\sigma_C, \cap \gamma \Sigma^*)^w + \{0\} \times \mathbb{R}^+ \]
\[ = \text{co}\bigcup_{i \in I} (\text{epi}\sigma_C, \cap \gamma \Sigma^*)^w. \]

Therefore (iii) \( \Rightarrow \) (ii). Since (ii) \( \Rightarrow \) (iii) is obvious, the proof is complete. \( \square \)

We give a simple application of our new characterization of the linear regularity in Theorem 4.4. The following theorem includes an important characterization of the linear regularity of finitely many closed convex sets in a Banach space, given in [30, Theorem 4.2].

**Theorem 4.5.** Let \( \gamma > 0 \) and suppose that \( X \) is a Banach space. Consider the following statements:

(i) For all \( x \in X \), \( d_C(x) \leq \gamma \sup_{i \in I} d_{C_i}(x) \).

(ii) For all \( x \in C \), \( N_C(x) \cap B^* \subseteq \text{co}\bigcup_{i \in I} (N_{C_i}(x) \cap \gamma B^*)^w. \)

Then (ii) implies (i). If we assume further that \( I \) is a compact metric space and \( i \mapsto C_i \) is lower semicontinuous, then (i) and (ii) are equivalent. In particular, when \( I \) is finite, (i), (ii), (iii), and (iii) are equivalent, where (ii) and (iii) are defined in the following:

(iii) For all \( x \in C \), \( N_C(x) \cap B^* \subseteq \text{co}\bigcup_{i \in I} (N_{C_i}(x) \cap \gamma B^*). \)

(iii) For all \( x \in C \) and for all \( x^* \in N_C(x) \cap B^* \), there exist \( x^*_i \in N_{C_i}(x), i \in I, \) such that \( \sum_{i \in I} \|x^*_i\| \leq \gamma \) and \( x^* = \sum_{i \in I} x^*_i. \)

**Remark 4.1.** Let \( \rho > 0 \) and recall from [3, 30] that the collection \( \{D_1, \ldots, D_m\} \) in \( X \) is said to have property \( (G_\rho) \) if

\[ \left( \sum_{i=1}^m D_i \right) \cap B \subseteq \bigcup_{i=1}^m \left( D_i \cap \frac{1}{\rho} B \right). \]

Clearly, when \( I \) is finite, there exists \( \gamma > 0 \) such that condition (iii) above holds if and only if the strong CHIP holds for all \( x \in C \) and there exists \( \rho > 0 \) such that, for each \( x \in C \), \( \{N_{C_i}(x) : i \in I\} \) has the property \( (G_\rho) \) in \( X^*. \)

**Proof.** (ii) \( \Rightarrow \) (i). In view of Theorem 4.4, to establish (i), it is sufficient to show that

\[ \text{gph}\sigma_C \cap \Sigma^* \subseteq \text{co}\bigcup_{i \in I} (\text{epi}\sigma_C, \cap \gamma \Sigma^*)^w. \]

To do this, let \( (y^*, \sigma_C(y^*)) \in \text{gph}\sigma_C \cap \Sigma^* \). We have to show that

\[ (y^*, \sigma_C(y^*)) \in \text{co}\bigcup_{i \in I} (\text{epi}\sigma_C, \cap \gamma \Sigma^*)^w. \]

Since the set on the right-hand side of (4.2) obviously contains the origin, we may suppose without loss of generality that \( y^* \neq 0 \).

Consider first the case when \( y^* \in \text{Im}\partial h_C \). Then \( y^* \in N_C(x) \cap B^* \) for some \( x \in C \) by (2.3). Thus one can apply (ii) to find a net \( \{\tilde{y}^*_i\} \) with \( w^* \)-limit \( y^* \) such that for each \( V, \tilde{y}^*_i \) is representable as

\[ \tilde{y}^*_i = \sum_{i \in J_V} \lambda_i y^*_i \]
for some finite index set $J_V \subseteq I$, $y^*_i \in N_{C_i}(x) \cap B^*$, $i \in J_V$, and $\lambda_i \in [0, 1]$ with $\sum_{i \in J_V} \lambda_i = 1$. Using (2.4) again, we obtain $(y^*_i, (y^*_i, x)) \in \text{epi} \sigma_{C_i}$ for each $i \in J_V$. In $w^*$-limits, it follows that
\[
(y^*, (y^*, x)) = \lim_{V} (\tilde{y}^*_V, (\tilde{y}^*_V, x)) = \lim_{V} \gamma \sum_{i \in J_V} \lambda_i (y^*_i, (y^*_i, x));
\]
hence,
\[
(4.3) \quad (y^*, (y^*, x)) \in \operatorname{co} \bigcup_{i \in I} (\text{epi} \sigma_{C_i} \cap \gamma \Sigma^*)_w^*;
\]
and, in particular, (4.2) holds provided that $y^* \in \text{Im} \partial \delta_B$ and $\|y^*\| \leq 1$. For the general case (that is, we do not assume that $y^* \in \text{Im} \partial \delta_B$), by [35, Theorem 3.1.4(ii)], there exists a sequence $(y^*_n, y^*_n) \in \text{gph} \partial \delta_B$ such that $y^*_n$ converges to $y^*$ in norm and $\sigma_C(y^*_n)$ converges to $\sigma_C(y^*)$. Note that by (2.4), we have $(y^*_n, (y^*_n, y^*_n)) \in \text{gph} \sigma_C$. If $\|y^*_n\| \leq 1$ for all but finitely many $n \in \mathbb{N}$, then one can apply (4.3) to $(y^*_n, \sigma_C(y^*_n))$ in place of $(y^*, \sigma_C(y^*))$ to conclude that
\[
(4.4) \quad (y^*_n, \sigma_C(y^*_n)) \in \operatorname{co} \bigcup_{i \in I} (\text{epi} \sigma_{C_i} \cap \gamma \Sigma^*)_w^*.
\]
On the other hand, if it happens that $\|y^*_n\| > 1$ for infinitely many $n \in \mathbb{N}$, then we must have $\|y^*\| = 1$ and $\|y^*_n\| \to 1$ as $n \to \infty$. Since $\text{Im} \partial \delta_B$ is a cone, we see that $\frac{y^*_n}{\|y^*_n\|} \in \text{Im} \partial \delta_B$. Applying (4.3) to $(\frac{y^*_n}{\|y^*_n\|}, \sigma_C(\frac{y^*_n}{\|y^*_n\|}))$ in place of $(y^*, \sigma_C(y^*))$, we obtain
\[
(4.5) \quad \left( \frac{y^*_n}{\|y^*_n\|}, \sigma_C\left( \frac{y^*_n}{\|y^*_n\|} \right) \right) \in \operatorname{co} \bigcup_{i \in I} (\text{epi} \sigma_{C_i} \cap \gamma \Sigma^*)_w^*.
\]
Taking limits in (4.4) and (4.5), we get (4.2), as required in both cases. This completes the proof of (ii)$\Rightarrow$(i).

