SOLUTION The inequalities

\[ 1 \leq x^2 + y^2 + z^2 \leq 4 \]

can be rewritten as

\[ 1 \leq \sqrt{x^2 + y^2 + z^2} \leq 2 \]

so they represent the points \((x, y, z)\) whose distance from the origin is at least 1 and at most 2. But we are also given that \(z \leq 0\), so the points lie on or below the \(xy\)-plane. Thus, the given inequalities represent the region that lies between (or on) the spheres \(x^2 + y^2 + z^2 = 1\) and \(x^2 + y^2 + z^2 = 4\) and beneath (or on) the \(xy\)-plane. It is sketched in Figure 11.

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**Exercises**

1. Suppose you start at the origin, move along the \(x\)-axis a distance of 4 units in the positive direction, and then move downward a distance of 3 units. What are the coordinates of your position?

2. Sketch the points \((3, 0, 1), (-1, 0, 3), (0, 4, -2)\), and \((1, 1, 0)\) on a single set of coordinate axes.

3. Which of the points \(P(6, 2, 3), Q(-5, -1, 4)\), and \(R(0, 3, 8)\) is closest to the \(xz\)-plane? Which point lies in the \(yz\)-plane?

4. What are the projections of the point \((2, 3, 5)\) on the \(xy\)-, \(yz\)-, and \(xz\)-planes? Draw a rectangular box with the origin and \((2, 3, 5)\) as opposite vertices and with its faces parallel to the coordinate planes. Label all vertices of the box. Find the length of the diagonal of the box.

5. Describe and sketch the surface in \(\mathbb{R}^3\) represented by the equation \(x + y = 2\).

6. (a) What does the equation \(x = 4\) represent in \(\mathbb{R}^3\)? What does it represent in \(\mathbb{R}^2\)? Illustrate with sketches.

(b) What does the equation \(y = 3\) represent in \(\mathbb{R}^3\)? What does \(z = 5\) represent? What does the pair of equations \(y = 3, z = 5\) represent? In other words, describe the set of points \((x, y, z)\) such that \(y = 3\) and \(z = 5\). Illustrate with a sketch.

7. Find the lengths of the sides of the triangle with vertices \(A(3, -4, 1), B(5, -3, 0),\) and \(C(6, -7, 4)\). Is \(ABC\) a right triangle? Is it an isosceles triangle?

8. Find the distance from \((3, 7, -5)\) to each of the following.

   (a) The \(xy\)-plane

   (b) The \(yz\)-plane

   (c) The \(xz\)-plane

   (d) The \(x\)-axis

   (e) The \(y\)-axis

   (f) The \(z\)-axis

9. Determine whether the points lie on a straight line.

   (a) \(A(5, 1, 3), B(7, 9, -1), C(1, -15, 11)\)

   (b) \(K(0, 3, -4), L(1, 2, -2), M(3, 0, 1)\)

10. Find an equation of the sphere with center \((6, 5, -2)\) and radius \(\sqrt{7}\). Describe its intersection with each of the coordinate planes.

11. Find an equation of the sphere that passes through the point \((4, 3, -1)\) and has center \((3, 8, 1)\).

12. Find an equation of the sphere that passes through the origin and whose center is \((1, 2, 3)\).

13–14 ■ Show that the equation represents a sphere, and find its center and radius.

13. \(x^2 + y^2 + z^2 = x + y + z\)

14. \(4x^2 + 4y^2 + 4z^2 - 8x + 16y = 1\)

15. (a) Prove that the midpoint of the line segment from \(P_1(x_1, y_1, z_1)\) to \(P_2(x_2, y_2, z_2)\) is

\[
\left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)
\]

(b) Find the lengths of the medians of the triangle with vertices \(A(1, 2, 3), B(-2, 0, 5),\) and \(C(4, 1, 5)\).

