Multi-degree Reduction of Disk Bézier Curves in $L_2$ Norm

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Abstract

A planar Bézier curve whose control points are disks is called a disk Bézier curve. It can be looked as a parametric curve with tolerance in the plane. It is an effective tool to measure or control the error. Based on minimum mean square error, this paper presents an algorithm for optimal multi-degree reduction of disk Bézier curves in $L_2$ norm. First, applying the orthogonal property of Jacobi polynomials, we provide the optimal multi-degree reduced polynomial approximation of the center curve of the original disk Bézier curve in $L_2$ norm, and regard it as the center curve of the degree-reduced one; Next, we transform the other task, optimally multi-degree reduced approximating the error radius curve of the disk Bézier curve, into solving a constrained quadratic programming (QP) problem. Also this paper gives error estimation for this algorithm, and shows some numerical examples to illustrate the correctness and the validity of theoretical reasoning.

Keywords: Disk Bézier Curve; Multi-degree Reduction; Tolerance; Best Mean Square Approximation; $L_2$ Norm; Constrained Quadratic Programming

1 Introduction

In geometric and solid modeling systems, lack of robustness is a fundamental problem. Just as Patrikalakis pointed out [11], “Such errors could drastically change models or even destroy validity of models, no matter how small geometric errors actually are.” In order to solve this problem, in the mid eighty, Mudur [10] and Toth [13] applied the interval arithmetic in geometric process. In 1992, Sederberg et al [12] first put forward the notion of interval Bézier curves, and formally introduced interval analysis into CAD. This kind of interval curve gives a complete description of approximating error. Based on it, Hu et al [5, 6, 7] improved the algorithms for curve/curve or surface/surface intersection, solid modeling, and stability of visualization, and gained some fruits. However, interval curves are lacking in affine invariability, because rectangular intervals are not...
rotationally symmetric. Considering these drawbacks, rectangles may naturally be replaced by disks as control points, and the corresponding Bézier curves are called disk Bézier curves [9]. A disk Bézier curve is also looked as a Bézier curve with tolerance, and it is more useful than interval Bézier curve. On the other hand, when all the control radii of the disk Bézier curve are equal, its boundary curve is just an offset curve of its center curve. Then the construction and properties of disk Bézier curve are tied up with the traditional offset theory.

As for disk Bézier curves, there are some research works. One of the important aspects is degree reduction [3]. Research on degree reduction of ordinary parametric curves started from the data points fitting in model design and shape imitation. With quick development of the techniques of digitizing objects, more and more data points need to fit. And then curves/surfaces of high degree are always generated. To improve the effectiveness of design system, we need to compress shape information. And after degree-reduced processing, the amount of stored information is reduced. In the past ten years, the treatises about degree reduction of parametric curves in $L_2$ norm have greatly come forth [1, 2, 4, 8, 15]. Additionally, degree reduction of disk Bézier curves is also important, especially when it is regarded as an effective tool to measure and control error in manufacture design and test.

Note that the most commonly used measurement standards are $L_\infty$ and $L_2$ in function approximation theory and their corresponding function approximations are uniform approximation and square approximation respectively. Therefore, the functional approximation whose measurement standard is to get the minimum mean square error would be more efficient. So in this paper we discuss the problem of multi-degree reduction of disk Bézier curve in $L_2$ norm. The main idea is divided into two steps. First, applying the orthogonal property of Jacobi polynomials, we provide an optimal multi-degree reduced square approximation of the center curve of a disk Bézier curve; secondly, transform the problem of degree reduction of the error radius curve of the disk Bézier curve into a constrained quadratic programming (QP) problem, and it guarantees that its approximation is also optimal. Combining both steps, we finally obtain the optimal multi-degree reduction of the disk Bézier curve in $L_2$ norm. Many numerical tests show that the algorithm is correct and reliable.

In this paper we first introduce preliminary knowledge: disk Bézier curves, Jacobi polynomials in Section 2. In Section 3, we present an efficient algorithm to solve the problem of optimal multi-degree reduction of disk Bézier curves in $L_2$ norm. Finally, the approximate effect is analyzed and some numerical examples are given in Section 4.

