Generalized derivatives of distance functions and the existence of nearest points

Jinsu He\textsuperscript{a,\texttrademark}, Chong Li\textsuperscript{b}

\textsuperscript{a} Department of Mathematics, Zhejiang Normal University, Jinhua, 321004, PR China
\textsuperscript{b} Department of Mathematics, Zhejiang University, Hangzhou 310027, PR China

\textbf{ARTICLE INFO}

Article history:
Received 8 February 2008
Accepted 19 March 2008

Keywords:
Generalized derivatives
Distance function
Nearest point
Locally uniformly convex

\textbf{ABSTRACT}

The relationships between the generalized directional derivative of the distance function and the existence of nearest points as well as some geometry properties in Banach spaces are studied. It is proved in the present paper that the condition that for each closed subset $G$ of $X$ and $x \in X \setminus G$, the Clarke, Michel-Penot, Dini or modified Dini directional derivative of the distance function is 1 or $-1$ implying the existence of the nearest points to $x$ from $G$ is equivalent to $X$ being compactly locally uniformly convex. Similar results for uniqueness of the nearest point are also established.

© 2008 Elsevier Ltd. All rights reserved.

1. Introduction

Let $X$ be a Banach space endowed with the norm $\| \cdot \|$. We use $S$ to denote the unit sphere of $X$. Let $G$ be a nonempty closed subset of $X$. Then the distance function associated with $G$ is defined as

$$d_G(x) := \inf\{\|x - z\| : z \in G\}, \quad \forall x \in X.$$ 

Let $x \in X$ and $z_0 \in G$, $z_0$ is called a nearest point (or best approximation) to $x$ from $G$ if $\|x - z_0\| = d_G(x)$. The set of all nearest points to $x$ from $G$ is denoted by $P_G(x)$, i.e.,

$$P_G(x) = \{z \in G : \|x - z\| = d_G(x)\}.$$ 

If $P_G(x) \neq \emptyset$ for each $x \in X$, then $G$ is said to be proximinal.

Recall that a minimizing sequence for $x$ is a sequence $\{x_n\} \subset G$ satisfying $\lim_{n \to \infty} \|x_n - x\| = d_G(x)$, and recall also that $G$ is approximatively compact for $x$ if each minimizing sequence for $x \in X$ has a converging subsequence. Thus $G$ is called approximatively compact if $G$ is approximatively compact for each $x \in X$. Clearly, if $G$ is approximatively compact for $x$, then $P_G(x) \neq \emptyset$.

Moreover, for any $v \in X$, the one-sided directional derivative of $d_G$ at $x$ is defined by

$$d'_G(x; v) = \lim_{t \to 0^+} \frac{d_G(x + tv) - d_G(x)}{t}.$$ 

Then $|d'_G(x; v)| \leq 1$ for each $v \in S$ if it exists.

\textsuperscript{\texttrademark} Supported in part by the National Natural Science Foundation of China (grant 10671175) and Program for New Century Excellent Talents in University.

\textsuperscript{\textdagger} Corresponding author. Fax: +86 579 82298188.

E-mail addresses: hejinsu@zjnu.cn (J. He), cli@zju.edu.cn (C. Li).

0362-546X/$ – see front matter © 2008 Elsevier Ltd. All rights reserved.
doi:10.1016/j.na.2008.03.043
The relationships between the differentiability property of the distance function and the existence of nearest points have been studied extensively; see for example, [4,6.8–10,15]. In particular, in [11], Fitzpatrick explored the relationships between the one-sided directional derivative of the distance function and the existence of nearest points in Banach spaces. It was proved in [11] that \( X \) is compactly locally uniformly convex (resp. locally uniformly convex), if and only if for each closed subset \( G \) of \( X \) and each \( x \in X \setminus G \), there exists \( v \in S \) such that \( d'_c(x; v) = 1 \) implies that \( G \) is approximatively compact for \( x \) (resp. \( G \) is approximatively compact for \( x \) and \( P_c(x) \) is a singleton), if and only if for each closed subset \( G \) of \( X \) and each \( x \in X \setminus G \), there exists \( v \in S \) such that \( d'_c(x; v) = -1 \) implies that \( P_c(x) \neq \emptyset \). Later in 2002, Li and Ni extended in [12] these results to the case of generalized distance functions.