16. Find an equation of a sphere if one of its diameters has endpoints \((2, 1, 4)\) and \((4, 3, 10)\).

17. Find equations of the spheres with center \((2, -3, 6)\) that touch (a) the \(xy\)-plane, (b) the \(yz\)-plane, (c) the \(xz\)-plane.

18. Find an equation of the largest sphere with center \((5, 4, 9)\) that is contained in the first octant.

19–28 ■ Describe in words the region of \(\mathbb{R}^3\) represented by the equation or inequality.

19. \(y = -4\)

20. \(x = 10\)

21. \(x > 3\)

22. \(y \geq 0\)

23. \(0 \leq z \leq 6\)

24. \(y = z\)
34. Consider the points $P$ such that the distance from $P$ to $A(-1, 5, 3)$ is twice the distance from $P$ to $B(6, 2, -2)$. Show that the set of all such points is a sphere, and find its center and radius.

35. Find an equation of the set of all points equidistant from the points $A(-1, 5, 3)$ and $B(6, 2, -2)$. Describe the set.

36. Find the volume of the solid that lies inside both of the spheres $x^2 + y^2 + z^2 + 4x - 2y + 4z + 5 = 0$ and $x^2 + y^2 + z^2 = 4$.

9.2 Vectors

The term vector is used by scientists to indicate a quantity (such as displacement or velocity or force) that has both magnitude and direction. A vector is often represented by an arrow or a directed line segment. The length of the arrow represents the magnitude of the vector and the arrow points in the direction of the vector. We denote a vector by printing a letter in boldface ($\mathbf{v}$) or by putting an arrow above the letter ($\vec{v}$).

For instance, suppose a particle moves along a line segment from point $A$ to point $B$. The corresponding displacement vector $\mathbf{v}$, shown in Figure 1, has initial point $A$ (the tail) and terminal point $B$ (the tip) and we indicate this by writing $\mathbf{v} = \overrightarrow{AB}$. Notice that the vector $\mathbf{u} = \overrightarrow{CD}$ has the same length and the same direction as $\mathbf{v}$ even though it is in a different position. We say that $\mathbf{u}$ and $\mathbf{v}$ are equivalent (or equal) and we write $\mathbf{u} = \mathbf{v}$. The zero vector, denoted by $\mathbf{0}$, has length 0. It is the only vector with no specific direction.

Combining Vectors

Suppose a particle moves from $A$ to $B$, so its displacement vector is $\overrightarrow{AB}$. Then the particle changes direction and moves from $B$ to $C$, with displacement vector $\overrightarrow{BC}$ as in
So the magnitudes of the tensions are

\[ |T_1| = \frac{100}{\sin 50^\circ + \tan 32^\circ \cos 50^\circ} \approx 85.64 \text{ lb} \]

and

\[ |T_2| = \frac{|T_1| \cos 50^\circ}{\cos 32^\circ} = 64.91 \text{ lb} \]

Substituting these values in (5) and (6), we obtain the tension vectors

\[ T_1 \approx -55.05 \hat{i} + 65.60 \hat{j} \]

\[ T_2 \approx 55.05 \hat{i} + 34.40 \hat{j} \]

### Exercises

1. Are the following quantities vectors or scalars? Explain.
   (a) The cost of a theater ticket
   (b) The current in a river
   (c) The initial flight path from Houston to Dallas
   (d) The population of the world

2. What is the relationship between the point (4, 7) and the vector \( \langle 4, 7 \rangle \)? Illustrate with a sketch.