2 Preliminaries

2.1 Disk Bézier Curves

Denote $\mathbb{R}$ as the set of all real numbers and $\mathbb{R}^+$ as the set of all nonnegative real numbers. Then there are some definitions as follows:

**Definition 1** A limited close set $\alpha = [p, r] = \{x \in \mathbb{R}^2 \mid \|x - p\| \leq r, p \in \mathbb{R}^2, r \in \mathbb{R}^+\}$ is called a disk whose centric point is $p$ and radius is $r$.

**Definition 2** For two arbitrary disks $\alpha = [p_1; r_1], \beta = [p_2; r_2]$, the two operations are defined as

$$\alpha + \beta = [p_1 + p_2; r_1 + r_2],$$
\[ k \alpha = k[p_1; r_1] = [kp_1; k| r_1], \quad \forall k \in \mathbb{R}. \]

These two above formulae can be generalized as
\[
\sum_{i=1}^{n} k_i [p_i; r_i] = \left[ \sum_{i=1}^{n} k_i p_i; \sum_{i=1}^{n} |k_i| r_i \right],
\]

**Definition 3** A disk Bézier curve corresponding to \( n + 1 \) disks \([P_i] = [p_i; r_i] (i = 0, 1, \cdots, n)\)
\[ [P](t) = \sum_{i=0}^{n} B_i^n(t)[P_i], \quad 0 \leq t \leq 1, \]
is called a degree \( n \) disk Bézier curve, where \( B_i^n(t) = \binom{n}{i} (1-t)^{n-i}t^i \) is Bernstein base function, \([P_i](i = 0, 1, \cdots, n)\) are called disk control points.

By Definition 2, it is easy to get the expression of the disk Bézier curve as
\[
[P](t) = [P(t); r(t)] = \left[ \sum_{i=0}^{n} B_i^n(t)p_i; \sum_{i=0}^{n} B_i^n(t)r_i \right], \quad 0 \leq t \leq 1. \quad (1)
\]

According to (1), the disk Bézier curve \([P](t)\) can be divided into two parts: the center curve \( \sum_{i=0}^{n} B_i^n(t)p_i (0 \leq t \leq 1) \) and the error radius curve \( \sum_{i=0}^{n} B_i^n(t)r_i \geq 0 (0 \leq t \leq 1) \). The former is a planar Bézier curve \( P(t) \) of degree \( n \) with control points \( p_i (i = 0, 1, \cdots, n) \), and the later is a Bernstein polynomial \( r(t) \) of degree \( n \) with Bézier ordinates \( r_i (i = 0, 1, \cdots, n) \). The disk Bézier curve (1) is virtually a scanning area generated by a disk with a variational radius \( r(t) \) moving its centre along the curve \( P(t) \).

### 2.2 Jacobi Polynomials

A Jacobi polynomial \( J_n^{(u,v)}(x) \) of degree \( n \) is an orthogonal polynomial in the interval \([-1,1]\) about the weight function
\[
\rho(x) = (1-x)^u(1+x)^v, x \in [-1,1] (u > -1, v > -1),
\]
so that
\[
\int_{-1}^{1} (1-x)^u(1+x)^v J_n^{(u,v)}(x) J_m^{(u,v)}(x) dx = \begin{cases} 
0, & n \neq m; \\
\delta_n^{(u,v)}, & n = m.
\end{cases}
\]
where
\[
\delta_n^{(u,v)} = \frac{2^{u+v+1} (n+1) \cdots (n+v)}{2n + u + v + 1 (n + u + 1) \cdots (n + u + v)}. \quad (2)
\]

### 3 Optimal Multi-degree Reduced Approximation of Disk Bézier Curve in \( L_2 \) Norm

In this section, we discuss the problem of optimal multi-degree reduction of disk Bézier curves in \( L_2 \) norm including for its center curve and its error radius curve.
3.1 Description of Approximation Problem

Multi-degree reduction of the disk Bézier curve (1) \([\mathbf{P}](t)\) in \(L_2\) norm is to find a degree \(m(<n)\) disk Bézier curve

\[
[\tilde{\mathbf{P}}](t) = [\mathbf{P}(t), \bar{r}(t)] = \left[ \sum_{i=0}^{m} B_i^m(t)\bar{p}_i; \sum_{i=0}^{m} B_i^m(t)\bar{r}_i \right], \quad 0 \leq t \leq 1,
\]

such that \([\tilde{\mathbf{P}}](t)\) can bound \([\mathbf{P}](t)\), i.e.,

\[
[\mathbf{P}](t) \subseteq [\tilde{\mathbf{P}}](t), \quad 0 \leq t \leq 1,
\]

and \([\tilde{\mathbf{P}}](t)\) bounds \([\mathbf{P}](t)\) as tight as possible in \(L_2\) norm.

