Ruin Probabilities in a Discrete Time Risk Model with Dependent Risks of Heavy Tail

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Abstract

This paper establishes some asymptotic results for both finite and ultimate ruin probabilities in a discrete time risk model with constant interest rates, and individual net losses in $R_{-\alpha}$, the class of regular variation with index $\alpha > 0$. The individual net losses are allowed to be generally dependent while they have zero index of upper tail dependence, so that our results partially generalize the counterparts in Tang (2004). The procedure of deriving our results also demonstrates a new approach of achieving asymptotic formulation for ruin probabilities when the individual risks are dependent.

Keywords: Discrete time risk model, ruin probability, regular variation distributions, index of upper tail dependence, copula, net loss.

1. Introduction

Let \( \{X_i, \ i = 1, 2, \cdots\} \) be a sequence of random variables (r.v.’s) and \( r \) be a nonnegative real number. Consider the following discrete time process:

\[
U_0 = x, \ U_i = U_{i-1}(1 + r) - X_i, \ i = 1, 2, \ldots.
\]

In the context of insurance risk modeling, \( U_i \) stands for the insurance company’s surplus at the end of period \( i \), \( x(\geq 0) \) represents its initial capital at time 0, \( r(\geq 0) \) denotes the constant interest rate, and \( X_i \) captures the (individual) net loss (or risk), i.e., the total claim amount minus the total premium incomes, during period \( i \). An interesting and important problem arising from the above insurance risk model is analyzing the ruin probabilities of an insurance company. Formally, the ruin probability within finite time horizon \([0, n]\) is defined as

\[
\psi_r(x, n) = \Pr\{\min_{0 \leq i \leq n} U_i < 0 | U_0 = x\}, \ x, r \geq 0,
\]

while the ultimate ruin probability in infinite time horizon is defined as

\[
\psi_r(x) = \Pr\{\min_{0 \leq i < \infty} U_i < 0 | U_0 = x\}, \ x, r \geq 0.
\]

The study of the above two forms of ruin probabilities associated with model (1.1) has been a classical area of research among actuaries and probabilists. More recently, the classical insurance risk model (1.1) has been generalized in a number of important directions to better reflect the market practice and empirical evidence. Here we will broadly summarize three of these generalizations. The first generalization is to allow the interest rate \( r \) to be stochastic. See Nyrhinen (1999, 2001), Cai (2002a, 2002b), Cai and Dickson (2004), and Gao, et al. (2007), for example. Most of these papers analyze the ruin probabilities by establishing the corresponding Lundberg’s bounds for the ruin probabilities. The second generalization is to examine the ruin-related problems when the individual risks are assumed to be heavy-tailed. See, for example, Klüppelberg and Stadtmüller (1998), and Kalashnikov and Konstantinides (2000) for related results on the continuous time risk models, and Tang and Tsitsiashvilib
Tang (2004), and Chen and Su (2006) for the discrete time risk models. Many of these results are devoted to establishing certain asymptotic formulation of the ruin probabilities.

The third generalization is to incorporate dependence among the individual net losses. See for example Albrecher (1998), Cossette and Marceau (2000), Cossette, et al. (2003, 2004) for related contributions. These papers induce dependence by only considering some very special dependency among the individual net losses. More specifically, Albrecher (1998) defines the dependence structure through mixing technique and investigates the effects of the dependency on ruin probabilities. Cossette and Marceau (2000) examine the discrete time risk models with classes of correlated business modelled by, respectively, Poisson model and negative binomial model with common shock. Under these settings, they discuss the effects of dependence on the finite time ruin probabilities as well as the adjustment coefficients. Cossette, et al. (2003, 2004) derive certain recursive formulas and Lundberg’s bound for the compound Markov binomial model, which is a special case of the discrete time risk model.

In this paper, we similarly investigate the related ruin probabilities under the discrete time risk model (1.1). Our assumed model, however, has the main advantage of allowing the individual net risks to be generally dependent with zero index of upper tail dependence, in addition to being heavy-tailed. More precisely, by assuming (i) the net losses $X_i, i = 1, 2, \ldots$, to be identically distributed as a generic r.v. $X$ with cumulative distribution function (c.d.f.) $F(\cdot)$, (ii) the individual net losses to be generally dependent from the regularly varying class with index $\alpha > 0$ (see Section 2 for the definition), (iii) the c.d.f. $F(\cdot)$ satisfies