In the present paper, motivated by the notions of the different kinds of generalized directional derivatives from nonsmooth analysis, we investigate the relationships between these generalized directional derivatives, instead of the one-sided directional derivative, and the existence of nearest points. Similar results are established and some new results are obtained.

2. Preliminaries

We begin with the following known notions for generalized derivatives in nonsmooth analysis; see for example [1,5,13]. Let \( x \in X \) and \( v \in X \).

- The Clarke derivatives of \( d_c \) at \( x \) in the direction \( v \) are
  \[
d'_c(x; v) := \limsup_{y \to x \atop t \to 0^+} \frac{d_c(y + tv) - d_c(y)}{t};
\]
  \[
(d'_c)_0(x; v) := \liminf_{y \to x \atop t \to 0^+} \frac{d_c(y + tv) - d_c(y)}{t}.
\]

- The Michel-Penot derivatives of \( d_c \) at \( x \) in the direction \( v \) are
  \[
d'_c(x; v) := \sup_y \limsup_{t \to 0^+} \frac{d_c(x + ty + tv) - d_c(x + ty)}{t}.
\]
  \[
(d'_c)_0(x; v) := \liminf_y \liminf_{t \to 0^+} \frac{d_c(x + ty + tv) - d_c(x + ty)}{t}.
\]

- The Dini derivatives of \( d_c \) at \( x \) in the direction \( v \) are
  \[
d'_c(x; v) := \limsup_{t \to 0^+} \frac{d_c(x + tv) - d_c(x)}{t};
\]
  \[
(d'_c)_0(x; v) := \liminf_{t \to 0^+} \frac{d_c(x + tv) - d_c(x)}{t}.
\]

- The modified Dini derivatives of \( d_c \) at \( x \) in the direction \( v \) are
  \[
d'_c(x; v) := \sup_{u \in X} (d'_c(x; v + u) - d'_c(x; u));
\]
  \[
(d'_c)_0(x; v) := \inf_{u \in X} [(d'_c)_0(x; v + u) - (d'_c)_0(x; u)].
\]

It is easy to see that
\[
(d'_c)_0(x; v) \leq (d'_c)(x; v) \leq (d'_c)'(x; v) \leq (d'_c)_0(x; v), \quad (2.1)
\]
and in the case when \( d'_c(x; v) \) exists
\[
(d'_c)_0(x; v) = (d'_c)(x; v) = (d'_c)'(x; v) = (2.3)
\]
Furthermore,
\[
-1 \leq (d'_c)_0(x; v) \leq d'_c(x; v) \leq 1, \quad \forall v \in S. \quad (2.4)
\]
For convenience, we set \( \mathfrak{N} = \{0, \circ, +\} \). Then, it follows from (2.1), (2.2) and (2.4) that
\[
d'_c(x; v) = 1 \iff (d'_c)(x; v) = 1 \forall \alpha \in \mathfrak{N}, \forall v \in S \quad (2.5)
\]
and
\[
(d'_c)_0(x; v) = -1 \iff ((d'_c)_0(x; v) = -1 \forall \alpha \in \mathfrak{N}, \forall v \in S. \quad (2.6)
\]
Consequently,
\[
d'_{c}(x; v) = 1 \implies (d_{c}^{2}(x; v) = 1 \forall \alpha \in \mathfrak{R}), \quad \forall v \in S
\] (2.7)
\[
\text{and}
\]
\[
d'_{c}(x; v) = -1 \implies ((d_{c})_{x}(x; v) = -1 \forall \alpha \in \mathfrak{R}), \quad \forall v \in S.
\] (2.8)

**Definition 2.1.** Let \( x \in S \). Then \( X \) is called

(i) compactly locally uniformly convex at \( x \) if, for any sequence \( \{x_{n}\} \subset S \), the condition \( \lim_{n \to \infty} \|x_{n} + x\| = 2 \) implies that \( \{x_{n}\} \) has a converging subsequence;