The above question can be decomposed to two parts as follows:

- As for the center curve \(\mathbf{P}(t)\) of the disk Bézier curve (1), find an optimal \(n-m\) degree reduced approximating curve \(\tilde{\mathbf{P}}(t)\) in \(L_2\) norm to be the center curve of the disk Bézier curve (3).

- As for the error radius curve \(r(t)\) of the disk Bézier curve (1), find an optimal \(n-m\) degree reduced approximating function \(\bar{r}(t)\) to be the error radius curve of the disk Bézier curve (3). Here the error between \(\bar{r}(t)\) and \(r(t)\) should be minimum in \(L_2\) norm under the precondition that the disk Bézier curve (3) bounds the disk Bézier curve (1).

Based on the above analysis, the problem of optimal \(n-m(m<n)\) degree reduced approximation of the degree \(n\) disk Bézier curve (1) can be described by mathematical formulae (A) \(\cup\) (B), where

\[
\text{Task (A) min } ||\tilde{\mathbf{P}}(t) - \mathbf{P}(t)||_{L_2};
\]

\[
\text{Task (B) } \min_{\bar{r}(t) \geq r(t) + \text{dist}(\tilde{\mathbf{P}}(t), \mathbf{P}(t))} ||\bar{r}(t) - r(t)||_{L_2};
\]

where \(\text{dist}(\tilde{\mathbf{P}}(t), \mathbf{P}(t)) (0 \leq t \leq 1)\) is the Hausdorff distance between the curve \(\tilde{\mathbf{P}}(t)\) and the curve \([\mathbf{P}](t)\) (See Fig. 1).

3.2 Multi-degree Reduced Approximation of Center Curve

Applying the orthogonal property of Jacobi polynomials, [14] presents an explicit expression of optimal multi-degree reduced approximation of a Bézier curve. We directly applied this algorithm for multi-degree reduced approximation of center curve.

**Lemma 1** [14] Let \(R_n(t)\) be a Bernstein polynomial of degree \(n\) in the interval \([0,1]\), and let \(b_i (i = 0, 1, \ldots, n)\) be its Bézier ordinates. Then \(R_n(t)\) can be optimally multi-degree reduced approximate to an \(m\)-degree \((m<n)\) one \(\bar{R}_m(t)\) in \(L_2\) norm, which has the same derivatives up to \(v-1(\geq 0)\)-th and \(u-1(\geq 0)\)-th orders respectively at the endpoints as the polynomial \(R_n(t)\). And the Bézier ordinates \(\bar{b}_i (i = 0, 1, \ldots, m)\) of the Bernstein polynomial \(\bar{R}_m(t)\) can be expressed as

\[
(\bar{b}_0, \bar{b}_1, \ldots, \bar{b}_m)^T = G (b_0, b_1, \ldots, b_n)^T,
\]

\[
G = FB_1 - F_2A_1A_2^{-1}B_2.
\]
where

\[ A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} c_0^{m-u-v+1} & c_0^{m-u-v+2} & \ldots & c_0^{n-u-v-1} & c_0^{n-u-v} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{m-v+1}^{m-u-v+1} & c_{m-v+1}^{m-u-v+2} & \ldots & c_{m-v+1}^{n-u-v-1} & c_{m-v+1}^{n-u-v} \\ 0 & c_{m-v+2}^{m-u-v+2} & \ldots & c_{m-v+2}^{n-u-v-1} & c_{m-v+2}^{n-u-v} \\ 0 & 0 & \ldots & c_{m-v+3}^{n-u-v-1} & c_{m-v+3}^{n-u-v} \end{pmatrix}, \]