For the converse implication, let us begin by noting that if (i) holds, then by the definition of subdifferential, we have
\[
(4.6) \quad \partial d_C(x) \subseteq \gamma \partial \sup_{i \in I} d_{C_i}(x) \quad \text{for each } x \in C.
\]
Now we suppose further that $I$ is compact and that $i \mapsto C_i$ is lower semicontinuous. Then by [1, Corollary 1.4.17], $i \mapsto d_{C_i}(\cdot)$ is upper semicontinuous. Hence one can apply [35, Theorem 2.4.18] to get the inclusion $\partial \left( \sup_{i \in I} d_{C_i}(x) \right) \subseteq \text{co} \bigcup_{i \in I} \partial d_{C_i}(x) w^*$, and it follows from (4.6) that $\partial d_C(x) \subseteq \gamma \text{co} \bigcup_{i \in I} \partial d_{C_i}(x) w^*$; hence (ii) holds for all $x \in C$ thanks to the standard result that $\partial d_C(x) = N_C(x) \cap B^*$ and $\partial d_{C_i}(x) = N_{C_i}(x) \cap B^*$ (cf. [35, Proposition 3.8.3]).

Next, we consider the case when $I$ is finite. We need only show that (ii)$\iff$(iii) in this case. For any $x \in C$, we note that by the Banach–Alaoglu theorem, $N_{C_i}(x) \cap B^*$ is $w^*$-compact for each $i \in I$; thus $\text{co} \bigcup_{i \in I} (N_{C_i}(x) \cap B^*)$ is $w^*$-closed as $I$ is finite. Hence (ii) and (iii) are the same when $I$ is finite.

Finally, we turn to prove that (iii)$\iff$(iv). The forward implication is obvious. For the converse implication, fix $x \in C$. Let $x^* \in N_{C_i}(x) \cap B^*$; we wish to show that $x^* \in \text{co} \bigcup_{i \in I} (N_{C_i}(x) \cap \gamma B^*)$. By (iii), there exist $x^*_i \in N_{C_i}(x)$, $i \in I$, with
\[ \sum_{i \in I} \| x_i^* \| \leq \gamma \text{ and } x^* = \sum_{i \in I} x_i^*. \] If all the \( x_i^* \)'s are zero, then the inclusion holds trivially. Otherwise, set \( \lambda := \sum_{i \in I} \| x_i^* \| > 0. \) Then \( \lambda \leq \gamma. \) Thus we see that

\[ x^* = \lambda \left( \sum_{i \in I, x_i^* \neq 0} \frac{\| x_i^* \|}{\| x_i^* \|} x_i^* + \left( 1 - \sum_{i \in I, x_i^* \neq 0} \frac{\| x_i^* \|}{\| x_i^* \|} \right) 0 \right) \in \text{co} \bigcup_{i \in I} (N_{C_i}(x) \cap \gamma B^*), \]

which completes the proof. \( \Box \)

**Theorem 4.6.** Suppose that

\[ (4.7) \]

\[ \text{co} \bigcup_{i \in I} (\text{epi} \sigma_{C_i} \cap \Sigma^*)^{w^*} \subseteq \sum_{i \in I} \text{epi} \sigma_{C_i}, \]

and that \( \{ C_i : i \in I \} \) is linearly regular. Then it satisfies the SECQ.

Proof. By the assumption, one can combine (4.7) with Theorem 4.4 to conclude that \( \text{epi} \sigma_{C} \cap \Sigma^* \subseteq \sum_{i \in I} \text{epi} \sigma_{C_i}, \) and hence that \( \text{epi} \sigma_{C} \subseteq \sum_{i \in I} \text{epi} \sigma_{C_i} \) for each \( \text{epi} \sigma_{C_i} \) is a cone. \( \Box \)

In the next theorem, we shall provide some sufficient conditions for (4.7). We first prove a simple lemma. We shall prove it in a bit more general context for later use. Recall that \( \{ C_i : i \in I \} \) is a CCS-system with 0 \( \in C. \)

**Lemma 4.7.** Let \( I \) be a metric space. Suppose that \( Z \) is a linear subspace of \( X \) and \( i \mapsto Z \cap C_i \) is lower semicontinuous. Consider elements \( i_0 \in I, (x_0^*, \alpha_0) \in X^* \times \mathbb{R} \) and nets \( \{ i_k \} \subseteq I, \{ (x_k^*, \alpha_k) \} \subseteq X^* \times \mathbb{R} \) with each \( (x_k^*, \alpha_k) \in \text{epi} \sigma_{C_{i_k}}. \) Suppose further that \( i_k \to i_0, \alpha_k \to \alpha_0, \) and \( x_k \mid Z \to w^* x_0^* \mid Z. \) If \( \{ x_k \mid Z \} \) is bounded, then \( (x_0^*, \alpha_0) \in \text{epi} \sigma_{Z \cap C_{i_0}}. \)

Proof. Let \( x \in Z \cap C_{i_0}. \) We have to prove that \( \langle x_0^*, x \rangle \leq \alpha_0. \) By the assumption, there exists a net \( \{ x_k \} \subseteq X \) with each \( x_k \in Z \cap C_{i_k} \) such that \( x_k \to x. \) Since

\[ \langle x_0^*, x \rangle = \langle x_0^*, x_k \rangle + \langle x_k^*, x - x_k \rangle + \langle x_k^*, x_k \rangle, \]

where on the right-hand side the first two terms converge to zero and the last term \( \langle x_0^*, x_k \rangle \leq \alpha_k \) for each \( k, \) it follows by passing to the limits that \( \langle x_0^*, x \rangle \leq \alpha_0. \) \( \Box \)

**Theorem 4.8.** Let \( I \) be a compact metric space and \( i \mapsto C_i \) be lower semicontinuous on \( I. \) Suppose that either \( I \) is finite or there exists an index \( i_0 \in I \) such that \( \dim C_{i_0} < +\infty. \) Then (4.7) holds. Consequently, if \( \{ C_i : i \in I \} \) is, in addition, linearly regular, then it satisfies the SECQ.