3. Name all the equal vectors in the parallelogram shown.

4. Write each combination of vectors as a single vector.
   (a) \( \overrightarrow{PQ} + \overrightarrow{QR} \)
   (b) \( \overrightarrow{RP} + \overrightarrow{PS} \)
   (c) \( \overrightarrow{QS} - \overrightarrow{PS} \)
   (d) \( \overrightarrow{RS} + \overrightarrow{SP} + \overrightarrow{PQ} \)

5. Copy the vectors in the figure and use them to draw the following vectors.
   (a) \( \overrightarrow{u} + \overrightarrow{v} \)
   (b) \( \overrightarrow{u} - \overrightarrow{v} \)
   (c) \( \overrightarrow{v} + \overrightarrow{w} \)
   (d) \( \overrightarrow{w} + \overrightarrow{v} + \overrightarrow{u} \)

6. Copy the vectors in the figure and use them to draw the following vectors.
   (a) \( \overrightarrow{a} + \overrightarrow{b} \)
   (b) \( \overrightarrow{a} - \overrightarrow{b} \)
   (c) \( 2\overrightarrow{a} \)
   (d) \( -\frac{1}{3}\overrightarrow{b} \)
   (e) \( 2\overrightarrow{a} + \overrightarrow{b} \)
   (f) \( \overrightarrow{b} - 3\overrightarrow{a} \)

7-10 ■ Find a vector \( \overrightarrow{a} \) with representation given by the directed line segment \( \overrightarrow{AB} \). Draw \( \overrightarrow{AB} \) and the equivalent representation starting at the origin.

7. \( A(-1, -1), \quad B(-3, 4) \)  
8. \( A(-2, 2), \quad B(3, 0) \)
9. \( A(0, 3, 1), \quad B(2, 3, -1) \)
10. \( A(1, -2, 0), \quad B(1, -2, 3) \)

11-14 ■ Find the sum of the given vectors and illustrate geometrically.

11. \( \langle 3, -1 \rangle, \quad \langle -2, 4 \rangle \)  
12. \( \langle -1, 2 \rangle, \quad \langle 5, 3 \rangle \)
13. \( \langle 1, 0, 1 \rangle, \quad \langle 0, 0, 1 \rangle \)  
14. \( \langle 0, 3, 2 \rangle, \quad \langle 1, 0, -3 \rangle \)

15-18 ■ Find \( |\overrightarrow{a}| \), \( \overrightarrow{a} + \overrightarrow{b} \), \( \overrightarrow{a} - \overrightarrow{b} \), \( 2\overrightarrow{a} \), and \( 3\overrightarrow{a} + 4\overrightarrow{b} \).

15. \( \overrightarrow{a} = \langle -4, 3 \rangle, \quad \overrightarrow{b} = \langle 6, 2 \rangle \)
16. \( \overrightarrow{a} = 2\overrightarrow{i} - 3\overrightarrow{j}, \quad \overrightarrow{b} = \overrightarrow{i} + 5\overrightarrow{j} \)
17. \( \overrightarrow{a} = \overrightarrow{i} - 2\overrightarrow{j} + \overrightarrow{k}, \quad \overrightarrow{b} = \overrightarrow{j} + 2\overrightarrow{k} \)
18. \( \overrightarrow{a} = 3\overrightarrow{i} - 2\overrightarrow{k}, \quad \overrightarrow{b} = \overrightarrow{i} - \overrightarrow{j} + \overrightarrow{k} \)
19. Find a unit vector with the same direction as $8\mathbf{i} - \mathbf{j} + 4\mathbf{k}$.

20. Find a vector that has the same direction as $\langle -2, 4, 2 \rangle$ but has length 6.

21. If $\mathbf{v}$ lies in the first quadrant and makes an angle $\pi/3$ with the positive $x$-axis and $|\mathbf{v}| = 4$, find $\mathbf{v}$ in component form.

22. If a child pulls a sled through the snow with a force of 50 N at an airspeed (speed in still air) of 250 km/h. The wind is blowing from the direction N W 60°, and makes an angle with the horizontal. Find the true course and the ground speed of the plane.

23. Two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ with magnitudes 10 lb and 12 lb act on an object at a point $P$ as shown in the figure. Find the resultant force $\mathbf{F}$ acting at $P$ as well as its magnitude and its direction. (Indicate the direction by finding the angle $\theta$ shown in the figure.)