(ii) locally uniformly convex at \( x \) if, for any sequence \( \{x_{n}\} \subset S \), the condition \( \lim_{n \to \infty} \|x_{n} + x\| = 2 \) implies that \( \lim_{n \to \infty} \|x_{n} - x\| = 0 \);

(iii) strictly convex at \( x \) if, for each \( y \in S \), the condition \( \|y + x\| = 2 \) implies that \( y = x \);

(iv) compactly locally uniformly convex (resp. locally uniformly convex, strictly convex) if so is \( X \) at every point \( x \in S \).

The following two lemmas are known from [11,12], and will be useful in our study.

**Lemma 2.1.** Let \( G \) be a closed nonempty subset of \( X \). Then
\[
|d_{c}(x) - d_{c}(y)| \leq \|x - y\|, \quad \forall x, y \in X.
\]

**Lemma 2.2.** Let \( \{v_{n}\} \subset S \) and \( v \in S \) be such that \( \lim_{n \to \infty} \|v + v_{n}\| = 2 \). Define
\[
G_{0} = \left\{ g_{n} = -\left(1 + \frac{1}{n}\right) \|v_{n} + v\|^{-1}(v + v_{n}) : n = 1, 2, \ldots \right\}.
\]
Then
\[
d'_{c_{0}}(0; v) = 1 \quad \text{and} \quad d'_{c_{0}}(0; -v) = -1.
\]
Consequently,
\[
d'_{c_{0}}(x; v) = d'_{c_{0}}(0; v) = 1, \quad \forall \alpha \in \mathfrak{R}
\] (2.9)
\[
\text{and}
\]
\[
(d_{c_{0}})_{v}(x; -v) = d'_{c_{0}}(0; -v) = -1, \quad \forall \alpha \in \mathfrak{R}.
\] (2.10)

3. Main results

Let \( C(X) \) denote the set of all nonempty closed subsets of \( X \). Let \( x \in X \) and \( G \in C(X) \) be such that \( x \not\in G \). Then \( G \) is called a sun for \( x \) if, for each \( g_{0} \in G \), \( g_{0} \in P_{c}(x) \) (implies that \( g_{0} \in P_{c}(g_{0} + t(x - g_{0})) \) for each \( t \geq 0 \). We use \( \delta_{x}(X) \) to denote the set of all \( G \in C(X) \) such that \( x \not\in G \) and \( G \) is a sun for \( x \). In particular, if \( P_{c}(x) = \emptyset \), then \( G \in \delta_{x}(X) \).

The notions of suns introduced by Efimov and Stechkin in [7] play an important role in nonlinear approximation theory and have been investigated extensively; see, for example, [2,3,7,14]. As is well known (cf. [14]), \( G \) is a sun for \( x \) if and only if for each \( g_{0} \in G \),
\[
g_{0} \in P_{c}(x) \iff \max\{|x^{*}, g_{0} - g| : x^{*} \in E(x - g_{0})\} \geq 0, \quad \forall g \in G,
\]
where \( E(x - g_{0}) \) denotes the set of all extreme points \( x^{*} \) from the unit ball of the dual \( X^{*} \) such that \( \langle x^{*}, x - g_{0} \rangle = \|x - g_{0}\| \).

For convenience of the study in the remainder, we list the following known results taken from [11,12].

**Proposition 3.1.** Let \( v \in S \). Then the following statements are equivalent.

(i) For each \( G \in C(X) \) and \( x \in X \setminus G \), if \( d_{c}^{+}(x; v) = 1 \), then \( G \) is approximatively compact for \( x \).

(ii) For each \( G \in C(X) \) and \( x \in X \setminus G \), if \( (d_{c})_{+}(x; v) = -1 \), then \( P_{c}(x) \neq \emptyset \).

(iii) \( X \) is compactly locally uniformly convex at \( v \).