$$\lim_{x \to \infty} \frac{F(-x)}{F(x)} = 0,$$

where the survival function $\overline{F}(x) = 1 - F(x)$, and drawing a lemma from Davis and Resnick (1996), we derive a simple asymptotic result for the ultimate ruin probability. This result generalizes Tang (2004) in the sense that the individual net risks in our case are allowed to be generally dependent but with zero index of upper tail dependence. Moreover, we derive an asymptotic result for the finite ruin probability. We will argue in Remark 3.2 of Section 3 that condition (1.4) is quite mild from a practical viewpoint. We will also present some discussions on the assumption of zero index of
upper tail dependence for the individual net risks \( \{X_i, i \in I\} \) (see Remarks 3.3, 3.4, and 3.5). It is worth noting that our results are also applicable to varying constant interest rates in different time periods, as justified in Remark 3.1.

To conclude this section, we note that an equivalent way of expressing the surplus process (1.1) is

\[
U_0 = x, \quad U_i = x(1 + r)^i - \sum_{k=1}^{i} X_k(1 + r)^{i-k}, \quad i = 1, 2, \ldots,
\]

(1.5)

for all \( x \geq 0 \). This suggests that the ruin probabilities can be rewritten in terms of the discounted values of the surplus process; i.e.,

\[
\psi_r(x, n) = \Pr\{ \min_{0 \leq i \leq n} (1 + r)^{-i}U_i < 0 | U_0 = x \} = \Pr\{ \max_{0 \leq i \leq n} \sum_{k=0}^{i} X_k(1 + r)^{-k} > x \},
\]

(1.6)

and

\[
\psi_r(x) = \Pr\{ \min_{0 \leq i < \infty} (1 + r)^{-i}U_i < 0 | U_0 = x \} = \Pr\{ \max_{0 \leq i < \infty} \sum_{k=0}^{i} X_k(1 + r)^{-k} > x \},
\]

(1.7)

where \( X_0 = 0 \) by convention. Since in this paper we are confining the individual net losses within the regularly varying class with index \( \alpha > 0 \), the maximum (1.7) is a proper r.v. concentrated on \([0, \infty)\) provided \( r > 0 \). The proof parallels to that of Lemma 4.3 in Tang (2004).

The rest of this paper is organized as follows: Section 2 is the preliminary, recalling definitions, lemmas, and together with their properties that will be used in the subsequent analysis. Section 3 states our main results with some remarks and Section 4 collects the proofs of our results. Section 5 concludes the paper.

2. Preliminary

In this section, we recall various concepts, definitions, properties, and lemmas which are useful in our subsequent analysis. For two positive functions \( a(x), b(x) \), we write \( a(x) \lesssim b(x) \) if \( \limsup_{x \to \infty} \frac{a(x)}{b(x)} \leq 1 \), \( a(x) \gtrsim b(x) \) if \( \liminf_{x \to \infty} \frac{a(x)}{b(x)} \geq 1 \), and \( a(x) \sim b(x) \) if both. Moreover, we use the notations \( I_n \) and \( I \) to denote, respectively, the index sets of \( \{1, 2, \ldots, n\} \) and \( \{1, 2, \ldots\} \) for a sequence.

A r.v. \( X \) or its c.d.f. \( F(\cdot) \) satisfying \( F(x) > 0 \) for all \( x \in (-\infty, \infty) \) is heavy-tailed to the right, or simply heavy-tailed, if \( E[e^{\gamma X}] = \infty \) for all \( \gamma > 0 \). We recall here
the definitions of two important classes of heavy-tailed distributions; i.e. the regular variation class $\mathcal{R}_{-\alpha}$ and the long-tailed class $\mathcal{L}$:

**Definition 2.1** (i) *Regular variation class $\mathcal{R}_{-\alpha}$*: a r.v. $X$ or its c.d.f. $F(\cdot)$ on $(-\infty, +\infty)$ belongs to $\mathcal{R}_{-\alpha}$ if

$$\lim_{x \to \infty} \frac{F(xy)}{F(x)} = y^{-\alpha}$$

for some $\alpha > 0$ and any $y > 0$.

(ii) *Long-tailed class $\mathcal{L}$*: a r.v. $X$ or its c.d.f. $F(\cdot)$ on $(-\infty, +\infty)$ belongs to $\mathcal{L}$ if

$$\lim_{x \to \infty} \frac{F(x-y)}{F(x)} = 1$$

for all $y > 0$.