24. Velocities have both direction and magnitude and thus are vectors. The magnitude of a velocity vector is called speed. Suppose that a wind is blowing from the direction N45°W at a speed of 50 km/h. (This means that the direction from which the wind blows is 45° west of the northerly direction.) A pilot is steering a plane in the direction N E 60°, from which the wind blows is west of the northerly direction.) Find the true course and the ground speed of the plane.

25. A woman walks due west on the deck of a ship at 3 mi/h. The ship is moving north at a speed of 22 mi/h. Find the speed and direction of the woman relative to the surface of the water.

26. Ropes 3 m and 5 m in length are fastened to a holiday decoration that is suspended over a town square. The decoration has a mass of 5 kg. The ropes, fastened at different heights, make angles of 52° and 40° with the horizontal. Find the tension in each wire and the magnitude of each tension.

27. A clothesline is tied between two poles, 8 m apart. The line is quite taut and has negligible sag. When a wet shirt with a mass of 0.8 kg is hung at the middle of the line, the midpoint is pulled down 8 cm. Find the tension in each half of the clothesline.

28. The tension $\mathbf{T}$ at each end of the chain has magnitude 25 N. What is the weight of the chain?

29. (a) Draw the vectors $\mathbf{a} = \langle 3, 2 \rangle$, $\mathbf{b} = \langle 2, -1 \rangle$, and $\mathbf{c} = \langle 7, 1 \rangle$.
(b) Show, by means of a sketch, that there are scalars $s$ and $t$ such that $\mathbf{c} = s\mathbf{a} + t\mathbf{b}$.
(c) Use the sketch to estimate the values of $s$ and $t$.
(d) Find the exact values of $s$ and $t$.

30. Suppose that $\mathbf{a}$ and $\mathbf{b}$ are nonzero vectors that are not parallel and $\mathbf{c}$ is any vector in the plane determined by $\mathbf{a}$ and $\mathbf{b}$. Give a geometric argument to show that $\mathbf{c}$ can be written as $\mathbf{c} = s\mathbf{a} + t\mathbf{b}$ for suitable scalars $s$ and $t$. Then give an argument using components.

31. Suppose $\mathbf{a}$ is a three-dimensional unit vector in the first octant that starts at the origin and makes angles of 60° and 72° with the positive $x$- and $y$-axes, respectively. Express $\mathbf{a}$ in terms of its components.

32. Suppose a vector $\mathbf{a}$ makes angles $\alpha$, $\beta$, and $\gamma$ with the positive $x$-, $y$-, and $z$-axes, respectively. Find the components of $\mathbf{a}$ and show that
$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$
(The numbers $\cos \alpha$, $\cos \beta$, and $\cos \gamma$ are called the direction cosines of $\mathbf{a}$.)

33. If $\mathbf{r} = \langle x, y, z \rangle$ and $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$, describe the set of all points $(x, y, z)$ such that $|\mathbf{r} - \mathbf{r}_0| = 1$.

34. If $\mathbf{r} = \langle x, y \rangle$, $\mathbf{r}_1 = \langle x_1, y_1 \rangle$, and $\mathbf{r}_2 = \langle x_2, y_2 \rangle$, describe the set of all points $(x, y)$ such that $|\mathbf{r} - \mathbf{r}_1| + |\mathbf{r} - \mathbf{r}_2| = k$, where $k > |\mathbf{r}_1 - \mathbf{r}_2|$.

35. Figure 16 gives a geometric demonstration of Property 2 of vectors. Use components to give an algebraic proof of this fact for the case $n = 2$.

36. Prove Property 5 of vectors algebraically for the case $n = 3$. Then use similar triangles to give a geometric proof.

37. Use vectors to prove that the line joining the midpoints of two sides of a triangle is parallel to the third side and half its length.
So far we have added two vectors and multiplied a vector by a scalar. The question arises: Is it possible to multiply two vectors so that their product is a useful quantity? One such product is the dot product, which we consider in this section. Another is the cross product, which is discussed in the next section.