Proof. We first assume that \( I \) is finite, say \( I = \{ 1, 2, \ldots, m \}. \) Let \( (\tau^*, \tau) \in \text{co} \bigcup_{i=1}^m (\text{epi} \sigma_{C_i} \cap \Sigma^*)^{w^*}. \) Then there exists a net \( \{ (\tau_k^*, \tau_k) \} \) in \( \text{co} \bigcup_{i=1}^m (\text{epi} \sigma_{C_i} \cap \Sigma^*) \) such that \( (\tau_k^*, \tau_k) \) \( w^* \)-converges to \( (\tau^*, \tau). \) Without loss of generality, we assume that \( 0 \leq \tau_k \leq \tau + 1 \) for all \( k. \) Each \( (\tau_k^*, \tau_k) \) can be expressed as a convex combination

\[ (4.8) \]

\[ (\tau_k^*, \tau_k) = \sum_{i=1}^m \lambda_k,i (x_k^*, i, i, \alpha_k,i) \]

for some \( (x_k^*, i, i, \alpha_k,i) \in \text{epi} \sigma_{C_i} \cap \Sigma^* \) and \( \lambda_k,i \in [0, 1] \) with \( \sum_{i=1}^m \lambda_k,i = 1. \) Note that

\[ (4.9) \]

\[ \lambda_k,i (x_k^*, i, i, \alpha_k,i) \in \text{epi} \sigma_{C_i} \cap \Sigma^* \] for each \( k \) and \( i. \)

By considering subnets if necessary and by the \( w^* \)-compactness of the closed unit ball in Banach dual space \( X^* \) (the Banach–Alaoglu theorem), we may assume without loss of generality that for each \( i, \) there exist \( x_i^* \in B^* \) and \( \beta_i \in [0, \tau + 1] \) such that

\[ (4.10) \]

\[ \lambda_k,i x_k^* \to x_i^*, \quad \lambda_k,i \alpha_k,i \to \beta_i \]
(note that $\lambda_k, \alpha_k \leq \alpha + 1$ for all $k$). By the $w^*$-closedness of the set $\text{epi} \sigma_{C_i}$, we have from (4.10) and (4.9) that
\begin{equation}
(4.11) \quad (x_i^*, \beta_i) \in \text{epi} \sigma_{C_i} \quad \text{for each } i.
\end{equation}

Passing to the limits in (4.8), we arrive at
\begin{equation}
(\bar{x}^*, \bar{\alpha}) = \sum_{i=1}^{m} (x_i^*, \beta_i) \in \sum_{i=1}^{m} \text{epi} \sigma_{C_i},
\end{equation}

where the inclusion follows from (4.11).

Next we assume that there exists an index $i_0 \in I$ such that $\dim C_{i_0} < +\infty$. Let $Z_0 = \text{span} C_{i_0}$ and let $\bar{\sigma} = \text{co} \bigcup_{j \in I} (\text{epi} \sigma_{C_i} \cap \Sigma^*)^*$. Then there exists a net $\{ (\bar{x}_k, \bar{\sigma}_k) \}$ in $\text{co} \bigcup_{j \in I} (\text{epi} \sigma_{C_i} \cap \Sigma^*)$ such that $(\bar{x}_k, \bar{\sigma}_k) \rightarrow_{w^*} (\bar{x}^*, \bar{\alpha})$. Since $Z_0 \times \mathbb{R}$ is of dimension $m + 1$, one can apply the Carathéodory theorem to express each $(\bar{x}_k, \bar{\sigma}_k)$ as a convex combination of $m + 2$ many elements of $\bigcup_{j \in I} (\text{epi} \sigma_{C_i} \cap \Sigma^*)$. Hence there exist indices $i_k^j \in I$, nonnegative scalars $\lambda_{k,j}$, and pairs
\begin{equation}
(4.12) \quad (x_{k,j}^*, \lambda_{k,j}) \in \text{epi} \sigma_{C_{i_k^j}} \cap \Sigma^* \quad \text{for each } 1 \leq j \leq m + 2
\end{equation}

with the properties $\sum_{j=1}^{m+2} \lambda_{k,j} = 1$ and
\begin{equation}
(4.13) \quad \lambda_{k,j} (x_{k,j}^*, \lambda_{k,j}) \in \text{epi} \sigma_{C_{i_k^j}} \cap \Sigma^*.
\end{equation}

Since $\{ \bar{\sigma}_k \}$ is convergent, by passing to subnets if necessary, we may assume that $\bar{\sigma} + 1 \geq \bar{\sigma}_k \geq 0$. Then we also have $\{ \bar{\sigma}_k \}$ and $\{ \lambda_{k,j} \} \lambda_{k,j} \alpha_{k,j} \} bounded for $1 \leq j \leq m + 2$. Hence, considering subnets if necessary, we may assume that each of the nets $\{ \lambda_{k,j} x_{k,j}^* \}, \{ \bar{\sigma}_k \}, \{ \lambda_{k,j} \alpha_{k,j} \} \} for $1 \leq j \leq m + 2$ converges, say with limits,
\begin{equation}
x_{0,j}^*, \quad \bar{\sigma}, \quad \alpha_{0,j},
\end{equation}

and we can assume further that $i^j_k$ converges to some $i^j_0 \in I$ $(1 \leq j \leq m + 2)$, thanks to the compactness assumption of $I$. Making use of (4.13) and thanks to the assumption that $i \rightarrow C_i$ is lower semicontinuous, it follows from Lemma 4.7 (applied to $X$ in place of $Z$) that
\begin{equation}
(4.14) \quad (x_{0,j}^*, \alpha_{0,j}) \in \text{epi} \sigma_{C_{i_0^j}} \quad \text{for each } 1 \leq j \leq m + 2.
\end{equation}

Moreover, passing to the limits in (4.12), we have
\begin{equation}
(4.15) \quad (\bar{x}^*, \bar{\sigma}) = \sum_{j=1}^{m+2} (x_{0,j}^*, \alpha_{0,j}).
\end{equation}

Noting the trivial relation that $\text{epi} \sigma_{C_{i_0}}$ contains $Z_0^+ \times \mathbb{R}^+$, where $Z_0^+ := \{ x^* \in X^* : x^*|_{Z_0} = 0 \}$, it follows that
\begin{equation}
(4.16) \quad (\bar{x}^*, \bar{\sigma}) \in \sum_{j=1}^{m+2} (x_{0,j}^*, \alpha_{0,j}) + Z_0^+ \times \mathbb{R}^+ \subseteq \sum_{i \in I} \text{epi} \sigma_{C_i}.
\end{equation}
This shows that (4.7) holds.

Finally, in addition we assume that \{C_i : i \in I\} is linearly regular. Then it follows from Theorem 4.6 that this system satisfies the SECQ.

We intend to relate bounded linear regularity with the strong CHIP. We first provide a sufficient condition for a system to be linearly regular. The result is known when the ambient space is a Hilbert space [4, Theorem 4.2.6, Corollary 4.4.4] or a Banach space [34, Corollary 5]. The corresponding theorems in those references are derived from a lemma whose proof is based on the open mapping theorem and thus does not work in general normed linear spaces. As some preparatory work, we first state the following lemma, which is a generalization of [7, Proposition 3.1(i)] to a normed linear space (or even locally convex space) setting. We shall omit its proof, as it is a direct application of [29, Remarque 10.2] (alternatively, the proof given for [7, Proposition 3.1(i)], which was based on a result in [34], can easily be adopted here).

**Lemma 4.9.** Let \(E, F\) be two closed convex sets in \(X\) with \(E \cap \text{int } F \neq \emptyset\). Then \(\{E, F\}\) satisfies the SECQ.