\[ c_j^l = \begin{cases} \sum_{k=0}^{l} (-1)^{k-j} \binom{l+2u}{l-k} \binom{l+2u+2v+k}{k} \binom{k+u}{j}, & 0 \leq j \leq u; \\ \sum_{k=j-u}^{l} (-1)^{k-j} \binom{l+2u}{l-k} \binom{l+2u+2v+k}{k} \binom{k+u}{j}, & u+1 \leq j \leq l+u; \end{cases} \]

\[ l = m - u - v + 1, m - u - v + 2, \ldots, n - u - v. \]
\( A_1, A_2 \) are the \((m - v + 1) \times (n - m)\) matrix and \((n - m) \times (n - m)\) matrix, respectively;

\[
B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = (b_{i,j})_{(n+1)(n+1)}, \quad b_{i,j} = \begin{cases} 0, & i < j, \\ (-1)^{i+j} \binom{n}{i} \binom{i}{j}, & i \geq j, \end{cases} i, j = 0, 1, \ldots, n.
\]

\( B_1, B_2 \) are the \((m + 1) \times (n + 1)\) matrix and \((n - m) \times (n + 1)\) matrix, respectively;

\[
F = (F_1, F_2) = (f_{i,j})_{(m+1) \times (m+1)}, \quad f_{i,j} = \begin{cases} 0, & i < j, \\ \binom{m - j}{i-j} / \binom{m}{i}, & i \geq j, \end{cases} i, j = 0, 1, \ldots, m.
\]

\( F_1, F_2 \) are the \((m + 1) \times v\) matrix and \((m + 1) \times (m - v + 1)\) matrix, respectively, and the error is

\[
Er = 2^{-u-v-\frac{1}{2}} \sqrt{b_{2n,u,v}^2 + b_{n-u-v}^2 + \cdots + b_{m-u-v+1}^2}
\]

where \(\delta^{(2n,v)}_v(i = m + 1, \ldots, n)\) are shown as (2) and \(\delta^{(2n,v)}_v(i = m + 1, \ldots, n)\) satisfy

\[
\begin{pmatrix} \tilde{b}_{m-u-v+1}, \tilde{b}_{m-u-v+2}, \ldots, \tilde{b}_{n-u-v} \end{pmatrix} = A_2^{-1} B_2 (b_0, b_1, \ldots, b_n)^T.
\]

Applying Lemma 1, we can obtain the optimal multi-degree reduced approximation curve of the degree \(n\) center curve \(P(t) = (x_n(t), y_n(t))\) of the disk Bézier curve (1), i.e., to perform Task (A). The concrete steps are as follows: for the coordinate functions \(x_n(t)\) and \(y_n(t)\), we obtain the optimal multi-degree reduced approximation polynomials \(\bar{x}_m(t)\) and \(\bar{y}_m(t)\), respectively. And then \((\bar{x}_m(t), \bar{y}_m(t))\) are regarded as the degree \(m\) center curve \(P(t)\), of the degree-reduced disk Bézier curve (3). Further, the accurate approximating error in the norm \(L_2\) is

\[
\varepsilon_c = \| \bar{P}(t) - P(t) \|_{L_2} = \sqrt{\| \bar{x}_m(t) - x_n(t) \|_{L_2}^2 + \| \bar{y}_m(t) - y_n(t) \|_{L_2}^2} = \sqrt{E_{x}^2 + E_{y}^2},
\]

where \(E_{x}, E_{y}\) are represented as the accurate minimum error of degree-reduced approximation of \(x_n(t)\) and \(y_n(t)\), and are shown as (6).

On the other hand, the maximum distance \(\text{dist}^\ast\) between the original center curve and degree-reduced center curve is the Hausdorff distance between the two curves, and it satisfies \(\text{dist} \leq \| \bar{P}(t) - P(t) \|_{\infty} = \sqrt{\| (\bar{x}_m(t) - x_n(t))^2 + (\bar{y}_m(t) - y_n(t))^2 \|_{\infty}}\). According to Eq. (12) in [14], for the degree-reduced approximation of every component, its remainder function can be written as

\[
R_n(t) - \bar{R}_m(t) = (1 - t)^n t^v R_{n-u-v}^R (2t - 1) = (t^v, \ldots, t^n) A \begin{pmatrix} \tilde{b}_{m-u-v+1}^R, \tilde{b}_{m-u-v+2}^R, \ldots, \tilde{b}_{n-u-v}^R \end{pmatrix}^T.
\]