**Work and the Dot Product**

An example of a situation in physics and engineering where we need to combine two vectors occurs in calculating the work done by a force. In Section 6.5 we defined the work done by a constant force in moving an object through a distance as 

\[ W = Fd, \]

but this applies only when the force is directed along the line of motion of the object. Suppose, however, that the constant force is a vector \( \vec{F} \) pointing in some other direction, as in Figure 1. If the force moves the object from \( P \) to \( Q \), then the displacement vector is \( D = \vec{PQ} \). So here we have two vectors: the force \( \vec{F} \) and the displacement \( D \). The work done by \( \vec{F} \) is defined as the magnitude of the displacement, \( |D| \), multiplied by the magnitude of the applied force in the direction of the motion, which, from Figure 1, is

\[ |\overrightarrow{PS}| = |\vec{F}| \cos \theta \]

So the work done by \( \vec{F} \) is defined to be

\[ W = |D| \left( |\vec{F}| \cos \theta \right) = |\vec{F}| |D| \cos \theta \]

Notice that work is a scalar quantity; it has no direction. But its value depends on the angle between the force and displacement vectors.

We use the expression in Equation 1 to define the dot product of two vectors even when they don’t represent force or displacement.

**Definition** The dot product of two nonzero vectors \( \vec{a} \) and \( \vec{b} \) is the number

\[ \vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta \]

where \( \theta \) is the angle between \( \vec{a} \) and \( \vec{b} \), \( 0 \leq \theta \leq \pi \). (So \( \theta \) is the smaller angle between the vectors when they are drawn with the same initial point.) If either \( \vec{a} \) or \( \vec{b} \) is 0, we define \( \vec{a} \cdot \vec{b} = 0 \).
9.3  Exercises

1. Which of the following expressions are meaningful? Which are meaningless? Explain.
   (a) \((a \cdot b) \cdot c\)  
   (b) \((a \cdot b)c\)
   (c) \(|a| (b \cdot c)\)  
   (d) \(a \cdot (b + c)\)
   (e) \(a \cdot b + c\)  
   (f) \(|a| \cdot (b + c)\)

2. Find the dot product of two vectors if their lengths are 6 and \(\frac{1}{2}\) and the angle between them is \(\pi/4\).

3–8  Find \(a \cdot b\).

3. \(|a| = 12, \ |b| = 15\), the angle between \(a\) and \(b\) is \(\pi/6\)

4. \(a = \langle \frac{1}{2}, 4 \rangle, \ b = \langle -8, -3 \rangle\)

5. \(a = \langle 5, 0, -2 \rangle, \ b = \langle 3, -1, 10 \rangle\)

6. \(a = \langle s, 2s, 3s \rangle, \ b = \langle -t, -t, 5t \rangle\)

7. \(a = i - 2j + 3k, \ b = 5i + 9k\)

8. \(a = 4j - 3k, \ b = 2i + 4j + 6k\)

9–10  If \(\mathbf{u}\) is a unit vector, find \(\mathbf{u} \cdot \mathbf{v}\) and \(\mathbf{u} \cdot \mathbf{w}\).

9.

10.

11. (a) Show that \(\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0\).
   (b) Show that \(\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1\).

12. A street vendor sells \(a\) hamburgers, \(b\) hot dogs, and \(c\) soft drinks on a given day. He charges \$2 for a hamburger, \$1.50 for a hot dog, and \$1 for a soft drink. If \(A = \langle a, b, c \rangle\) and \(P = \langle 2, 1.5, 1 \rangle\), what is the meaning of the dot product \(A \cdot P\)?

13–15  Find the angle between the vectors. (First find an exact expression and then approximate to the nearest degree.)