We now give a sufficient condition for a system to be linearly regular.

**Lemma 4.10.** Let \(E\) be a closed convex set in \(X\) containing the origin and let \(r > 0\). Then

\[
d_{E \cap rB}(x) = 4 \max\{d_E(x), drB(x)\} \quad \text{for each } x \in X.\tag{4.14}
\]

**Proof.** We first show that

\[
gph \sigma_{E \cap rB} \cap \Sigma^* \subseteq \text{co}((\text{epi } \sigma_E \cap 4\Sigma^*) \cup (\text{epi } \sigma_{rB} \cap 4\Sigma^*)).\tag{4.15}
\]

Take \((y^*, \sigma_{E \cap rB}(y^*)) \in \text{gph } \sigma_{E \cap rB} \cap \Sigma^*\). By Lemma 4.9, there exist \((y_1^*, \alpha_1) \in \text{epi } \sigma_E\) and \((y_2^*, \alpha_2) \in \text{epi } \sigma_{rB}\) such that

\[
(y^*, \sigma_{E \cap rB}(y^*)) = (y_1^*, \alpha_1) + (y_2^*, \alpha_2).
\]

This implies that

\[
\sigma_{E \cap rB}(y^*) = \alpha_1 + \alpha_2.\tag{4.16}
\]

Since \(0 \in E\), we have \(0 \leq \sigma_E(y_1^*) \leq \alpha_1\) and hence \(\alpha_2 \leq \sigma_{E \cap rB}(y^*) \leq r\) thanks to (4.16). It follows that \(r\|y_2^*\| = \sigma_{rB}(y_2^*) \leq \alpha_2 \leq r\), and thus \(\|y_1^*\| \leq \|y^*\| + \|y_2^*\| \leq 2\).

Therefore,

\[
(y^*, \sigma_{E \cap rB}(y^*)) = \frac{1}{2} [(2y_1^*, 2\alpha_1) + (2y_2^*, 2\alpha_2)] \in \text{co}((\text{epi } \sigma_E \cap 4\Sigma^*) \cup (\text{epi } \sigma_{rB} \cap 4\Sigma^*))
\]

and (4.15) is established. By the implication (iii) \(\Rightarrow\) (i) of Theorem 4.4 (with \(\gamma = 4\)), it follows that (4.14) holds.

The following proposition on a relationship between bounded linear regularity and the linear regularity was shown in [4, Theorem 4.2.6(ii)] for the special case when \(X\) is a Hilbert space.

**Proposition 4.11.** Let \(\{A_i : i \in I\}\) be a system of closed convex sets in \(X\) containing the origin, and suppose that \(\{A_i : i \in I\}\) is boundedly linearly regular. Then for all \(r > 0\), the system \(\{rB, A_i : i \in I\}\) is linearly regular.

**Proof.** Write \(A = \cap_{i \in I} A_i\) and let \(r > 0\). By assumption, there exists \(k_r > 0\) such that

\[
d_A(x) \leq k_r \sup_{i \in I} d_{A_i}(x) \quad \text{for each } x \in rB.\tag{4.17}
\]
Let $f$ be defined by $f(x) := k_r \sup_{i \in I} d_{A_i}(x) - d_A(x)$ for each $x \in X$. From (4.17), we see that $f(x) \geq 0$ for all $x \in rB$, and the equality holds for all $x \in \bigcap_{i \in I} A_i \cap rB$. Since $f$ is clearly Lipschitz with modulus $k_r + 1$, it follows from [8, Proposition 2.4.3] that $f(x) + (k_r + 1)d_{rB}(x) \geq 0$ for all $x \in X$. This implies

$$d_A(x) \leq (2k_r + 1) \max \left\{ d_{rB}(x), \sup_{i \in I} d_{A_i}(x) \right\} \quad \text{for each } x \in X.$$  

It follows from Lemma 4.10 that

$$d_{A \cap rB}(x) \leq 4 \max \{ d_{rB}(x), d_A(x) \} \leq 4(2k_r + 1) \max \left\{ d_{rB}(x), \sup_{i \in I} d_{A_i}(x) \right\} \quad \text{for each } x \in X.$$  

This completes the proof. \[\square\]

For the following corollary, we need a lemma, which will also be used in the next section.

**Lemma 4.12.** Let $\{D, C_i : i \in I\}$ be a family of closed convex sets with nonempty intersection. Let $A$ be a closed subset of $X$ such that

\[
D \cap \left( \bigcap_{i \in I} C_i \right) \cap \text{int } A \neq \emptyset.
\]  

If $\{D, C_i : i \in I\}$ has the strong CHIP, then so does $\{D \cap A, C_i : i \in I\}$. As a partial converse result, if $\{D \cap A, C_i : i \in I\}$ has the strong CHIP at some point $a \in D \cap \bigcap_{i \in I} C_i \cap \text{int } A$, so does $\{D, C_i : i \in I\}$.

**Proof.** Set $E := D \cap \bigcap_{i \in I} C_i$. By hypothesis, $N_E(x) = N_D(x) + \sum_{i \in I} N_{C_i}(x)$ for each $x \in E$. Since $E \cap \text{int } A \neq \emptyset$ and $D \cap \text{int } A \neq \emptyset$,

$$N_{(A \cap D) \cap \bigcap_{i \in I} C_i}(x) = N_{A \cap E}(x) = N_A(x) + N_E(x) = N_A(x) + N_D(x) + \sum_{i \in I} N_{C_i}(x) = N_{A \cap D}(x) + \sum_{i \in I} N_{C_i}(x) \quad \text{for each } x \in A \cap E.$$  

This proves the first part. For the second part, observe that $N_A(a) = \{0\}$; hence $N_{A \cap D}(a) = N_D(a)$ and $N_{A \cap E}(a) = N_E(a)$. The conclusion is then immediate. \[\square\]

**Corollary 4.13.** Suppose that $i$ is a compact metric space and the set-valued function $i \mapsto C_i$ is lower semicontinuous. Suppose that either $I$ is finite or there exists an index $i_0 \in I$ such that $\dim C_{i_0} < +\infty$. If $\{C_i : i \in I\}$ is boundedly linearly regular, then it has the strong CHIP.