Substituting (7) into the above formula, we have

\[
R_n(t) - \bar{R}_m(t) = (t^v, \ldots, t^n) A A_2^{-1} B_2 (b_0^R, b_1^R, \ldots, b_n^R)^T,
\]

(8)
where \( R_n(t) = x_n(t), \) or \( y_n(t); R = x, \) or \( y; 0 \leq t \leq 1; \) and \( \{ b_i^R \}_{i=0}^n \) are the Bézier ordinates of \( R_n(t). \)

So according to (8) and applying the mathematical software Matlab, we can compute the maximums \( \| \bar{x}_m(t) - x_n(t) \|_\infty, \) \( \| \bar{y}_m(t) - y_n(t) \|_\infty \) in \( x \) and \( y \) directions respectively, and hence

\[
\text{dist} \leq d = \sqrt{\| \bar{x}_m(t) - x_n(t) \|_\infty + \| \bar{y}_m(t) - y_n(t) \|_\infty}
\] (9)

It should be pointed out that in order to approximate the error radius function of the disk Bézier curve (1) as good as possible, we usually choose the constrained condition as \( u = v = 1 \) when degree-reduced approximating the center curve.

### 3.3 Multi-degree Reduced Approximation of Error Radius Curve

First, we elevate the degree of the error radius curve \( \bar{r}(t) \) from \( m \) to \( n. \) Then the curve can be represented as

\[
\bar{r}(t) = \sum_{i=0}^{m} B_i^m(t) \bar{r}_i = \sum_{i=0}^{n} B_i^n(t) \bar{r}_i,
\] (10)

where

\[
\bar{r}_i = \sum_{j=\max(0,i+m-n)}^{\min(i,m)} \binom{i}{j} \binom{n-i}{m-i} / \binom{n}{m}, \quad i = 0, 1, \ldots, n.
\]

We first introduce the following simple sufficient condition to satisfy the precondition of Task (B) for degree-reduced approximating the disk Bézier curve (1), so that the disk Bézier curve (3) can bound the disk Bézier curve (1).

**Lemma 2** A sufficient condition for the precondition of Task (B) is the following two formulae holding simultaneously

\[
\hat{r}_i \geq r_i + d, \quad i = 0, 1, \ldots, n,
\] (11)

\[
\Delta^i \hat{r}_0 = 0, \quad i = m + 1, \ldots, n,
\] (12)

where \( \Delta^i \hat{r}_0 = \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} \hat{r}_j, \) and \( d \) is shown as in (9).

**Proof** Suppose (11) holds, for any \( t \in [0, 1], \) there is

\[
\bar{r}(t) - r(t) \geq d = \sqrt{\| \bar{x}_m(t) - x_n(t) \|_\infty + \| \bar{y}_m(t) - y_n(t) \|_\infty}
\]

\[
\geq \sqrt{\| \bar{x}_m(t) - x_n(t) \|_2^2 + \| \bar{y}_m(t) - y_n(t) \|_2^2}
\]

\[
\geq \max_{0 \leq s_0 \leq 1} \min_{0 \leq t_0 \leq 1} \| \bar{P}(t_0) - P(s_0) \| = \text{dist} \left( \bar{P}(t), P(t) \right).
\]
In addition, (12) guarantees that the highest degree of the curve $\bar{r}(t)$ is $m$. So the precondition of Task (B) holds, and Lemma 2 is proved.

As for (11), without loss of generality, suppose
\[
\varepsilon_i = \hat{r}_i - r_i - d, \quad i = 0, 1, \ldots, n, \tag{13}
\]
then the problem of multi-degree reduction of error radius function, i.e., degree reduction Task (B), can be transformed to find the optimal solution of the following problem:
\[
\begin{align*}
\min & \left\| \sum_{i=0}^{n} B_i^n(t) \varepsilon_i \right\|_{L_2} \\
\text{s.t.} & \Delta^i \varepsilon_0 + \Delta^i r_0 = 0, \quad i = m + 1, \ldots, n, \\
& \varepsilon_i \geq 0, \quad i = 0, 1, \ldots, n.
\end{align*} \tag{14}
\]

Next we show that the object function in (14) can be transformed to a quadratic form. This is the academic base of the optimal algorithm of the error radius curve.