**Proposition 3.2.** Let \( v \in S \). Then \( X \) is locally uniformly convex at \( v \) if and only if for each \( G \in C(X) \) and \( x \in X \setminus G \), if \( d_{c}^{+}(x; v) = 1 \), then \( G \) is approximatively compact for \( x \) and \( P_{c}(x) \) is a singleton.

The first theorem of the present paper is now as follows.

**Theorem 3.1.** Let \( v \in S \) and \( \beta \in \mathfrak{R} \setminus \{+\} \). Then the following statements are equivalent.

(i) For each \( G \in C(X) \) and \( x \in X \setminus G \), if there exists \( \alpha \in \mathfrak{R} \) such that \( d_{c}^{+}(x; v) = 1 \), then \( P_{c}(x) \neq \emptyset \).
For each $G \in C(X)$ and $x \in X \setminus G$, if $d_C^G(x; v) = 1$, then $P_C(x) \neq \emptyset$.

(iii) For each $G \in C(X)$ and $x \in X \setminus G$, if $d_C^G(x; v) = 1$ for each $\alpha \in \mathcal{R}$, then $P_C(x) \neq \emptyset$.

(iv) For each $G \in C(X)$ and $x \in X \setminus G$, if $d_C^G(x; v) = 1$ for each $\alpha \in \mathcal{R}$, then $G$ is approximatively compact for $x$.

(v) $X$ is compactly locally uniformly convex at $v$.

**Proof.** By (2.1) and (2.4), it is easy to see that (i)⇒(ii)⇒(iii). In view of (2.5), it follows from Proposition 3.1 that (iv) and (v) are equivalent. Hence we only need to prove that (iii)⇒(v) and (v)⇒(i).

To show (iii)⇒(v), we suppose on the contrary that (v) does not hold. Then there exists a sequence $\{v_n\} \subseteq S$ such that $\lim_{n \to \infty} \|v_n + v\| = 2$ but $\{v_n\}$ has no converging subsequences. Let $x \in X$ and define

$$G = \left\{ x - \left(1 + \frac{1}{n}\right)v + v_n : n = 1, 2, \ldots \right\}. \quad (3.1)$$

Then $G \in C(X)$ and $d_C(x) = \inf_{z \in G} \|x - z\| = 1$. Note that, for each $z \in G$, there exists $n = 1, 2, \ldots$ such that $z = x - \left(1 + \frac{1}{n}\right) \left(\frac{1}{n-1}\right)$. Therefore, $\|x - z\| = 1 + \frac{1}{n} > d_C(x)$. This means that $x$ has no nearest point from $G$, i.e., $P_C(x) = \emptyset$.

However, by Lemma 2.2, we have that $d_C^G(x; v) = d_{\mathcal{R}}^G(0; v) = 1$ for each $\alpha \in \mathcal{R}$, which contradicts (iii). Hence (iii)⇒(v) is proved.

To show (v)⇒(i), let $G \in C(X)$ and $x \in X \setminus G$ be such that $d_C^G(x; v) = 1$. Then there exist $\{t_n\} \subseteq \mathbb{R}$ and $\{y_n\} \subseteq X$ such that $t_n \to 0^+$, $y_n \to x$ and

$$\lim_{n \to \infty} \frac{d_C(y_n + t_nv) - d_C(y_n)}{t_n} = 1.$$ 

Without loss of generality, we may assume that $0 < t_n < d_C(x)$. Now take $\{z_n\} \subseteq G$ such that $\|z_n - y_n\| < d_C(y_n) + t_n^2$.

For each $n$, since the function $h_n$ defined by

$$h_n(\tau) := \frac{\|y_n - z_n + t\nu - y_n - z_n\|}{\tau} \quad \text{for each } \tau > 0$$

is monotonically increasing on $(0, +\infty)$, it follows that

$$\frac{\|y_n - z_n + t_nv - y_n - z_n\|}{t_n} \leq \frac{\|y_n - z_n + \|x - z_n\| \cdot \nu - y_n - z_n\|}{\|x - z_n\|}.$$ 