**Proof.** Fix any $x \in \bigcap_{i \in I} C_i$. Let $r = \|x\| + 1$. Since $\{C_i : i \in I\}$ is boundedly linearly regular, we obtain from Proposition 4.11 that $\{rB, C_i : i \in I\}$ is linearly regular. Taking an index $i_0 \notin I$, set $I_\infty = I \cup \{i_\infty\}$ and $C_{i_\infty} = rB$. Clearly the map $i \mapsto C_i$ is lower semicontinuous on $I_\infty$. It now follows from the assumptions and Theorem 4.8 that $\{C_i : i \in I_\infty\}$ satisfies the SECQ and so does $\{rB, C_i : i \in I\}$; thus $\{rB, C_i : i \in I\}$ has the strong CHIP (thanks to Theorem 3.1). Then it follows from Lemma 4.12 (with $D = X$ and $A = rB$) that $\{C_i : i \in I\}$ has the strong CHIP at $x$ because $x \in \text{int}(rB)$. The proof is complete. \[\square\]

Corollary 4.13 and the implication (i) \(\Rightarrow\) (ii) in Theorem 4.5 were established under the following conditions: (a) $I$ is compact, and (b) the set-valued function
Then, in each case,\( (4.19) \sup_{x \in I} d_{C_i}(x) = \|x\| = d(x, 0) = d_{\cap_{i \in I} C_i}(x). \)

Thus the system \( \{ C_i : i \in I \} \) is linearly regular in each of (a), (b), and (c). Note also that (c) satisfies both (a) and (\( \beta \)) and hence that Theorem 4.5(ii) and the conclusion of Corollary 4.13 hold (and these can be verified directly too). For each of (a) and (b), we have \( C = \{0\} \) and \( C_i(0) = \{0\} \) for each \( i \in I \), so \( C_i(0) \cap B = B \) but \( \text{co} \cup_{i \in I} (N_{C_i}(0) \cap B) = \{0\} \); thus the corresponding system does not have the strong CHIP (nor the SECQ, by Theorem 3.1), and Theorem 4.5(ii) does not hold. This failure of (a) and (b) is because each satisfies only one condition of (a), (\( \beta \)) (I is not compact in (a) and the set-valued function \( i \mapsto C_i \) is not lower semicontinuous in (b)).

5. Interior-point conditions and the SECQ. Recall that \( I \) is an index-set and \( C = \cap_{i \in I} C_i \subseteq X \). As in [27], the family \( \{ D, C_i : i \in I \} \) is called a closed convex set system with base-set \( D \) (CCS-system with base-set \( D \)) if \( D \) and all \( C_i \)'s are closed convex subsets of \( X \). Furthermore, throughout the remainder of this section, we always assume that \( I \) is a compact metric space and that \( 0 \in D \cap C \). Thus, \( \sigma_D \) and \( \sigma_{C_i} \) are nonnegative functions on \( X^* \) for all \( i \in I \).

Let \( |J| \) denote the cardinality of the set \( J \).

**Definition 5.1.** Let \( \{ D, C_i : i \in I \} \) be a CCS-system with base-set \( D \). Let \( m \) be a positive integer. Then the CCS-system \( \{ D, C_i : i \in I \} \) is said to satisfy

(i) the \( m \)-D-interior-point condition if, for any subset \( J \) of \( I \) with \( |J| \leq m \),

\[
D \cap \left( \bigcap_{i \in J} \text{int}_D C_i \right) \neq \emptyset;
\]

(ii) the \( m \)-interior-point condition if, for any subset \( J \) of \( I \) with \( |J| \leq m \),

\[
D \cap \left( \bigcap_{i \in J} \text{int} C_i \right) \neq \emptyset.
\]

Before proving our main theorems, we first give the following lemma. Recall that, for a linear subspace \( Z \) of \( X \), \( y^*|_Z \in Z^* \) is the restriction to \( Z \) of \( y^* \).

**Lemma 5.2.** Let \( m \) be a positive integer and let \( \{ D, C_i : i \in I \} \) be a CCS-system with the base-set \( D \). Let \( Z := \text{span} D \) and suppose that the following conditions are satisfied:

(a) \( D \) is finite dimensional.
(b) The set-valued mapping \( i \mapsto Z \cap C_i \) is lower semicontinuous on \( I \).

e(c) The system \( \{D, C_i : i \in I\} \) satisfies the \( m \)-\( D \)-interior-point condition.

Let \( (y^*, \alpha) \in X^* \times \mathbb{R} \) and let \( \{(y^*_k, \alpha_k)\} \subseteq X^* \times \mathbb{R} \) be a sequence such that

\[
(y^*_k|_Z, \alpha_k) \text{ converges to } (y^*_|Z, \alpha),
\]

where each \( (y^*_k|_Z, \alpha_k) \) can be expressed in the form

\[
(y^*_k|_Z, \alpha_k) = (v^*_k|_Z, \beta_k) + \sum_{j=1}^m \left( x^*_i |_Z, \alpha^*_i \right)
\]

with

\[
(v^*_k, \beta_k) \in \text{epi } \sigma_D, \quad \left( x^*_i, \alpha^*_i \right) \in \text{epi } \sigma_{C_{i}}
\]

for some \( i^*_1, \ldots, i^*_m \in I \). Then

\[
(y^*, \alpha) \in \text{epi } \sigma_D + \sum_{i \in I} \text{epi } \sigma_{Z \cap C_i}.
\]

**Proof.** Since \( I \) is compact, by considering subsequences if necessary we may assume that there exists \( i_j \in I \) such that \( i_j^* \to i_j \) for each \( j = 1, \ldots, m \). By assumption (c), there exist \( z \in D \) and \( \delta' > 0 \) such that

\[
B(z, \delta') \cap Z \subseteq C_i \cap Z \quad \text{for each } j = 1, 2, \ldots, m.
\]

Set for convenience \( B := B(z, \delta) \cap Z \), where \( \delta = \frac{\delta'}{2} \). Then \( B \) is compact, thanks to assumption (a). For each \( j = 1, 2, \ldots, m \), we make use of assumption (b) and apply Proposition 2.9 at the point \( t_0 := i_j \) of the lower semicontinuous function \( i \mapsto C_i \cap Z \) to conclude from (5.5) that \( B \subseteq C_{i_j} \cap Z \) for all large enough \( k \). Do this for each \( j = 1, 2, \ldots, m \) and take \( k_0 \in \mathbb{N} \) large enough such that

\[
B \subseteq C_{i_j} \cap Z \quad \text{for each } 1 \leq j \leq m \text{ and } k \geq k_0.
\]

Note that, for each \( 1 \leq j \leq m \) and \( k \in \mathbb{N} \),

\[
\sigma_B \left( x^*_i \right) = \sup_{x \in B} \left\{ x^*_i, x \right\} = \sup_{x \in \delta B \cap Z} \left\{ x^*_i, x \right\} \leq \delta \left\| x^*_i \right\|_Z + \left\langle x^*_i, z \right\rangle.
\]

It follows from (5.6) that

\[
\alpha_{i_j} \geq \sigma_{C_{i_j}} \left( x^*_i \right) \geq \sigma_{C_{i_j} \cap Z} \left( x^*_i \right) \geq \sigma_B \left( x^*_i \right) = \delta \left\| x^*_i \right\|_Z + \left\langle x^*_i, z \right\rangle,
\]

provided that \( k \geq k_0 \). Moreover, since \( z \in D \) and \( (v^*_k, \beta_k) \in \text{epi } \sigma_D \), (5.2) establishes that

\[
\alpha_k - \langle y^*_k, z \rangle = \beta_k - \langle v^*_k, z \rangle + \sum_{j=1}^m \left( \alpha_{i_j} - \left\langle x^*_i, z \right\rangle \right) \geq \sum_{j=1}^m \left( \alpha_{i_j} - \left\langle x^*_i, z \right\rangle \right).
\]