13. \(a = \langle 3, 4 \rangle, \ b = \langle 5, 12 \rangle\)

14. \(a = \langle 6, -3, 2 \rangle, \ b = \langle 2, 1, -2 \rangle\)

15. \(a = \mathbf{j} + \mathbf{k}, \ b = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}\)

16. Find, correct to the nearest degree, the three angles of the triangle with the vertices \(P(0, -1, 6), Q(2, 1, -3),\) and \(R(5, 4, 2)\).

17. Determine whether the given vectors are orthogonal, parallel, or neither.
   (a) \(a = \langle -5, 3, 7 \rangle, \ b = \langle 6, -8, 2 \rangle\)
   (b) \(a = \langle 4, 6 \rangle, \ b = \langle -3, 2 \rangle\)
   (c) \(a = -\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}, \ b = 3\mathbf{i} + 4\mathbf{j} - \mathbf{k}\)
   (d) \(a = 2\mathbf{i} + 6\mathbf{j} - 4\mathbf{k}, \ b = -3\mathbf{i} - 9\mathbf{j} + 6\mathbf{k}\)

18. For what values of \(b\) are the vectors \(\langle -6, b, 2 \rangle\) and \(\langle b, b^2, b \rangle\) orthogonal?

19. Find a unit vector that is orthogonal to both \(\mathbf{i} + \mathbf{j}\) and \(\mathbf{i} + \mathbf{k}\).

20. For what values of \(c\) is the angle between the vectors \(\langle 1, 2, 1 \rangle\) and \(\langle 1, 0, c \rangle\) equal to 60°?

21–24  Find the scalar and vector projections of \(\mathbf{b}\) onto \(\mathbf{a}\).

21. \(a = \langle 2, 3 \rangle, \ b = \langle 4, 1 \rangle\)

22. \(a = \langle 3, -1 \rangle, \ b = \langle 2, 3 \rangle\)

23. \(a = \langle 4, 2, 0 \rangle, \ b = \langle 1, 1, 1 \rangle\)

24. \(a = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}, \ b = \mathbf{i} + 6\mathbf{j} - 2\mathbf{k}\)

25. Show that the vector \(\text{orth}_a \mathbf{b} = \mathbf{b} - \text{proj}_a \mathbf{b}\) is orthogonal to \(\mathbf{a}\). (It is called an orthogonal projection of \(\mathbf{b}\)).

26. For the vectors in Exercise 22, find \(\text{orth}_a \mathbf{b}\) and illustrate by drawing the vectors \(\mathbf{a}, \mathbf{b}, \text{proj}_a \mathbf{b},\) and \(\text{orth}_a \mathbf{b}\).

27. If \(a = \langle 3, 0, -1 \rangle\), find a vector \(\mathbf{b}\) such that \(\text{comp}_a \mathbf{b} = 2\).

28. Suppose that \(a\) and \(b\) are nonzero vectors.
   (a) Under what circumstances is \(\text{comp}_a \mathbf{b} = \text{comp}_b \mathbf{a}\)?
   (b) Under what circumstances is \(\text{proj}_a \mathbf{b} = \text{proj}_b \mathbf{a}\)?

29. A constant force with vector representation \(\mathbf{F} = 10\mathbf{i} + 18\mathbf{j} - 6\mathbf{k}\) moves an object along a straight line from the point \((2, 3, 0)\) to the point \((4, 9, 15)\). Find the work done if the distance is measured in meters and the magnitude of the force is measured in newtons.

30. Find the work done by a force of 20 lb acting in the direction N50°W in moving an object 4 ft due west.

31. A woman exerts a horizontal force of 25 lb on a crate as she pushes it up a ramp that is 10 ft long and inclined at an angle of 20° above the horizontal. Find the work done on the box.

32. A wagon is pulled a distance of 100 m along a horizontal path by a constant force of 50 N. The handle of the wagon is held at an angle of 30° above the horizontal. How much work is done?

33. Use a scalar projection to show that the distance from a point \(P(x_1, y_1)\) to the line \(ax + by + c = 0\) is
\[
\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}
\]
Use this formula to find the distance from the point $(-2, 3)$ to the line $3x - 4y + 5 = 0$.