Note that a c.d.f $F(\cdot)$ concentrated on $(-\infty, \infty)$ belongs to the class $\mathcal{R}_{-\alpha}$ if and only if there exists some positive slowly varying function $L(x)$ such that

$$F(x) = x^{-\alpha}L(x).$$

(2.1)

In addition to these two classes, there are two other important classes of heavy-tailed distributions known as the Dominant variation class $\mathcal{D}$ and the Subexponential class $\mathcal{S}$. These classes satisfy the following inclusion relations:

$$\mathcal{R}_{-\alpha} \subset \mathcal{D} \cap \mathcal{L} \subset \mathcal{S} \subset \mathcal{L}. \quad (2.2)$$

For a comprehensive review on heavy-tailed distributions and their applications in insurance and finance, see Bingham, et al. (1987), Cline and Samorodnitsky (1994), and Embrechts, et al. (1997).

The following two lemmas play a critical role in the development of our asymptotic results on ruin probabilities. Their proofs can be found, respectively, in Resnick (1987) and Davis and Resnick (1996).

**Lemma 2.1** Suppose $F(x)$ is a c.d.f. on $(-\infty, \infty)$. If $F(\cdot) \in \mathcal{R}_{-\alpha}$, $\alpha > 0$, then for any $0 < \beta < \alpha < \theta < \infty$ there exists positive constants $p_1$ and $x_0$ such that

$$y^{-\theta} \leq \frac{F(xy)}{F(x)} \leq y^{-\beta} \quad (2.3)$$
holds uniformly for $xy > x > x_0$ and

$$F(x) \leq p_1x^{-\beta} \quad (2.4)$$

holds uniformly for $x > x_0$.

**Lemma 2.2** Let $X_i$ be a nonnegative r.v. with c.d.f $F_i(\cdot) \in \mathcal{R}_{-\alpha}$ for some real number $\alpha > 0$ and $i \in I$, and $F(\cdot) \in \mathcal{R}_{-\alpha}$. If

$$\lim_{x \to \infty} \frac{\Pr\{X_i > x\}}{F(x)} = c_i, \quad \forall \ i \in I, \quad (2.5)$$

and

$$\lim_{x \to \infty} \frac{\Pr\{X_i > x, X_j > x\}}{F(x)} = 0, \quad \forall \ i \neq j, i,j \in I, \quad (2.6)$$

then

$$\lim_{x \to \infty} \frac{\Pr\{\sum_{i=1}^{n} X_i > x\}}{F(x)} = \sum_{i=1}^{n} c_i. \quad (2.7)$$

Using the properties of the regularly varying distribution, we also obtain the following trivial lemma. Its proof is omitted for brevity:

**Lemma 2.3** For individual net losses $X_i \in \mathcal{R}_{-\alpha}$, where $\alpha > 0$, $i \in I$, conditions (1.4) and (2.6) are, respectively, equivalent to the following two conditions:

$$\lim_{x \to \infty} \frac{F(-\delta x)}{F(x)} = 0, \quad (2.8)$$

and

$$\lim_{x \to \infty} \frac{\Pr\{X_i > \delta_1 x, X_j > \delta_2 x\}}{F(x)} = 0, \quad (2.9)$$

for arbitrary positive constants $\delta, \delta_1$ and $\delta_2$, $i \neq j, i,j \in I$.

### 3. Main Results and Remarks

The following theorem states the main results of this paper. We delay the proof to Section 4.
Theorem 3.1 For model (1.1), suppose individual net losses \( \{X_i, i \in I\} \) are identically distributed as a generic c.d.f. \( F(\cdot) \in \mathcal{R}_{-\alpha} \), \( \alpha > 0 \), satisfying conditions (1.4) and (2.6), then

\[
(a) \quad \psi_r(x, n) \sim \Pr \left\{ \sum_{k=1}^{n} X_k (1 + r)^{-k} > x \right\} \sim \Pr \left\{ \sum_{k=1}^{n} X_k^+ (1 + r)^{-k} > x \right\} \sim F(x) \sum_{k=1}^{n} (1 + r)^{-\alpha k}, \tag{3.1}
\]

where \( X^+ = X 1_{\{X > 0\}} \). If additionally \( r > 0 \), then

\[
(b) \quad \psi_r(x) \sim \Pr \left\{ \sum_{k=1}^{\infty} X_k^+ (1 + r)^{-k} > x \right\} \sim \frac{F(x)}{(1 + r)^{\alpha - 1}}. \tag{3.2}
\]