Set $v_n = \frac{\|z_n - \nu\|}{\|z_n - x\|}$ for each $n$. Then $\{v_n\} \subseteq S$ and

$$\frac{d_C(y_n + t_nv) - d_C(y_n)}{t_n} \leq \frac{\|y_n - z_n + t_nv - y_n - z_n\| + t_n^2}{t_n} \leq \frac{\|y_n - z_n + \|x - z_n\| \cdot \nu - y_n - z_n\|}{\|x - z_n\|} + t_n = \|v_n + v - x\|_{\|x - z_n\|} - \|x - z_n\|_{\|x - z_n\|} + t_n.$$ 

This implies that $\lim_{n \to \infty} \|v_n + v\| = 2$ because $t_n \to 0$, $y_n \to x$ and, for each $n$, $\|x - z_n\| \geq d_C(x) > 0$ as assumed. By (v), $\{v_n\}$ has a converging subsequence denoted by $\{v_n\}_m$. Because

$$d_C(x) \leq \|x - x_n\| \leq \|x - y_n\| + \|y_n - z_n\| \leq \|x - y_n\| + d_C(y_n) + t_n^2,$$

we have $\lim_{n \to \infty} \|x - z_m\| = d_C(x)$. Noting that $z_m = x - \|x - z_m\| v_{m, \{v_n\}}$, $\{z_m\}$ converges to some point $z_0 \in G$ and $\|x - z_0\| = d_C(x)$. Hence (i) is seen to hold and the proof is complete.

The following corollary, which is a global version of Theorem 3.1, is a direct consequence of Theorem 3.1 and Lemma 2.2.

**Corollary 3.1.** Let $\beta \in \mathcal{R} \setminus \{+\}$. Then the following statements are equivalent.

(i) For each $G \in C(X)$ and $x \in X \setminus G$, if there exist $\alpha \in \mathcal{R}$ and $v \in S$ such that $d_C^G(x; v) = 1$, then $P_C(x) \neq \emptyset$.

(ii) For each $G \in C(X)$ and $x \in X \setminus G$, if there exists $v \in S$ such that $d_C^G(x; v) = 1$, then $P_C(x) \neq \emptyset$.

(iii) For each $G \in C(X)$ and $x \in X \setminus G$, if there exists $v \in S$ such that $d_C^G(x; v) = 1$ for each $\alpha \in \mathcal{R}$, then $P_C(x) \neq \emptyset$.

(iv) For each $G \in C(X)$ and $x \in X \setminus G$, if there exists $v \in S$ such that $d_C^G(x; v) = 1$ for each $\alpha \in \mathcal{R}$, then $G$ is approximatively compact for $x$.

(v) $X$ is compactly locally uniformly convex.

**Theorem 3.2.** Let $\beta \in \mathcal{R}$. Then the following statements are equivalent.

(i) For each $x \in X$ and $G \in \delta_\beta(X)$, if there exists $v \in S$ and $\alpha \in \mathcal{R}$ such that $d_C^G(x; v) = 1$, then $P_C(x) \neq \emptyset$.
(i) For each \( x \in X \) and \( G \in \mathcal{S}_c(X) \), if there exists \( v \in S \) such that \( d_c^G(x; v) = 1 \), then \( P_c(x) \neq \emptyset \).

(ii) For each \( x \in X \) and \( G \in \mathcal{S}_c(X) \), if there exists \( v \in S \) such that \( d_c^G(x; v) = 1 \) for each \( \alpha \in \mathbb{R} \), then \( P_c(x) \neq \emptyset \).

(iii) \( X \) is compactly locally uniformly convex.

Moreover, the conclusion remains true if the condition that \( P_c(x) \neq \emptyset \) is replaced with a stronger one that \( G \) is approximatively compact for \( x \).