Combining (5.7) and (5.8) yields that

\[
\alpha_k - \langle y^*_k, z \rangle \geq \sum_{j=1}^m \left( \alpha_{i_j} - \left\langle x^*_i, z \right\rangle \right) \geq \sum_{j=1}^m \delta \left\| x^*_i \right\|_Z.
\]
This implies that \( \{ x^*_i j \mid j, k \in \mathbb{N} \} \) is bounded for each \( 1 \leq j \leq m \) thanks to (5.1). Consequently \( \{ v^*_k \mid k \in \mathbb{N} \} \) is bounded as, by (5.2), \( (v^*_k) \) is the sum of \( m + 1 \) bounded sequences. Since \( Z \) is finite dimensional (and by passing to subsequences if necessary) we may assume that for each \( j = 1, 2, \ldots, m \), there exist \( \tilde{x}^*_i j \) and \( \tilde{v}^* \in Z^* \) such that

\[
x^*_i j \big|_Z \to \tilde{x}^*_i j \quad \text{and} \quad v^*_k \big|_Z \to \tilde{v}^* \quad \text{as} \quad k \to \infty.
\]

Now, observe from (5.1)–(5.3) that \( \{ \alpha_i j \} \) and \( \{ \beta_k \} \) are bounded. Thus we may also assume that, for each \( j \), \( \alpha_i j \to \hat{\alpha}_i \) for some \( \hat{\alpha}_i \in \mathbb{R} \) and that \( \beta_k \to \hat{\beta} \) for some \( \hat{\beta} \in \mathbb{R} \). Then, by (5.1) and (5.2),

\[
y^* \big|_Z = \tilde{v}^* + \sum_{j=1}^{m} \tilde{x}^*_i j \quad \text{and} \quad \alpha = \hat{\beta} + \sum_{j=1}^{m} \hat{\alpha}_i.
\]

Let \( x^*_i j \in X^* \) be an extension of \( \tilde{x}^*_i j \) to \( X \) and \( v^* \in X^* \) be an extension of \( \tilde{v}^* \) to \( X \).

It follows from Lemma 4.7 that \( (x^*_i j, \hat{\alpha}_i) \in \text{epi } \sigma_{(C_i \cap Z)} \) and \( (v^*, \hat{\beta}) \in \text{epi } \sigma_{D} \). Write \( \tilde{y}^* = y^* - v^* - \sum_{j=1}^{m} x^*_i j \). Then by (5.9), \( \tilde{y}^* \in Z^\perp \) and

\[
(y^*, \alpha) = (\tilde{y}^*, 0) + (v^*, \hat{\beta}) + \sum_{j=1}^{m} (x^*_i j, \hat{\alpha}_i) \in Z^\perp \times \{0\} + \text{epi } \sigma_{D} + \sum_{j=1}^{m} \text{epi } \sigma_{C_i \cap Z}.
\]

Thus, (5.4) holds, as \( Z^\perp \times \{0\} \) is clearly contained in \( \text{epi } \sigma_{D} \).

\[ \Box \]

**Remark 5.1.** If, for (a) of Lemma 5.2, \( \dim D \leq m - 1 \), then the following implication is valid:

\[ (a) \land (b) \land (c) \Rightarrow \{ D, (\text{span } D) \cap C_i : i \in I \} \text{ satisfies the SECQ.} \]

(This can be seen from (i) of Theorem 5.3 below, but with \( m \) replaced by \( m - 1 \).)

**Theorem 5.3.** Let \( m \in \mathbb{N} \) and let \( \{ D, C_i : i \in I \} \) be a CCS-system with the base-set \( D \). We consider the following conditions:

(a) \( D \) is of finite dimension \( m \).

(b) The set-valued mapping \( i \mapsto (\text{span } D) \cap C_i \) is lower semicontinuous on \( I \).

(c) The system \( \{ D, C_i : i \in I \} \) satisfies the \( (m + 1) \)-interior-point condition.

(d) For each \( i \in I \), the pair \( \{ D, C_i \} \) has the property

\[
\text{epi } \sigma_{(\text{span } D) \cap C_i} \subseteq \text{epi } \sigma_{D} + \text{epi } \sigma_{C_i},
\]

\( \text{e.g., } \{ D, C_i \} \) satisfies the SECQ.

(c*) The system \( \{ D, C_i : i \in I \} \) satisfies the \( m \)-interior-point condition.

(d*) For each finite subset \( J \) of \( I \) with \( |J| = \min \{ m + 1, |I| \} \), the subsystem \( \{ D, C_j : j \in J \} \) satisfies the SECQ.

Then the following assertions hold:

(i) If (a), (b), (c) are satisfied, then \( \{ D, (\text{span } D) \cap C_i : i \in I \} \) satisfies the SECQ.

(ii) If (a), (b), (c), (d) are satisfied, then \( \{ D, C_i : i \in I \} \) satisfies the SECQ.