Remark 3.1 The above Theorem 3.1 also holds true for varying but deterministic interest rates. To see this, let \( v_i \) be the discount factor between time 0 and time \( i \) satisfying \( v_i \geq v_{i+1} > 0, i = 1, 2, \cdots \). Then under the same conditions imposed on individual net losses as in Theorem 3.1, the theorem can be re-stated as:

\[
(a) \quad \psi_r(x, n) \sim \Pr \left\{ \sum_{k=1}^{n} X_k v_k > x \right\} \sim \Pr \left\{ \sum_{k=1}^{n} X_k^+ v_k > x \right\} \sim F(x) \sum_{k=1}^{n} v_k^\alpha, \tag{3.3}
\]

and furthermore, provided \( \sum_{k=1}^{\infty} v_k^\delta < \infty \), for some \( 0 < \delta \leq \min(\alpha, 1) \), we have

\[
(b) \quad \psi_r(x) \sim \Pr \left\{ \sum_{k=1}^{\infty} X_k^+ v_k > x \right\} \sim F(x) \sum_{k=1}^{\infty} v_k^\alpha. \tag{3.4}
\]

The proofs for (3.3) and (3.4) parallel to that for Theorem 3.1 and we therefore omit them.

Remark 3.2 By defining nonnegative r.v.’s \( Y_i \) and \( Z_i \) as, respectively, the total claim amount and the total premium amount in period \( i \), we decompose each individual net loss \( X_i \) such that

\[
X_i = Y_i - Z_i, \quad i \in I. \tag{3.5}
\]

Furthermore, assume that \( Y_i \) and \( Z_i \) are identically distributed as two generic r.v.’s \( Y \) and \( Z \) respectively, and the sequences \( \{Y_i, i \in I\} \) and \( \{Z_i, i \in I\} \) are mutually independent. If we further assume that \( Y \in \mathcal{R}_{-\alpha} \) and \( \mathbb{E}[Z^\beta] < \infty \) with \( \beta > \alpha \), then
\( X \in \mathcal{R}_{-\alpha} \) and condition (1.4) is automatically satisfied. The following justifies this claim.

Denote the c.d.f. of the generic r.v. \( Z \) as \( F_Z(\cdot) \). If \( Y \in \mathcal{R}_{-\alpha} \), then \( Y \in \mathcal{L} \) by (2.2). Hence it follows from the dominated convergence theorem that

\[
\lim_{x \to \infty} \frac{\Pr\{X > x\}}{\Pr\{Y > x\}} = \int_0^\infty \left( \lim_{x \to \infty} \frac{\Pr\{Y > x + z\}}{\Pr\{Y > x\}} \right) dF_Z(z) = 1, \tag{3.6}
\]

which implies \( X \in \mathcal{R}_{-\alpha} \). Moreover, we have

\[
\Pr\{X \leq -x\} = \Pr\{Y - Z \leq -x\} = \int_0^\infty \Pr\{Z \geq x + y\} dF_Y(y) \leq \frac{E[Z^\beta]}{x^\beta}, \tag{3.7}
\]

where the Markov inequality is applied. Consequently, there exists a slowly varying function \( L(x) \) such that

\[
\lim_{x \to \infty} \frac{\Pr\{X \leq -x\}}{\Pr\{X > x\}} \leq \lim_{x \to \infty} \frac{x^{\alpha - \beta}}{L(x)} = 0, \tag{3.8}
\]

which implies (1.4).

In practice, it is reasonable to assume that the premium amount \( Z_i \) is light-tail distributed, and even perhaps bounded, while the claim amount \( Y_i \) tends to be heavy-tail distributed. This suggests that condition (1.4) is quite mild, particularly from a practical point of view.

**Remark 3.3** Condition (2.6) is relevant to the concept of the index of upper tail dependence, which can be defined through the copula function. Let \( U = (U_1, U_2) \) be a bivariate random vector with marginal distributions \( F_1(\cdot) \) and \( F_2(\cdot) \), then Sklar’s Theorem (see Nelsen(2006) or Joe (1997)) states that the dependence structure of \( U_1 \) and \( U_2 \) is completely determined by a bivariate copula function \( C(u,v) \), and the joint distribution function of \( X \) is given by \( C(F_1(\cdot), F_2(\cdot)) \). For a bivariate copula \( C \), the index of upper tail dependence is defined by

\[
\lambda_U = \lim_{v \to 1} \frac{1 - 2v + C(v, v)}{1 - v}. \tag{3.9}
\]

Note that the net losses \( X_i, i \in I \) in our model are identically distributed with common c.d.f. \( F(\cdot) \). Hence, setting \( v = F(x) \) in the above definition (3.9), we have

\[
\lambda_U = \lim_{x \to \infty} \frac{\Pr\{X_i > x, X_j > x\}}{F(x)}, \quad i, j \in I, \tag{3.10}
\]
so that the condition of zero index of upper tail dependence for the net loss sequence \( \{X_k, k \in I\} \) is equivalent to (2.6).