**Proof.** The implications (i) \( \implies \) (ii) \( \implies \) (iii) are clear. Note that for each \( x \in X \) and \( v \in S \), the subset \( G \) defined by (3.1) belongs to \( \mathcal{S}_c(X) \). Hence, by the argument for the proof of (iii) \( \implies \) (v) in Theorem 3.1, one has that \( X \) is compactly locally uniformly convex at \( v \). This proves the implication (iii) \( \implies \) (iv). Furthermore, (iv) \( \implies \) (i) follows from Corollary 3.1. Thus, to complete the proof, it suffices to show that, for each \( x \in X \) and \( G \in \mathcal{S}_c(X) \), if \( G \) is compactly locally uniformly convex and \( P_c(x) \neq \emptyset \), then \( G \) is approximatively compact for \( x \). To this end, let \( \{z_n\} \subseteq G \) be a minimizing sequence for \( x \) and let \( z_0 \in P_c(x) \). Then \( z_0 \in P_c(z_n + 2(x - z_n)) \) and

\[
2\|x - z_n\| \leq \|z_0 + 2(x - z_n) - z_n\| = \|x - z_n + x - z_0\| \leq \|x - z_0\| + \|x - z_n\|.
\]

Taking limits, we have that \( \lim_n \|x - z_n + x - z_0\| = 2\|x - z_0\| \). This together with the compactly locally uniform convexity implies that \( \{x - z_n\} \) and so \( \{z_n\} \) has a converging subsequence. The proof is complete. \( \square \)

The following theorem can be proved with a similar argument to that for Theorem 3.1 and so the proof is omitted here.

**Theorem 3.3.** Let \( v \in S \). Then the following statements are equivalent.

(i) For each \( G \in \mathcal{C}(X) \) and \( x \in X \setminus G \), if there exists \( \alpha \in \mathbb{R} \) such that \( (d_c)_G(x; v) = -1 \), then \( P_c(x) \neq \emptyset \).

(ii) For each \( G \in \mathcal{C}(X) \) and \( x \in X \setminus G \), if \( (d_c)_G(x; v) = -1 \) for each \( \alpha \in \mathbb{R} \), then \( P_c(x) \neq \emptyset \).

(iii) \( X \) is compactly locally uniformly convex at \( v \).

The global version of Theorem 3.3 is as follows.

**Corollary 3.2.** The following statements are equivalent.

(i) For each \( G \in \mathcal{C}(X) \) and \( x \in X \setminus G \), if there exists \( \alpha \in \mathbb{R} \) and \( v \in S \) such that \( (d_c)_G(x; v) = -1 \), then \( P_c(x) \neq \emptyset \).

(ii) For each \( G \in \mathcal{C}(X) \) and \( x \in X \setminus G \), if \( (d_c)_G(x; v) = -1 \) for each \( \alpha \in \mathbb{R} \), then \( P_c(x) \neq \emptyset \).

(iii) \( X \) is compactly locally uniformly convex.

**Lemma 3.1.** Let \( G \in \mathcal{C}(X) \) and \( x \in X \setminus G \). Let \( g_0 \in P_c(x) \) and \( v = \frac{g_0 - x}{\|g_0 - x\|} \). Then \( d'_c(x; v) = -1 \).

**Proof.** Without loss of generality, assume that \( d_c(x) = \|x - g_0\| = 1 \). By definition, \( g_0 \in P_c(g_0 + t(x - g_0)) \) for each \( t \in [0, 1] \). Hence \( d_c(x + tv) = d_c(g_0 + (1 - t)(x - g_0)) = 1 - t \). Consequently, \( d'_c(x; v) = -1 \). \( \square \)

Combining Lemma 3.1 and Theorem 3.3 gives the following corollary.

**Corollary 3.3.** Let \( G \in \mathcal{C}(X) \) and suppose that \( X \) is compactly locally uniformly convex. Then the following statements are equivalent.

(i) \( G \) is proximinal.

(ii) For each \( x \in X \setminus G \), there exist \( \alpha \in \mathbb{R} \) and \( v \in S \) such that \( (d_c)_G(x; v) = -1 \).

(iii) For each \( x \in X \setminus G \), there is \( v \in S \) such that \( (d_c)_G(x; v) = -1 \) for each \( \alpha \in \mathbb{R} \).