**Theorem 1** Problem (14) can be transformed to a constrained quadratic programming problem as follows:
\[
\begin{align*}
\min & \varepsilon^T H \varepsilon \\
\text{s.t.} & \Delta^i \varepsilon_0 + \Delta^i r_0 = 0, \quad i = m + 1, \ldots, n, \\
& \varepsilon_i \geq 0, \quad i = 0, 1, \ldots, n.
\end{align*} \tag{15}
\]
where
\[
H = (h_{i,j})_{n \times n}, \quad h_{i,j} = \left( \begin{array}{c} n \\ i \\ \end{array} \right) \left( \begin{array}{c} n \\ j \\ \end{array} \right) / \left( \begin{array}{c} 2n \\ i+j \\ \end{array} \right).
\]

**Proof** First we consider the object function in (15). It is obvious that
\[
\left( \sum_{i=0}^{n} B_i^n(t) \varepsilon_i \right)^2 = \left[ (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n) (B_0^n(t), B_1^n(t), \ldots, B_n^n(t))^T \right] \\
\times \left[ (B_0^n(t), B_1^n(t), \ldots, B_n^n(t)) (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n)^T \right] \\
= \varepsilon^T \left[ (B_0^n(t), B_1^n(t), \ldots, B_n^n(t))^T (B_0^n(t), B_1^n(t), \ldots, B_n^n(t)) \right] \varepsilon \\
= \varepsilon^T \left( \begin{array}{cccc}
B_0^n(t) & B_0^n(t) & \cdots & B_0^n(t) \\
B_0^n(t) & B_1^n(t) & \cdots & B_0^n(t) \\
\vdots & \vdots & \ddots & \vdots \\
B_0^n(t) & B_n^n(t) & \cdots & B_0^n(t) \\
\end{array} \right) \varepsilon.
\]
Then

\[
\left\| \sum_{i=0}^{n} B^n_i(t) \varepsilon_i \right\|_{L_2} = \varepsilon^T \begin{pmatrix}
\int_{0}^{1} B^n_0(t) B^n_0(t) dt & \int_{0}^{1} B^n_0(t) B^n_1(t) dt & \cdots & \int_{0}^{1} B^n_0(t) B^n_n(t) dt \\
\int_{0}^{1} B^n_1(t) B^n_0(t) dt & \int_{0}^{1} B^n_1(t) B^n_1(t) dt & \cdots & \int_{0}^{1} B^n_1(t) B^n_n(t) dt \\
\vdots & \vdots & \ddots & \vdots \\
\int_{0}^{1} B^n_n(t) B^n_0(t) dt & \int_{0}^{1} B^n_n(t) B^n_1(t) dt & \cdots & \int_{0}^{1} B^n_n(t) B^n_n(t) dt
\end{pmatrix} \varepsilon,
\]

where for \(i, j = 0, 1, \ldots, n\)

\[
\int_{0}^{1} B^n_i(t) B^n_j(t) dt = \int_{0}^{1} \binom{n}{i} \binom{n}{j} (1 - t)^{2n-i-j} t^{i+j} dt
\]

\[
= \binom{n}{i} \binom{n}{j} \frac{1}{(2n+1)} \int_{0}^{1} B^{2n}_{i+j}(t) dt = \binom{n}{i} \binom{n}{j} \frac{2n}{(i+j)^{2n+1}}.
\]

By the above two formulae, we have

\[
\left\| \sum_{i=0}^{n} B^n_i(t) \varepsilon_i \right\|_{L_2} = \frac{1}{(2n+1)} \varepsilon^T H \varepsilon.
\]

This completes the proof.

As for a constrained quadratic programming problem like (15), we can use mathematical software Matlab to solve the solution, and have the following theorem.

**Theorem 2** The Bézier ordinates of the multi-degree reduced error radius curve (7) of the disk Bézier curve (1), \(\bar{r}_j(j = 0, 1, \ldots, m)\), satisfy the following system of linear equations as follows:

\[
\sum_{j=\max(0,i+m-n)}^{\min(i,m)} \bar{r}_j \binom{i}{j} \binom{n-i}{m-i} / \binom{n}{m} = r_i + d + \varepsilon_i, \quad i = 0, 1, \ldots, n.
\]

where \(d\) and \(r_i\) are as in (7) and (1) respectively, and the approximating error is \(\varepsilon^T H \varepsilon/(2n+1)\).