Consequently, to verify condition (2.6), one may turn to model the bivariate dependence structure of these individual net losses via a bivariate copula, and to determine if the copula has zero index of upper tail dependence. This can easily be verified for some families of copulas. For example, if the generator \( \phi(t) \) for an Archimedean copula satisfies

\[
\lim_{t \to 0} \frac{d}{dt} \phi^{-1}(t) \neq -\infty,
\]

where \( \phi^{-1}(\cdot) \) is the inverse of \( \phi(\cdot) \), then the copula has zero index of upper tail dependence. See, for example, Panjer (2006, Chapter 8) for detailed discussion on the Archimedean copulas with respect to the index of upper (lower) tail dependence. Other important families of copulas with zero index of upper tail dependence include Gaussian copulas and Farlie-Gumbel-Morgenstern copulas.

**Remark 3.4**

(a) If the sequence of individual net losses \( \{X_i, i \in I\} \) are negatively quadrant dependent (NQD) (see Joe (1997, p20) or Nelsen (2006, p187) for the definition of NQD sequence), then condition (2.6) is automatically satisfied since in this case

\[
0 \leq \lim_{x \to \infty} \frac{\Pr\{X_i > \delta_1 x, X_j > \delta_2 x\}}{F(x)} \leq \lim_{x \to \infty} \frac{F(\delta_1 x)F(\delta_2 x)}{F(x)} = 0.
\]

(b) Moreover, if the individual net losses are independent, then condition (1.4) can also be dropped in Theorem 3.1. In this special case, we recover Corollary 3.1 of Tang (2004). See Subsection 4.3 for the proof.

**Remark 3.5** When applying Theorem 3.1 to time series data, it is important to verify condition (2.6). While in general the series generated by stochastic processes (such as an autoregressive-moving-average (ARMA) process or a Markov process) need
not necessarily satisfy condition (2.6), we now present an interesting example where this condition is satisfied.\footnote{The authors are grateful to the anonymous referee for pointing out this example.} Suppose the individual net risks $X_i$ can be decomposed as $X_i = Y_i + Z_i$ for $i \in I$, where $\{Y_i, i \in I\}$ and $\{Z_i, i \in I\}$ are two mutually independent processes with the former representing the large catastrophic claims, while the latter denoting the net losses in the ordinary business for an insurance company. Furthermore, suppose $Y_i, i \in I$, are independently and identically distributed from $\mathcal{R}_{-\alpha}$, while $Z_i, i \in I$ are identically distributed, and generated by a process, such as an ARMA process or a Markov process, with tails lighter than that of $Y_i, i \in I$, i.e.,

$$\lim_{x \to \infty} \frac{\Pr\{Z_1 > x\}}{\Pr\{Y_1 > x\}} = 0.$$  

Then, it is easy to show that $X_i \in \mathcal{R}_{-\alpha}$ and $\Pr\{X_i > x\} \sim \Pr\{Y_i > x\}$ for $i \in I$. Moreover, for $i \neq j$ and $x > 0$, we have

$$\Pr\{X_i > x, X_j > x\} = \Pr\{Y_i + Z_i > x, Y_j + Z_j > x\} \leq \Pr\{Y_i > x/2, Y_j > x/2\} + \Pr\{Y_i > x/2, Z_j > x/2\} + \Pr\{Z_i > x/2, Y_j > x/2\} + \Pr\{Z_i > x/2, Z_j > x\} \leq \Pr\{Y_i > x/2\} \Pr\{Y_j > x/2\} + \Pr\{Z_j > x/2\} + \Pr\{Z_i > x/2\} + \Pr\{Z_i > x/2\};$$

which implies condition (2.6).