**Theorem 3.4.** Let \( v \in S \) and \( \beta \in \mathbb{R} \setminus \{+\} \). Consider the following statements.

(i) For each \( G \in \mathcal{C}(X) \) and \( x \in X \setminus G \), if there exists \( \alpha \in \mathbb{R} \) such that either \( d_c^G(x; v) = 1 \) or \( (d_c)_{G\beta}(x; v) = -1 \), then \( P_c(x) \) is a singleton.

(ii) For each \( G \in \mathcal{C}(X) \) and \( x \in X \setminus G \), if either \( d_c^G(x; v) = 1 \) or \( (d_c)_{G\beta}(x; v) = -1 \), then \( P_c(x) \) is a singleton.

(iii) For each \( G \in \mathcal{C}(X) \) and \( x \in X \setminus G \), if \( (d_c)_{G\beta}(x; v) = -1 \), then \( P_c(x) \) is a singleton.

(iv) For each \( G \in \mathcal{C}(X) \) and \( x \in X \setminus G \), if \( d_c^G(x; v) = 1 \) for each \( \alpha \in \mathbb{R} \), then \( P_c(x) \) is a singleton.

(v) \( X \) is locally uniformly convex at \( v \).

(vi) For each \( G \in \mathcal{C}(X) \) and \( x \in X \setminus G \), if \( d_c^G(x; v) = 1 \) for each \( \alpha \in \mathbb{R} \), then \( G \) is approximatively compact for \( x \) and \( P_c(x) \) is a singleton.

Then the following implications hold.

\[
(i) \implies (ii) \implies (iii') \implies (iv') \downarrow \downarrow \downarrow \\
(iii) \implies (iv) \iff (v) \iff (vi).
\]
Proof. In view of (2.1), (2.2) and (2.4), it is easy to see that the implications (i) $\implies$ (ii) $\implies$ (iii) $\implies$ (iv), and (ii) $\implies$ (iii) $\implies$ (iv) hold, while the implication (vi) $\implies$ (iv) is trivial. By (2.5) and Proposition 3.2, (vi) $\iff$ (v). Furthermore, suppose that (iv) holds. By Theorem 3.1, for each $G \in C(X)$ and $x \in X \setminus G$, if $d_G^c(x; v) = 1$ for each $\alpha \in \mathcal{R}$, then $G$ is approximatively compact for $x$; hence (vi) holds and the equivalence (iv) $\iff$ (vi) is seen to hold.

Thus it remains to verify the implication (iv) $\implies$ (v). To this end, suppose that (iv) holds. Then, by Theorem 3.3, $X$ is compactly locally convex at $v$. Hence it suffices to show that $X$ is strictly convex at $v$. Let $y \in S$ be such that $\|y + v\| = 2$ and define $G = \{x + y, x + v\}$. Then $P_G(x) = G$. This together with Lemma 3.1 implies that $d_G^c(x; v) = −1$. Consequently, $(d_G^c)y(x; v) = −1$ for each $\alpha \in \mathcal{R}$ thanks to (2.8). Hence $P_G(x)$ is a singleton by (iv'), which means that $G$ is a singleton because $P_G(x) = G$. Hence $y = x$ and the proof is complete. □

The following simple example illustrates that: (a) the condition that $P_G(x) \neq \emptyset$ in (i) and (ii) of Theorems 3.1 and 3.3 (and so Corollary 3.2) cannot be improved to the stronger one that $G$ is approximatively compact for $x$; (b) neither the implication (v) $\implies$ (iii) nor the implication (v) $\implies$ (iv') in Theorem 3.4 is valid.

Example 3.1. Let $X$ be an arbitrary locally uniformly convex Banach space of infinite dimension. Let $G = \{x \in X : \|x\| \geq 1\}$ and $x = 0$. Then $G$ is not approximatively compact for $x$ and $P_G(x) = \emptyset$. For $v \in S$, then, by Lemma 3.1, $d_G^c(x; ±v) = −1$, and so $(d_G^c)y(x; v) = −1$ for each $\alpha \in \mathcal{R}$ thanks to (2.8). Furthermore, $d_G^c(x; −v) = −1$ by (2.3). Let $u_0 = −v$. Then

$$d_G^c(x; u_0 + v) = d_G^c(x; 0) = 0.$$ Consequently,

$$d_G^c(x; v) \geq d_G^c(0; u_0 + v) − d_G^c(x; u_0) = 1;$$

hence, $d_G^c(0; v) = 1$.