**Proof** It can be directly proved from Theorem 1.
4 Error Estimation and Examples

For the multi-degree reduced approximation, the bounding error and relative bounding error are respectively defined as

\[
err([P](t), [\bar{P}](t)) := \|\bar{r}(t) - r(t)\|_\infty + \|\text{dist} (\bar{P}(t), P(t))\|_\infty,
\]

\[
\bar{err}([P](t), [\bar{P}](t)) := \frac{\|\bar{r}(t) - r(t)\|_\infty + \|\text{dist} (\bar{P}(t), P(t))\|_\infty}{\|P(t)\|_\infty},
\]

where

\[
\|P(t)\|_\infty = \max \left(\max_{0 \leq i \leq n} x_i - \min_{0 \leq i \leq n} x_i, \max_{0 \leq i \leq n} y_i - \min_{0 \leq i \leq n} y_i\right).
\]

By the proof of Lemma 2, it follows that

\[
\|\text{dist} (\bar{P}(t), P(t))\|_\infty \leq \sqrt{\|\bar{x}_m(t) - x_n(t)\|_\infty + \|\bar{y}_m(t) - y_n(t)\|_\infty} = d.
\]

And by (10) and (13), noting that \(\varepsilon_i \geq 0 (i = 0, 1, \ldots, n)\), we have

\[
\|\bar{r}(t) - r(t)\|_\infty \leq d + \left\| \sum_{i=0}^{n} B_i^n(t) \varepsilon_i \right\|_\infty \leq d + \max_{0 \leq i \leq n} \{\varepsilon_i\} (16)
\]

**Example 1.** Given a disk Bézier \([P](t)\) of degree five. Its control disks are \([p_0, r_0] = [(96, 141, 1)], [p_1, r_1] = [(104, 271, 10)], [p_2, r_2] = [(178, 363, 15)], [p_3, r_3] = [(331, 378, 15)], [p_4, r_4] = [(486, 285, 10)], [p_5, r_5] = [(486, 140, 6)]. We will present its one-degree and two-degree reduced disk Bézier curves in the norm \(L_2\) respectively.

We first consider the case of one-degree reduction. For the center curve \(P(t)\) of this disk Bézier curve, according to Lemma 1, we obtain the matrix \(G\) as follows:

\[
G = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
1/12 & -5/12 & 5/6 & 5/6 & -5/12 & 1/12 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix},
\]

and hence the control points of the center curve after degree reduction are

\[
G \times ((96, 141), (104, 271), (178, 363), (331, 378), (486, 285), (486, 140))^T = ((105.7917, 301.8542), (226.8333, 409.25), (486.2083, 322.8958), (486, 140))^T.
\]

By (8), we know that the distance between the original center curve and the degree-reduced center curve satisfies \(dist \leq d = \sqrt{\max_{0 \leq t \leq 1} |\dot{x}(t)| + \max_{0 \leq t \leq 1} |\dot{y}(t)|}\), where \(\dot{x}(t) = 0.8333t^4 - 7.5t^2 + 21.6667t^3 - 25t^4 + 10t^5, \dot{y}(t) = 6.5833t^5 - 59.25t^2 + 171.1667t^3 - 197.5t^4 + 79t^5\).

Then applying the mathematical software Matlab, we can calculate \(d = 0.2291\), and the solution of the problem (14) is \(\varepsilon = (0, 0, 0.5, 0, 0)\). Then the control radii of the degree-reduced error radius curve are 1.2291, 12.4791, 17.8958, 11.2291, 6.2291 respectively. (See Fig. 2)
Finally, by (16), the corresponding absolute error is $err = 0.7291$, and the relative error is $e\bar{err} = 0.19\%$.

Next the case of two-degree reduction is discussed. Analogizing to the above one-degree reduced, for the center curve $P(t)$ of this disk Bézier curve, applying the algorithm in Section 3.2, straightforward computation gives the control points of the center curve after degree reduction are $(96, 141), (78.6984, 372.4365), (455.9201, 400.8254), (486, 140)$ respectively.