4. The proof of the main results

4.1 Proof of Theorem 3.1(a)

Since $X_0 = 0$ and

$$\sum_{k=1}^{n} X_k (1 + r)^{-k} \leq \max_{0 \leq m \leq n} \sum_{k=1}^{m} X_k (1 + r)^{-k} \leq \sum_{k=1}^{n} X_k^+ (1 + r)^{-k}, \quad (4.1)$$

it suffices to verify

$$\Pr \left\{ \sum_{k=1}^{n} X_k^+ (1 + r)^{-k} > x \right\} \sim \overline{F}(x) \sum_{k=1}^{n} (1 + r)^{-\alpha k} \quad (4.2)$$
and
\[ \Pr \left\{ \sum_{k=1}^{n} X_k(1+r)^{-k} > x \right\} \geq \overline{F}(x) \sum_{k=1}^{n} (1+r)^{-\alpha k}. \tag{4.3} \]

Expression (4.2) follows immediately from Lemma 2.2 and Lemma 2.3, along with the fact that for \( k = 1, 2, \cdots, \)
\[ \lim_{x \to \infty} \Pr \left\{ X_k + k(1+r)^{-k} > x \right\} = \overline{F}(x) \]
\[ = \lim_{x \to \infty} \frac{\Pr \left\{ X_k > x(1+r)^k \right\}}{\overline{F}(x)} = (1+r)^{-\alpha k}. \tag{4.4} \]

We now establish (4.3). If \( n = 1, \) (4.3) holds trivially since \( F(x) \in \mathcal{R}_{-\alpha}. \) Now suppose \( n \geq 2. \) Let \( v > 1 \) be a constant, \( y = (v-1)/(n-1) > 0, \) and define the following two sets:
\[ A_k = \{ X_k \leq v(1+r)^k \}, \quad A_k^- = \{ X_k \leq -y(1+r)^k \}, \quad k \in I. \tag{4.5} \]

We first analyze the left-hand-side of (4.3):
\[ \Pr \left\{ \sum_{s=1}^{n} X_s(1+r)^{-s} > x \right\} \]
\[ \geq \Pr \left\{ \sum_{s=1}^{n} X_s(1+r)^{-s} > x, \max_{1 \leq k \leq n} X_k(1+r)^{-k} > vx \right\} \]
\[ = \Pr \left\{ \bigcup_{k=1}^{n} \left[ \left( \sum_{s=1}^{n} X_s(1+r)^{-s} > x \right) \cap \left( X_k(1+r)^{-k} > vx \right) \right] \right\} \]
\[ \geq \sum_{k=1}^{n} \Pr \left\{ \sum_{s=1}^{n} X_s(1+r)^{-s} > x, X_k(1+r)^{-k} > vx \right\} \]
\[ - \sum_{1 \leq k \neq l \leq n} \Pr \left\{ \sum_{s=1}^{n} X_s(1+r)^{-s} > x, X_k(1+r)^{-k} > vx, X_l(1+r)^{-l} > vx \right\} \]
\[ = \Delta_1 - \Delta_2. \tag{4.6} \]

Each summand in \( \Delta_1 \) satisfies the following bound:
\[ \Pr \left\{ \sum_{s=1}^{n} X_s(1+r)^{-s} > x, X_k(1+r)^{-k} > vx \right\} \]
\[ \geq \Pr \left\{ \sum_{s=1}^{n} X_s(1+r)^{-s} > x, X_k(1+r)^{-k} > vx, X_s > -y(1+r)^s x, 1 \leq s \leq n, s \neq k \right\} \]
\[ \geq \Pr \left\{ X_k(1+r)^{-k} > vx, X_k(1+r)^{-k} > vx, X_s(1+r)^{-s} > -yx, 1 \leq s \leq n, s \neq k \right\} \]
\[ = \Pr \left\{ X_k(1+r)^{-k} > vx, X_s(1+r)^{-s} > -yx, 1 \leq s \leq n, s \neq k \right\} \]
\[ \geq 1 - \left[ \Pr \{ A_k \} + \sum_{s=1, s \neq k}^{n} \Pr \{ A_s^- \} \right]. \tag{4.7} \]
Moreover, by Lemma 2.3 and condition (1.4), we have for all \( j \in I \),
\[
\frac{\Pr \{ A_k^{-j} \}}{F(x)} \to 0 \text{ a.s. as } x \to \infty.
\]  
(4.8)

Hence, combining (4.7) and (4.8) yields
\[
\lim_{x \to \infty} \frac{\Pr \left\{ \sum_{s=1}^{n} X_s (1 + r)^{-s} > x, X_k (1 + r)^{-k} > vx \right\}}{F(x)} \\
\geq \lim_{x \to \infty} \frac{1 - \Pr \{ A_k \}}{F(x)} \\
= \lim_{x \to \infty} \frac{\Pr \{ X_k > v(1 + r)^k x \}}{F(x)} \\
= v^{-\alpha} (1 + r)^{-\alpha k}, \quad \text{for } k \in I_n,
\]  
(4.9)

and consequently,
\[
\lim_{x \to \infty} \frac{\Delta_1 F(x)}{F(x)} = \sum_{k=1}^{n} v^{-\alpha} (1 + r)^{-\alpha k}.
\]  
(4.10)

