Hodograph computation and bound estimation for rational B-spline curves

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Abstract It is necessary to compute the derivative and estimate the bound of rational B-spline curves in design systems which has not been studied to date. To improve the function of computer aided design\(1\) CAD\(1\) system\(1\) and to enhance the efficiency of different algorithms of rational B-spline curves\(1\) the representation of scaled hodograph and bound of derivative magnitude of uniform planar rational B-spline curves are derived by applying Dir function\(1\) which indicates the direction of Cartesian vector between homogeneous points\(1\) discrete B-spline theory and the formula of translating the product into a summation of B-spline functions. As an application of the result above\(1\) upper bound of parametric distance between any two points in a uniform planar rational B-spline curve is further presented.

Keywords computer aided design\(1\) CAD\(1\) rational B-spline\(1\) discrete B-spline\(1\) hodograph\(1\) bound of derivative\(1\) distance between two points.

Derivation of parametric curves and surfaces is of great importance in computer graphics and geometric modeling. Although the derivative at any single point on a curve or surface can be calculated by subdivision\(1\) in many cases\(1\) assessing the bound of derivative\(1\) but not the value of derivative\(1\) is usually our concern. Estimation of the bound of derivative direction can help detecting intersections among curves or surfaces\(1\) and estimation of the bound of derivative magnitude also can help improving different algorithms’ efficiency of curves or surfaces. Many authors have studied the derivative of parametric curves and surfaces. For example\(1\) Hermann\(1\)\(^{1}\) derived the derivative formula about lower degree rational Bézier curves\(1\) Sederberg et al.\(1\)\(^{24}\) and Floate\(1\)\(^{30}\) obtained the derivative formula for rational Bézier curves of higher degree and their bounds\(1\) and Selimovic\(1\)\(^{40}\) and Wang et al.\(1\)\(^{50}\) presented the derivative of rational Bézier surfaces of degree \(m \times n\). However\(1\) as the product of B-splines and knot vector analysis are full of difficulties\(1\) there seldom have results about the derivative of rational B-spline curves being the most common tools in geometric design\(1\) which seriously affects the efficiency of algorithms of the B-spline design system.

To deal with the problem above\(1\) firstly\(1\) the hodograph representation of uniform rational B-splines is derived\(1\) and then the bound of derivative magnitude of uniform rational B-splines is calculated by applying Dir function\(1\)\(^{607}\) which indicates the direction of Cartesian vector between homogeneous points\(1\) discrete B-spline theory\(1\)\(^{61}\) and the formula of translating the product into a summation of B-spline functions\(1\)\(^{607}\). The derivation process is of concision\(1\) brilliant geometry signification and rigorous bound estimation. Finally\(1\) using the formula of the bound of derivative magnitude\(1\) upper bound of parametric distance between any two points in a uniform planar rational B-spline curve is presented. The results above undoubtedly promote the function of the design system of rational B-spline curves.

1 Preliminaries and notations

Definition 1. Let \(k\) be a positive integer\(1\) and \(N_{i,k}(t)\) \(x\) given by the following de Boor-Cox recurrence formula\(1\) called the \(i\)th B-spline basis of order \(k\) \(\bar{t}\) degree \(k - 1\) defined on a nondecreasing sequence \(t = [t_0, t_1, \ldots, t_{k+1}]\).
\[
\begin{align*}
N_{i,t}^{(k)}(x) &= \begin{cases} 
1 & \text{if } x \in \mathcal{I}_i, 
0 & \text{otherwise}
\end{cases} \\
N_{i,t}^{(k)}(x) &= \omega_{i,t}^{(k)} N_{i+1,t-1}^{(k)}(x) + \omega_{i+1,t}^{(k)} N_{i,t-1}^{(k)}(x) & k \geq 2
\end{align*}
\]

where
\[
\begin{align*}
\omega_{i,t}^{(k)}(x) &= \omega_{i,t}^{(k)}(x) \\
&= \begin{cases} 
1 & \text{if } x \in \mathcal{I}_i, 
0 & \text{otherwise}
\end{cases} \\
&= \begin{cases} 
0 & t_i < t_{i+k-1}, 
1 & t_i \leq t_{i+k-1} 
\end{cases}
\end{align*}
\]

We denote by \( S^{(k)} \) the linear space spanned by these B-spline bases \( \omega_{i,t}^{(k)} \) and simultaneously present the differential-difference formula.

\[
N_{i,t}^{(k)}(x) = \frac{k-1}{t_{i+k-1} - t_i} N_{i+1,k-1}^{(k-1)}(x) - \frac{k-1}{t_{i+k} - t_{i+1}} N_{i+1,k-1}^{(k-1)}(x) \quad k \geq 2
\]