By (8), we know that the distance between the original center curve and the degree-reduced center curve satisfies $\text{dist} \leq d = \sqrt{\max_{0 \leq t \leq 1} |\tilde{x}(t)| + \max_{0 \leq t \leq 1} |\tilde{y}(t)|}$, where $\tilde{x}(t) = 91.9048t - 523.5714t^2 + 871.6667t^3 - 450t^4 + 10t^5$, $\tilde{y}(t) = 44.3095t - 229.1429t^2 + 303.8333t^3 - 40t^4 - 79t^5$.

Next, applying Matlab, we can calculate $d = 5.7071$, and the solution of the problem (14) is $\varepsilon = (0, 2/3, 0, 5/3, 0)$. Then the control radii of the degree-reduced error radius curve are $6.7071, 22.8183, 21.1514, 11.7076$ respectively. (See Fig. 3)

By (16), the corresponding absolute error is $err = 7.3738$, and the relative error is $e\bar{err} = 1.89\%$.

**Example 2.** Given a degree 8 disk Bézier curve $[P](t)$. Its control disks are $[p_0, r_0] = [(61, 149), 10], [p_1, r_1] = [(86, 303), 4], [p_2, r_2] = [(203, 449), 10], [p_3, r_3] = [(357, 430), 15], [p_4, r_4] = [(412, 328), 20], [p_5, r_5] = [(385, 115), 18], [p_6, r_6] = [(482, 81), 8], [p_7, r_7] = [(661, 102), 10], [p_8, r_8] = [(705, 237), 5]. We will present its three-degree reduced disk Bézier curves in the norm $L_2$.

For the center curve $P(t)$ of this disk Bézier curve, applying the algorithm in Section 3.2, we obtain the control points of the center curve after degree reduction are $(61, 149), (72.8536, 398.1963), (534.7231, 602.1497), (241.1259, 43.6434), (618.3361, 33.0606), (705, 237)$, respectively.
By (8), we know that the distance between the original and degree-reduced center curves satisfies
\[ \text{dist} \leq d = \sqrt{\max_{0 \leq t \leq 1} |\tilde{x}(t)| + \max_{0 \leq t \leq 1} |\tilde{y}(t)|}, \]
where \( \tilde{x}(t) = 140.7319t - 1924.1585t^2 + 8974.8252t^3 - 18828.7832t^4 + 1812.3385t^5 - 5544t^6 - 2200t^7 + 1258t^8 \), \( \tilde{y}(t) = -13.9814t + 228.4289t^2 - 1619.8322t^3 + 6592t^4 - 15872.6154t^5 + 21756t^6 - 15496t^7 + 4426t^8 \).

Next, applying Matlab, we can calculate \( d = 3.4374 \), and the solution of the problem (14) is \( \varepsilon = (0.0224, 0.06571, 2.5936, 0, 0, 5.9936, 0, 0) \). Then the control radii of the degree-reduced error radius curve are 13.4598, 3.8240, 29.6961, 21.5983, 16.5212, 7.6791. (See Fig. 4)

By (16), the corresponding absolute error is \( err = 9.4310 \), and the relative error is \( \bar{err} = 1.46\% \).

![Fig. 3: A degree 5 disk Bézier curve (with black) and its 2-degree reduced curve (with red) with their control disks.](image1)

![Fig. 4: A degree 8 disk Bézier curve (with black) and its 3-degree reduced curve (with red) with their control disks.](image2)

5 Conclusion

Degree reduction of disk Bézier curves is urgently required for error measurement and control in industry. In this paper, we propose a method based on Jacobi polynomials and quadratic programming to solve the problem of multi-degree reduction of disk Bézier curves in \( L^2 \) norm. And some examples show that our method is a multi-degree reduction and has good approximation. For center curve or error radius curve, the corresponding method for degree reduction is an optimal approximation because of theoretical guarantee. Therefore our method is credible and effective. In addition, the operations of this method are very convenient. For the center curve, the control points of the degree-reduced center curve can be obtained by multiplying an explicit matrix with the control points of the original center curve. This kind of matrix can be calculated beforehand according to its parameters \((n, m, u, v)\), stored in computer and used at any moment like the table of trigonometric functions. For the error radius curve, the degree-reduced error radius curve can be obtained by solving a constrained quadratic programming using Matlab.

In future, we will consider the method to combine constrained multi-degree reduction of disk Bézier curves with subdivision.

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