On the other hand, applying Lemma 2.3 to each summand in \( \Delta_2 \), we have
\[
\lim_{x \to \infty} \frac{\Pr \left\{ \sum_{s=1}^{n} X_s (1 + r)^{-s} > x, X_k (1 + r)^{-k} > vx, X_l (1 + r)^{-l} > vx \right\}}{F(x)} \\
\leq \lim_{x \to \infty} \frac{\Pr \{ X_k (1 + r)^{-k} > vx, X_l (1 + r)^{-l} > vx \}}{F(x)} \\
= 0, \quad \text{for } 1 \leq k \neq l \leq n,
\]  
(4.11)

which implies
\[
\lim_{x \to \infty} \frac{\Delta_2}{F(x)} = 0.
\]  
(4.12)

Finally combining (4.6), (4.10) and (4.12), we obtain
\[
\frac{\Pr \left\{ \sum_{k=1}^{n} X_k (1 + r)^{-k} > x \right\}}{F(x)} \geq \sum_{k=1}^{n} v^{-\alpha} (1 + r)^{-\alpha k}.
\]  
(4.13)

Letting \( v \to 1 \) in the above expression immediately leads to
\[
\frac{\Pr \left\{ \sum_{k=1}^{n} X_k (1 + r)^{-k} > x \right\}}{F(x)} \geq \sum_{k=1}^{n} (1 + r)^{-\alpha k},
\]  
(4.14)

and this implies (4.3). □
4.2 Proof of Theorem 3.1(b)

We need to show
\[
\psi_r(x) \equiv \Pr \left\{ \max_{0 \leq n < \infty} \sum_{k=1}^{n} X_k (1 + r)^{-k} > x \right\} \sim F(x) \sum_{k=1}^{\infty} (1 + r)^{-\alpha k}, \tag{4.15}
\]
and
\[
\Pr \left\{ \sum_{k=1}^{\infty} X_k^+ (1 + r)^{-k} > x \right\} \sim F(x) \sum_{k=1}^{\infty} (1 + r)^{-\alpha k}. \tag{4.16}
\]

We will only present the proof for (4.15) as (4.16) can be proved similarly.

Using the results of Theorem 3.1(a), we have for each \( n = 1, 2, \cdots \),
\[
\frac{\psi_r(x)}{F(x)} \gtrsim \frac{\Pr \left\{ \max_{1 \leq m \leq n} \sum_{k=1}^{m} X_k (1 + r)^{-k} > x \right\}}{F(x)} = \sum_{k=1}^{\infty} (1 + r)^{-\alpha k} - \sum_{k=n+1}^{\infty} (1 + r)^{-\alpha k}. \tag{4.17}
\]
Since \( r > 0 \), we have \( \sum_{k=1}^{\infty} (1 + r)^{-\alpha k} < \infty \). By letting \( n \to \infty \) in the above expression, we immediately have
\[
\psi_r(x) \gtrsim F(x) \sum_{k=1}^{\infty} (1 + r)^{-\alpha k}. \tag{4.18}
\]