Let \( \mathbf{t} = [t_0, t_1, \ldots, t_n] \) be a subsequence of \( \mathbf{t} \) then \( S^{(k)} \in S^{(k)} \). And thus the B-splines \( N_{i,t}^{(k)} \) in \( S^{(k)} \) are linear combinations of the B-splines \( N_{i,t}^{(k)} \) i.e.

\[
N_{i,t}^{(k)} = \sum a_{i,t}^{(k)} i N_{i,t}^{(k)}
\]

The coefficients \( a_{i,t}^{(k)} \) above are called discrete B-splines of order \( k \). Discrete B-splines are provided with a recurrence relation similar to the one \( \mathbf{I}_i = [1, \ldots, 1] \) for B-spline bases \( \mathbf{N}_{i,t}^{(k,t)} \).

\[
a_{i,t}^{(k,t)} = a_{i,t}^{(k,t)} i \quad a_{i,t}^{(k,t)} = \omega_{i,t}^{(k,t)} t_{i+k-1} - t_i a_{i,t}^{(k,t)} i + \omega_{i+1,t}^{(k,t)} t_{i+k} - t_{i+1} a_{i+1,t}^{(k,t)} i \quad 4
\]

where \( \omega_{i,t}^{(k,t)} \) is given by \( \mathbf{I}_i \) and \( a_{i,t}^{(k)} i = N_{i,t}^{(k)} t_i \).

**Definition 2.** Let \( k = k_1 + k_2 - 1 \) where \( \mathbf{k} = [1, \ldots, k_1, \ldots, 1, \ldots, 1] \) are positive integers. Suppose \( P_j = [p_{j,1}, p_{j,2}, \ldots, p_{j,k_1-1}] \) is a selection of \( k_1 \) positive integers from the set \( \mathbf{I}_{k_1-1} = [1, \ldots, k_1, \ldots, 1] \) and denote the remaining \( k - k_1 \) positive integers by \( Q_j \) so that \( Q_j = I_{k_1-1} \setminus P_j \), \( j = 1, \ldots, C_{k_1-1}^{k_1-1} \). For a given integer \( i \) we define the corresponding knot vector:

\[
t_i^{P_j} = [t_0, \ldots, t_{i-1}, t_{i+p_{j,1}}, t_{i+p_{j,2}}, \ldots] \\
t_i^{Q_j} = [t_0, \ldots, t_{i-1}, t_{i+q_{j,1}}, t_{i+q_{j,2}}, \ldots]
\]

Let \( \prod_{k=1}^{k-1} \) be the digital set consisting of all the integer subsets based on the above-mentioned principle i.e.

\[
\bigcup_{i=1}^{k-1} \prod_{k=1}^{k-1} = \sum_{i=1}^{k-1} P_i \prod_{Q_i}
\]

**Definition 3.** Suppose that a homogeneous point \( \mathbf{r} = X_i Y_i Z_i \mathbf{w}_i \) has the Cartesian coordinate \( \mathbf{r} = \mathbf{x}_i \mathbf{y}_i \mathbf{z}_i \mathbf{w}_i \). Based on the equality \( \mathbf{D} \mathbf{r} \mathbf{r}_i = \mathbf{w}_i \mathbf{r}_i \mathbf{r}_2 - \mathbf{r}_2 \mathbf{r}_1 \mathbf{w}_1 \mathbf{w}_2 \neq 0 \) we define

\[
\mathbf{D} \mathbf{r}_i \mathbf{r}_j = \mathbf{w}_1 \mathbf{r}_1 \mathbf{w}_2 \mathbf{r}_2 - \mathbf{r}_2 \mathbf{r}_1 \mathbf{w}_1 \mathbf{w}_2 \mathbf{w}_3 \quad 5
\]

with the direction of Cartesian vector between two homogenous points being \( \mathbf{r}_i = X_i Y_i Z_i \mathbf{w}_i \mathbf{j}_i = 1 \mathbf{d}_i \).