Theorem 3.5. Let $\beta \in \mathcal{R}$ and $v \in S$. Then the following statements are equivalent.

(i) For each $x \in X$ and $G \in \delta_s(X)$, if there exists $\alpha \in \mathcal{R}$ such that $d_G^c(x; v) = 1$ or $−1$, then $P_G(x) \neq \emptyset$ and/or $P_G(x)$ is a singleton and/or $G$ is approximatively compact for $x$.

(ii) For each $x \in X$ and $G \in \delta_s(X)$, if $d_G^c(x; v) = 1$ or $−1$, then $P_G(x) \neq \emptyset$ and/or $P_G(x)$ is a singleton and/or $G$ is approximatively compact for $x$.

(iii) For each $x \in X$ and $G \in \delta_s(X)$, if $d_G^c(x; v) = 1$ or $−1$ for each $\alpha \in \mathcal{R}$, then $P_G(x) \neq \emptyset$ and/or $P_G(x)$ is a singleton and/or $G$ is approximatively compact for $x$.

(iv) $X$ is locally uniformly convex at $v$.

Proof. We only prove the case when the generalized derivative at $v$ is $1$. As before, the implications (i) $\implies$ (ii) $\implies$ (iii) are clear. The proof of the implication (iii) $\implies$ (iv) is almost the same as that for (iv) $\implies$ (v) of Theorem 3.4. To verify the implication (iv) $\implies$ (i), suppose that $X$ is locally uniformly convex at $v$, and let $x \in X, G \in \delta_s(X)$ be such that there is some $\alpha \in \mathcal{R}$ satisfying $d_G^c(x; v) = 1$. We have to show that $P_G(x) \neq \emptyset, P_G(x)$ is a singleton and $G$ is approximatively compact for $x$. Without loss of generality, assume that $\alpha = 0$, that is, $d_G^c(x; v) = 1$. As in the proof for (iv) $\implies$ (i) of Theorem 3.1, there exists a minimizing sequence $(z_n) \subseteq G$ such that $\lim_{n \to \infty} \|v + v_n\| = 2$, where $v_n = \frac{z_n - z_0}{\|z_n - z_0\|}$. Then by the assumed locally uniform convexity one has that $\lim_{n \to \infty} \|v_n - v\| = 0$ and so $\lim_{n \to \infty} z_n = x - d_G^c(x; v)v$. This means that $z_0 := x - d_G^c(x; v)v \in P_G(x)$. Since $G \in \delta_s(X)$, it follows that $z_0 \in P_G(2x - z_0)$. To show that $P_G(x)$ is a singleton, let $z \in P_G(x)$. Then

$$\|x - z + x - z_0\| = \|z_0 + 2(x - z_0)\| \geq 2\|x - z_0\|. \tag{3.2}$$

Without loss of generality, assume that $d_G^c(x) = \|x - z_0\| = 1$. Then $z - z_0 = v = x - z_0$ thanks to the strict convexity at $v$. Hence $z = z_0$ and $P_G(x)$ is a singleton. Finally, to show that $G$ is approximatively compact for $x$, let $(z_n) \subseteq G$ be a minimizing sequence for $x$. Noting that $d_G^c(x) = \|x - z_0\| = 1$ as assumed, we have that

$$\|x - z_0 + v\| = \|z_0 - x + x - z_0\| = \|z_0 + 2(x - z_0) - z_n\| \geq 2.$$

Taking limits gives that $\lim_{n \to \infty} \|x - z_0 + v\| = 2$. Since $\lim_{n \to \infty} \|x - z_0\| = 1$, it follows from the locally uniform convexity at $v$ that $\lim_{n \to \infty} \|x - z_0 - v\| = 0$ and so $(z_n)$ converges and the proof is complete. □

References