(iii) If \( D \) is bounded and (a), (b), (c*), (d*) are satisfied, then \( \{ D, C_i : i \in I \} \) satisfies the SECQ.
Proof. (i) Write \( Z := \text{span} \, D \) as before. For a subset \( H \) of \( X^* \times \mathbb{R} \), we use 
\[
H|_Z = \{ (x^*|_Z, \beta) : (x^*, \beta) \in H \}.
\]
Let \( (y^*, \alpha) \in \text{epi} \sigma_D + \sum_{i \in I} \text{epi} \sigma_{C_i}^{m^*} \). Since \( Z \) is finite dimensional, there exists a sequence \( \{(y^*_k, \alpha_k)\} \subseteq X^* \times \mathbb{R} \) with 
\[
(y^*_k, \alpha_k) \in \text{epi} \sigma_D + \sum_{i \in I} \text{epi} \sigma_{C_i} \quad \text{for each } k \in \mathbb{N}
\]
such that \( (y^*_k|_Z, \alpha_k) \) converges to \( (y^*|_Z, \alpha) \). By (5.10) we express for each \( k \in \mathbb{N} \),
\[
(y^*_k, \alpha_k) = (v^*_k, \beta_k) + (u^*_k, \gamma_k),
\]
where \( (v^*_k, \beta_k) \in \text{epi} \sigma_D \) and \( (u^*_k, \gamma_k) \in \sum_{i \in I} \text{epi} \sigma_{C_i} \). Since \( (\sum_{i \in I} \text{epi} \sigma_{C_i})|_Z \) is a convex cone in the \((m + 1)\)-dimensional space \( Z^* \times \mathbb{R} \), it follows from [32, Theorem 3.15] that, for each \( k \), there exist indices \( \{i^k_1, \ldots, i^k_{m+1}\} \subseteq I \) and \( \{(x^*_{i^k_1}, \alpha^k_{i^k_1}), \ldots, (x^*_{i^k_{m+1}}, \alpha^k_{i^k_{m+1}})\} \) with \( (x^*_{i^k_j}, \alpha^k_{i^k_j}) \in \text{epi} \sigma_{C^k_{i^k_j}} \) for each \( 1 \leq j \leq m + 1 \) such that 
\[
(u^*_k|_Z, \gamma_k) = \sum_{j=1}^{m+1} (x^*_{i^k_j}|_Z, \alpha^k_{i^k_j}) \quad \text{for each } k \in \mathbb{N}.
\]
Thus we have
\[
(y^*_k|_Z, \alpha_k) = (v^*_k|_Z, \beta_k) + \sum_{j=1}^{m+1} (x^*_{i^k_j}|_Z, \alpha^k_{i^k_j}) \quad \text{for each } k \in \mathbb{N}.
\]
By Lemma 5.2 and thanks to assumptions (a), (b), (c),
\[
(y^*, \alpha) \in \text{epi} \sigma_D + \sum_{i \in I} \text{epi} \sigma_{Z \cap C_i}.
\]
We have just proved the inclusion
\[
\text{epi} \sigma_D + \sum_{i \in I} \text{epi} \sigma_{C_i}^{m^*} \subseteq \text{epi} \sigma_D + \sum_{i \in I} \text{epi} \sigma_{Z \cap C_i}.
\]
Noting \( D \cap (\cap_{i \in I}(Z \cap C_i)) = D \cap (\cap_{i \in I} C_i) \), it follows from Proposition 2.4 and (5.14) that 
\[
\text{epi} \sigma_{D \cap (\cap_{i \in I}(Z \cap C_i))} = \text{epi} \sigma_D + \sum_{i \in I} \text{epi} \sigma_{C_i}^{m^*} \subseteq \text{epi} \sigma_D + \sum_{i \in I} \text{epi} \sigma_{Z \cap C_i}.
\]
Thus \( \{D, (\text{span} \, D) \cap C_i : i \in I\} \) satisfies the SECQ by Corollary 2.5. This proves assertion (i).

(ii) Now suppose in addition that (d) is also satisfied. Then (5.15) implies that 
\[
\text{epi} \sigma_{D \cap \cap_{i \in I} C_i} \subseteq \text{epi} \sigma_D + \sum_{i \in I} (\text{epi} \sigma_D + \text{epi} \sigma_{C_i}) \subseteq \text{epi} \sigma_D + \sum_{i \in I} \text{epi} \sigma_{C_i}.
\]
By Corollary 2.5 again, this implies that \{D, C_i : i \in I\} satisfies the SECQ; that is, (ii) holds.

(iii) Now suppose that (a), (b), (c*), (d*) are satisfied. Without loss of generality, we may assume that \(|I| > m + 1\) since, otherwise, the conclusion follows from assumption (d*). Consider \((\gamma^*, \alpha), (y^*_k, \alpha_k), (v^*_j, \beta_k), (u^*_k, \gamma_k)\) satisfying (5.10)–(5.13). Let \(k \in \mathbb{N}\) and set \(I^k = \{i^*_1, \ldots, i^*_m, i^*_m+1\}\). Then for any \(z \in D \cap \bigcap_{j \in I^k} C_j (\subseteq Z)\),

\[
\alpha_k = \beta_k + \sum_{j \in I^k} \alpha_{y^*_j} \geq \sigma_D(v^*_k) + \sum_{j=1}^{m+1} \sigma_{C_j} \left( x^*_{y^*_j} \right) \geq \left( v^*_k + \sum_{j=1}^{m+1} x^*_{y^*_j} \right) \langle z, z \rangle = \langle y^*_k, z \rangle,
\]

thanks to (5.13). Since \(D \cap (\bigcap_{j \in I^k} C_j)\) is compact, there exists \(x^k \in D \cap (\bigcap_{j \in I^k} C_j)\) such that

\[
\alpha_k \geq \langle y^*_k, x^k \rangle = \sigma_D(\bigcap_{j \in I^k} C_j)(y^*_k),
\]

i.e., \(y^*_k \in N_{D \cap (\bigcap_{j \in I^k} C_j)}(x^k)\). It follows from assumption (d*) and Theorem 3.1 that \(y^*_k \in N_D(x^k) + \sum_{j \in I^k} N_{C_j}(x^k)\). Applying [32, Theorem 3.15] to the \(m\)-dimensional linear subspace \(Z\), \(y^*_k|Z\) can be expressed in the form

\[
y^*_k|Z = d^*_k|Z + \sum_{j \in J^k} z^*_j|Z
\]

for some \(d^*_k \in N_D(x^k)\) and \(z^*_j \in N_{C_j}(x^k)\) (\(j \in J^k\)), where \(J^k\) is a subset of \(I^k\) with \(m\) elements. Evaluating (5.17) at \(x^k \in D \cap (\bigcap_{j \in I^k} C_j)\), and invoking (2.4) and (5.16), we have

\[
\alpha_k \geq \langle y^*_k, x^k \rangle = \sigma_D(d^*_k) + \sum_{j \in J^k} \sigma_{C_j}(z^*_j).
\]

Define

\[
\mu_k = \alpha_k - \sum_{j \in J^k} \sigma_{C_j}(z^*_j).
\]

Then \(\mu_k \geq \sigma_D(d^*_k)\) by (5.18). Denoting \(\sigma_{C_j}(z^*_j)\) by \(\gamma_j\), this and (5.17) imply that

\[
(\langle y^*_k|Z, \alpha_k \rangle) = (\langle d^*_k|Z, \mu_k \rangle + \sum_{j \in J^k} \langle z^*_j|Z, \gamma_j \rangle).
\]

Note that \((d^*_k, \mu_k) \in \text{epi } \sigma_D\) and \((z^*_j, \gamma_j) \in \text{epi } \sigma_{C_j}\) for each \(j \in J^k\). Since \(|J^k| = m\) and thanks to assumptions (a), (b), and (c*), Lemma 5.2 asserts that

\[
\text{epi } \sigma_D \cap \text{epi } \sigma_{C_j} \subseteq \text{epi } \sigma_{(D \cap (\bigcap_{j \in J} C_j))} \subseteq \text{epi } \sigma_D + \sum_{j \in J} \text{epi } \sigma_{C_j}.
\]

Let \(i \in I\) and let \(J\) be any subset of \(I\) such that \(i \in J\) and \(|J| = m + 1\). Then, by assumption (d*), one has that

\[
\text{epi } \sigma_{(Z \cap C_i)} \subseteq \text{epi } \sigma_{(D \cap (\bigcap_{j \in J} C_j))} \subseteq \text{epi } \sigma_D + \sum_{j \in J} \text{epi } \sigma_{C_j}.
\]
Therefore, by (5.19) and (5.20), \((y',\alpha) \in \text{epi} \sigma_D + \sum_{i \in I} \text{epi} \sigma_{C_i}\), and thus \(\text{epi} \sigma_D + \sum_{i \in I} \text{epi} \sigma_{C_i}\) is weakly* closed in the case when assumptions (a), (b), (c*), and (d*) are satisfied. By Corollary 2.5, this implies that \(\{D, C_i : i \in I\}\) satisfies the SECQ. The proof is complete. \(\Box\)

**Corollary 5.4.** Let \(m \in \mathbb{N}\) and let \(\{D, C_i : i \in I\}\) be a CCS-system with the base-set \(D\) satisfying the following conditions:

(a) \(D\) is of finite dimension \(m\).