2 Deduction of scaled hodograph of rational B-spline curves

Let \( \mathbf{r}_i = w_i x_i \mathbf{x}_i \mathbf{y}_i \mathbf{z}_i \mathbf{w}_i \mathbf{z}_i \mathbf{w}_i \mathbf{w}_i > 0 \) be \( n \) homogenous points in space \( \mathbb{R}^4 \). Suppose \( N_{i,t}^{(k)} \) \( \mathbf{t} \) be the B-spline basis of order \( k \) corresponding to the uniform knot vector \( \mathbf{T} \) defined by \( \mathbf{T} = [t_0, t_1, \ldots, t_n] \). Then the rational parametric curve of order \( k \)

\[
\mathbf{r}(t) = \sum_{i=1}^{n} N_{i,t}^{(k)} \mathbf{r}_i \\
t_k \leq t \leq t_{n+1} \\
n \geq k
\]

is called a uniform rational B-spline curve of order \( k \) defined on the knot vector \( \mathbf{T} \) and \( \mathbf{r}_i \) is its corresponding control point. Moreover in affine coordinate \( \mathbf{E} \mathbf{q} \) \( \mathbf{E} = \sum_{i=1}^{n} \mathbf{N}_{i,t}^{(k)} \mathbf{r}_i \mathbf{w}_i \mathbf{w}_i \mathbf{w}_i \mathbf{w}_i \mathbf{w}_i > 0 \) \( n \) corresponds to

\[
\mathbf{r}(t) = \sum_{i=1}^{n} \mathbf{r}_i \mathbf{w}_i \mathbf{w}_i \mathbf{w}_i \mathbf{w}_i \mathbf{w}_i
\]

\[
\mathbf{r}_i = x_i \mathbf{y}_i \mathbf{z}_i \mathbf{w}_i \mathbf{w}_i \quad t_k \leq t \leq t_{n+1} \\
n \geq k
\]

According to the differential-difference formula above\( \prod_{k=1}^{k-1} \) the derivative of the curve \( \mathbf{E} \) or \( \mathbf{E} \mathbf{d}_i \) in homogenous coordinates is

\[
\mathbf{r}(t) = \sum_{i=1}^{n} \frac{k-1}{t_{i+k-1} - t_i} N_{i,t}^{(k)} \mathbf{r}_i \\
- \sum_{i=1}^{n} \frac{k-1}{t_{i+k} - t_{i+1}} N_{i+1,t}^{(k)} \mathbf{r}_i
\]

Accordingly based on Definition 3 the derivative of...
the above-mentioned curve in Cartesian coordinates is
\[ d\hat{r} dt = \begin{bmatrix} X(t) \\ Y(t) \\ Z(t) \end{bmatrix} \]
and the derivative of a curve is often called hodograph in computer aided geometric design (CAGD).
Hence, Eqs. 12 and 14 are the scaled hodograph and hodograph of the uniform rational B-spline curve of order \( k \) defined by Eqs. 6 or Eq. 7 respectively. Eq. 12 multiplying Eq. 14 by \( \omega(t) \) is a B-spline curve of order \( 2k - 3 \) with the control points \( H_j \).

From all the above analyses we have

**Theorem 1.** When Dir function \( \omega(t) \) which indicates the direction of the Cartesian vector between homogeneous points is the same as Eq. 4. The valued field of \( s \) is the field that makes B-splines nonzero in the corresponding knot vector and

\[
\mathbf{\tau}^P_0 = \cdots \mathbf{\tau}_{i-1} \mathbf{\tau}_i \mathbf{\tau}_{i+p_1} \mathbf{\tau}_{i+p_2} \cdots
\]

\[
\mathbf{\tau}^Q_0 = \cdots \mathbf{\tau}_{j-1} \mathbf{\tau}_j \mathbf{\tau}_{j+q_1} \mathbf{\tau}_{j+q_2} \cdots
\]

Consequently we also have

\[
\mathbf{\tau}^P_0 = \cdots \mathbf{\tau}_{i-1} \mathbf{\tau}_i \mathbf{\tau}_{i+p_1} \mathbf{\tau}_{i+p_2} \cdots
\]

Consequently we can analyze properties of uniform rational B-spline curves.

Apply Eq. 8 Definition 1 and Definition 3 and note that (1) when \( i = j \) \( \mathbf{D}_i \mathbf{\hat{r}}_i \mathbf{\hat{r}}_j = 0 \) holds (2) \( t_j - t_i = t_{i+k} - t_{i+k+1} \) \( i \neq j \) (3) \( t_{i+k} - t_{i+k+1} = \) (4) for arbitrary integers \( i \) and \( j \) we have \( \mathbf{D}_i \mathbf{\hat{r}}_i \mathbf{\hat{r}}_j = - \mathbf{D}_j \mathbf{\hat{r}}_j \mathbf{\hat{r}}_i \). We can express \( \mathbf{D}_i \mathbf{\hat{r}}_i \mathbf{\hat{r}}_j \) \( i \neq j \) again by

\[
\mathbf{D}_i \mathbf{\hat{r}}_i \mathbf{\hat{r}}_j = \mathbf{N}_{i+k} \mathbf{t} \mathbf{N}_{i+k} \mathbf{t} + \mathbf{N}_{i+k+1} \mathbf{t} \mathbf{N}_{i+k+1} \mathbf{t}
\]