Next, we turn to verify
\[
\psi_r(x) \lesssim F(x) \sum_{k=1}^{\infty} (1 + r)^{-\alpha k} \tag{4.19}
\]
and conclude the proof by combining this and (4.18). First note that for all \( n \in I \),
\[
\max_{0 \leq m < \infty} \sum_{k=1}^{m} X_k (1 + r)^{-k} \leq \max_{0 \leq m \leq n} \sum_{k=1}^{m} X_k (1 + r)^{-k} + \sum_{k=n+1}^{\infty} X_k^+ (1 + r)^{-k} \triangleq A_n + B_n. \tag{4.20}
\]
Thus,
\[
\psi_r(x) \leq \Pr \left\{ A_n > (1 - \theta)x \right\} + \Pr \left\{ B_n > \theta x \right\} \tag{4.21}
\]
for all \( n \in I \) and constant \( \theta \) such that \( 0 < \theta < 1 \). Let us now define \( N_0 = \left\lfloor 2 \ln \left( \frac{1+r}{1-(1+r)^{-\alpha}} \right) \right\rfloor \) where \( \lfloor x \rfloor \) denotes the largest integer less than or equal to \( x \). By
assuming \( n \geq N_0 \), we have for all \( x > 0 \),
\[
\Pr\{B_n > \theta x\} \leq \Pr \left\{ \sum_{k=n+1}^{\infty} X_k^+ (1+r)^{-k} > \sum_{k=n+1}^{\infty} (1+r)^{-k/2} \theta x \right\}
\leq \sum_{k=n+1}^{\infty} \Pr\{X_k^+ > (1+r)^{k/2} \theta x\}
= \sum_{k=n+1}^{\infty} \Pr\{X_k > (1+r)^{k/2} \theta x\}. \tag{4.22}
\]
Since \( F(x) \in \mathcal{R}_{-\alpha} \), we have
\[
\lim_{x \to \infty} \frac{\Pr\{B_n > \theta x\}}{F(x)} \leq \lim_{x \to \infty} \sum_{k=n+1}^{\infty} \frac{\Pr\{X_k > (1+r)^{k/2} \theta x\}}{F(x)}
= \theta^{-\alpha} \sum_{k=n+1}^{\infty} (1+r)^{-\frac{\alpha}{2} k}
= \theta^{-\alpha} \frac{(1+r)^{-\frac{\alpha}{2} (n+1)}}{1 - (1+r)^{-\frac{\alpha}{2}}}. \tag{4.23}
\]
Moreover, it follows from Theorem 3.1(a) that
\[
\Pr\left\{ A_n > (1-\theta)x \right\} \sim (1-\theta)^{-\alpha} \sum_{k=1}^{n} (1+r)^{-\alpha k}. \tag{4.24}
\]
Combining (4.21), (4.23) and (4.24) yields
\[
\psi_r(x) \lesssim F(x) \left[ (1-\theta)^{-\alpha} \sum_{k=1}^{n} (1+r)^{-\alpha k} + \theta^{-\alpha} \frac{(1+r)^{-\frac{\alpha}{2} (n+1)}}{1 - (1+r)^{-\frac{\alpha}{2}}} \right]. \tag{4.25}
\]
Now letting \( n \to \infty \), we immediately obtain
\[
\psi_r(x) \lesssim F(x) \left[ (1-\theta)^{-\alpha} \sum_{k=1}^{\infty} (1+r)^{-\alpha k} \right], \tag{4.26}
\]
and further letting \( \theta \to 0 \) in (4.26) leads to (4.19). This completes the proof. \( \square \)

### 4.3 Proof of Remark 3.4(b)

Note that condition (1.4) is used only for deriving (4.9) in the proof of Theorem 3.1. Based on (4.7), we see that if the individual net losses \( \{X_i, i = 1, 2, \cdots\} \) are independent then
\[
\Pr\left\{ \sum_{s=1}^{n} X_s (1+r)^{-s} > x, X_k (1+r)^{-k} > v x \right\}
\geq \Pr\left\{ X_k (1+r)^{-k} > v x \right\} \cdot \prod_{s=1, s \neq k}^{n} \Pr\left\{ X_s (1+r)^{-s} > -y x \right\}. \tag{4.27}
\]
Now that
\[
\lim_{x \to \infty} \Pr\{X_s > -y(1+r)^s x\} = 1 \text{ for } s = 1, 2, \ldots, \tag{4.28}
\]
we conclude that
\[
\lim_{x \to \infty} \frac{\Pr\left\{ \sum_{s=1}^{n} X_s (1+r)^{-s} > x, X_k (1+r)^{-k} > vx \right\}}{F(x)}
\geq \lim_{x \to \infty} \frac{\Pr\{X_k (1+r)^{-k} > vx\} \cdot \prod_{s=1, s \neq k}^{n} \Pr\{X_s (1+r)^{-s} > -yx\}}{F(x)}
= \lim_{x \to \infty} \frac{\Pr\{X_k (1+r)^{-k} > vx\}}{F(x)}
= v^{-\alpha} (1+r)^{-\alpha k}, \text{ for } k \in I_n, \tag{4.29}
\]
which coincides with the last term in (4.9), and therefore completes the proof. □

5 Conclusion

This paper considers a discrete time risk model under constant interest rate and generally dependent individual net losses that have regular variation distribution and zero index of upper tail dependence. Some asymptotic results for both finite ruin probability and ultimate ruin probability are established. For future research, we will extend these results in three aspects: (1) incorporating stochastic interest rates with certain dependence structure into the model, (2) generalizing the results to heavy-tailed distribution classes for the individual net losses wider than the regular variation class, and (3) establishing similar results for other models, particularly the continuous time risk models.

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