(b) The set-valued mapping \(i \mapsto (\text{span} D) \cap C_i\) is lower semicontinuous on \(I\).

(c*) The system \(\{D, C_i : i \in I\}\) satisfies the \((m + 1)\)-interior-point condition.

Then \(\{D, C_i : i \in I\}\) satisfies the SECQ.

**Proof.** By Lemma 4.9, (c*) implies conditions (d) and (c) of Theorem 5.3. Thus, Theorem 5.3(ii) is applicable. \(\Box\)

The following corollary, which is a direct consequence of Theorem 5.3(i), is an improvement of Theorem 1.1.

**Corollary 5.5.** Let \(\{D, C_i : i \in I\}\) be a CCS-system with the base-set \(D\). Let \(m \in \mathbb{N}\) and let \(x_0 \in D \cap C\). Suppose that the following conditions are satisfied:

(a) \(D\) is of finite dimension \(m\).

(b) The set-valued mapping \(i \mapsto (\text{span} D) \cap C_i\) is lower semicontinuous on \(I\).

(c) The system \(\{D, C_i : i \in I\}\) satisfies the \((m + 1)\)-interior-point condition.

(d) For each \(i \in I\), the pair \(\{D, C_i\}\) has the property

\[N_{(\text{span} D) \cap C_i}(x_0) \subseteq N_D(x_0) + N_{C_i}(x_0).\]

Then the system \(\{D, C_i : i \in I\}\) has the strong CHIP at \(x_0\).

The following corollary is an important improvement of Theorem 1.2. Our main improvement lies in the fact that we need not require the upper semicontinuity of the set-valued map \(i \mapsto (\text{span} D) \cap C_i\) and that (d) can be weakened to require that only the subsystems \(\{D, C_j : j \in J\}\) with \(|J| = l + 1\) have the strong CHIP.

**Corollary 5.6.** Let \(m \in \mathbb{N}\) and let \(\{D, C_i : i \in I\}\) be a CCS-system with the base-set \(D\) satisfying the following conditions:

(a) \(D\) is of finite dimension \(m\).

(b) The set-valued mapping \(i \mapsto (\text{span} D) \cap C_i\) is lower semicontinuous on \(I\).

(c*) The system \(\{D, C_i : i \in I\}\) satisfies the \(m\)-interior-point condition.

(d) For each finite subset \(J\) of \(I\) with \(|J| = \min\{m + 1, |I|\}\), the subsystem \(\{D, C_j : j \in J\}\) has the strong CHIP.

Then the system \(\{D, C_i : i \in I\}\) has the strong CHIP.

**Proof.** If \(|J| < m + 1\), then \(\min\{m + 1, |J|\} = |I|\), so the result is trivially true by (d). Thus we may assume that \(|J| \geq m + 1\). Recall that \(C = \cap_{i \in J} C_i\) and let \(x \in D \cap C\). We have to show that the system has the strong CHIP at \(x\). To this end, let \(\bar{D} = D \cap B(x, r_x)\), where \(r_x = \|x\| + 1\). Consider the system \(\{\bar{D}, C_i: i \in I\}\). We claim that the following conditions hold:

(\(\bar{a}\)) \(\bar{D}\) is of finite dimension and \(\text{dim} \bar{D} = m\).

(\(\bar{b}\)) The set-valued mapping \(i \mapsto (\text{span} \bar{D}) \cap C_i\) is lower semicontinuous on \(I\).

(\(\bar{c}\)) The system \(\{\bar{D}, C_i : i \in I\}\) satisfies the \(m\)-interior-point condition.

(\(\bar{d}\)) For each finite subset \(J\) of \(I\) with \(|J| = m + 1\), the subsystem \(\{\bar{D}, C_j : j \in J\}\) satisfies the SECQ.

In fact, by assumption (c*), for each finite subset \(J\) of \(I\) with \(|J| = m\), there exist \(\bar{x} \in \bar{D}\) and \(\delta > 0\) such that \(B(\bar{x}, \delta) \cap \text{span} \bar{D} \subseteq \bar{D} \cap \cap_{j \in J} C_j\). Since \(0 \in \text{int} B(x, r_x)\), there exists \(\lambda \in (0, 1)\) such that \(\lambda B(\bar{x}, \delta) \subseteq B(x, r_x)\). Consequently,
\[
\lambda (5.21) \quad \Lambda B(x, \delta) \cap \text{span} D \subseteq \bar{D} \cap \bigcap_{j \in J} B(x, r_x) \subseteq \bar{D} \cap \bigcap_{i \in I} C_i.
\]

This implies that \( \text{int} B(x, \delta) \cap \text{ri} D \neq \emptyset \); hence
\[
\text{span} \bar{D} = \text{span} D. 
\]

Consequently, condition (c) holds by (5.21). Moreover, by (a), (b), and (5.22), it is seen that (a) and (b) hold. As to condition (d), let \( J \) be any subset of \( I \) with \( |J| = m + 1 \). By (d) the subsystem \( \{D, C_j : j \in J\} \) has the strong CHIP. Since \( x \in \text{int} B(x, r_x) \cap (D \cap (\bigcap_{j \in J} C_j)) \), and by applying Lemma 4.12 to the ball with center \( x \), radius \( r_x \), and \( J \) in place of \( A \) and \( I \), it follows that \( \{D, C_j : j \in J\} \) has the strong CHIP and consequently satisfies the SECQ, thanks to Corollary 3.2(i) because \( \bar{D} \cap (\bigcap_{j \in J} C_j) \) is compact. Thus (d) is established. Thus part (iii) of Theorem 5.3 is applicable for concluding that the system \( \{\bar{D}, C_i : i \in I\} \) satisfies the SECQ, which in turn implies that it has the strong CHIP at \( x \). Consequently, the system has the strong CHIP at \( x \) by Lemma 4.12 applied to the ball with center \( x \), radius \( r_x \), and \( J \) in place of \( A \) and \( J \). The proof is complete. \( \square \)

Acknowledgments. The authors would like to express their sincere thanks to the two referees for many helpful comments and for shorter proofs of Theorem 4.6 and of other minor results.

REFERENCES