Then using the formula (8) of translating the product into a summation of B-spline functions we have

\[
\mathbf{D}_i \mathbf{\hat{r}}_i \mathbf{\hat{r}}_j = \sum_s \mathbf{D}_s \mathbf{\hat{r}}_i \mathbf{\hat{r}}_j
\]

Theorem 1. When Dir function \( \omega(t) \) which indicates the direction of the Cartesian vector between homogeneous points is the same as that in Ref. 6 the scaled hodograph of the uniform rational B-spline curves of order \( k \) is defined by Eqs. 6 or Eq. 7 respectively. Eq. 12 multiplying Eq. 14 by \( \omega(t) \) is a B-spline curve of order \( 2k - 3 \) with the control points \( H_j \).

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hodograph.

3 Bound of derivative magnitude of the uniform rational B-spline curve

In what follows, the property of the bound of derivative magnitude of a uniform rational B-spline curve is presented.

**Theorem 2.** The uniform rational B-spline curve of order \( k \) degree \( k-1 \) is the same as Eq. 6 or Eq. 7. Then the bound of its derivative magnitude satisfies
\[
\| \frac{d}{dt} D(t) \| \leq \frac{1}{d} \max_{1 \leq i \leq n} \min_{1 \leq i \leq n} \max_{1 \leq i \leq n-1} \| \mathbf{r}_{j+i} \| \quad \text{where} \quad d \text{ indicates the accumulated tolerance of the knot vector } \mathbf{T}.
\]

**Proof.** Firstly, deduce an identity. In homogeneous coordinate, take a uniform rational B-spline curve of order \( k \) as \( \mathbf{b}(t) = t^{k-1}t + c_0 \mathbf{c} \) where \( c_0 \) is a constant and satisfies \( c_0 > 0 \) then the equality \( D(t) \| t^{k-1}t + c_0 \mathbf{c} \| = \| D(t) \| t^{k-1}t \| \) holds. From Eq. 12 it follows that \( \| \mathbf{H}_s \| \leq c_0 \| \mathbf{c} \| \) for \( s \) in the range on the other hand \( D(t) \| \mathbf{r}_i \| = \| \mathbf{r}_{i+1} \| \quad \text{where} \quad c_0 \| \mathbf{c} \| \| \mathbf{c} \| \) holds. Inserting the two equalities above into Eq. 13 we can obtain
\[
\| \mathbf{H}_s \| \leq \sum_{1 \leq i \leq n} \left( \sum_{1 \leq j \leq n} \| \mathbf{r}_j \| \right)
\]

So we have
\[
\| \mathbf{H}_s \| \leq \frac{1}{d} \sum_{1 \leq i \leq n} \left( \sum_{1 \leq j \leq n} \| \mathbf{r}_j \| \right) \quad \text{where} \quad d \text{ indicates the accumulated tolerance of the knot vector } \mathbf{T}.
\]

On the other hand, \( D(t) \| \mathbf{r}_i \| \) can be bounded as
\[
\| D(t) \| \mathbf{r}_i \| = \| w_a w_b \| \sum_{i=a}^{i+b-1} \| \mathbf{r}_{i+1} \| - \| \mathbf{r}_i \|
\]

From Eqs. 16 and 17 we can finally obtain
\[
\| \mathbf{H}_s \| \leq \frac{1}{d} \sum_{1 \leq i \leq n} \min_{1 \leq j \leq n} \max_{1 \leq n-1} \| \mathbf{r}_{j+i} \| \quad \text{where} \quad h \text{ is an arbitrary real number.}
\]

4 Upper bound estimate of parametric distance between any two points in a uniform rational B-spline curve

Applying mean value theorem and Theorem 2 we can easily obtain the corollary as follows

**Corollary** The uniform rational B-spline curve of order \( k \) degree \( k-1 \) is the same as Eq. 6 or Eq. 7. Then the upper bound of \( \| D(t) \| t + \mathbf{h} \| \| D(t) \| t \| \) which indicates the parametric distance between any two points in the curve \( \| \mathbf{h} \| \sum_{1 \leq i \leq n} \min_{1 \leq j \leq n} \max_{1 \leq n-1} \| \mathbf{r}_{j+i} \| \). Here \( h \) is an arbitrary real number.

5 Conclusion

In this paper, the representation of scaled hodograph and bound of derivative magnitude of uniform planar rational B-spline curves are derived. The result which is exacted in CAD software design system is novel, simple and applicable. Computing the derivative and the bound of derivative of non-uniform planar rational B-spline curves and surfaces are our future work.