34. If $\mathbf{r} = (x, y, z)$, $\mathbf{a} = (a_1, a_2, a_3)$, and $\mathbf{b} = (b_1, b_2, b_3)$, show that the vector equation $(\mathbf{r} - \mathbf{a}) \cdot (\mathbf{r} - \mathbf{b}) = 0$ represents a sphere, and find its center and radius.

35. Find the angle between a diagonal of a cube and one of its edges.

36. Find the angle between a diagonal of a cube and a diagonal of one of its faces.

37. A molecule of methane, CH₄, is structured with the four hydrogen atoms at the vertices of a regular tetrahedron and the carbon atom at the centroid. The bond angle is the angle formed by the H—C—H combination; it is the angle between the lines that join the carbon atom to two of the hydrogen atoms. Show that the bond angle is about $109.5^\circ$. [Hint: Take the vertices of the tetrahedron to be the points $(1, 0, 0), (0, 1, 0), (0, 0, 1)$, and $(1, 1, 1)$ as shown in the figure. Then the centroid is $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$]

38. If $\mathbf{c} = |\mathbf{a}| \mathbf{b} + |\mathbf{b}| \mathbf{a}$, where $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are all nonzero vectors, show that $\mathbf{c}$ bisects the angle between $\mathbf{a}$ and $\mathbf{b}$.

39. Prove Property 4 of the dot product. Use either the definition of a dot product (considering the cases $c > 0$, $c = 0$, and $c < 0$ separately) or the component form.

40. Suppose that all sides of a quadrilateral are equal in length and opposite sides are parallel. Use vector methods to show that the diagonals are perpendicular.

41. Prove the Cauchy-Schwarz Inequality:

$$|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$$

42. The Triangle Inequality for vectors is

$$|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$$

(a) Give a geometric interpretation of the Triangle Inequality.

(b) Use the Cauchy-Schwarz Inequality from Exercise 41 to prove the Triangle Inequality. [Hint: Use the fact that $|\mathbf{a} + \mathbf{b}|^2 = (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b})$ and use Property 3 of the dot product.]

43. The Parallelogram Law states that

$$|\mathbf{a} + \mathbf{b}|^2 + |\mathbf{a} - \mathbf{b}|^2 = 2|\mathbf{a}|^2 + 2|\mathbf{b}|^2$$

(a) Give a geometric interpretation of the Parallelogram Law.

(b) Prove the Parallelogram Law. (See the hint in Exercise 42.)

---

### 9.4 The Cross Product

The cross product $\mathbf{a} \times \mathbf{b}$ of two vectors $\mathbf{a}$ and $\mathbf{b}$, unlike the dot product, is a vector. For this reason it is also called the vector product. We will see that $\mathbf{a} \times \mathbf{b}$ is useful in geometry because it is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. But we introduce this product by looking at a situation where it arises in physics and engineering.

#### Torque and the Cross Product

If we tighten a bolt by applying a force to a wrench as in Figure 1, we produce a turning effect called a torque $\mathbf{\tau}$. The magnitude of the torque depends on two things:

- The distance from the axis of the bolt to the point where the force is applied. This is $|\mathbf{r}|$, the length of the position vector $\mathbf{r}$.
- The scalar component of the force $\mathbf{F}$ in the direction perpendicular to $\mathbf{r}$. This is the only component that can cause a rotation and, from Figure 2, we see that it is

$$|\mathbf{F}| \sin \theta$$

where $\theta$ is the angle between the vectors $\mathbf{r}$ and $\mathbf{F}$. 

## Exercises

1. State whether each expression is meaningful. If not, explain why. If so, state whether it is a vector or a scalar.
   (a) \( \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \)  
   (b) \( \mathbf{a} \times (\mathbf{b} \cdot \mathbf{c}) \)  
   (c) \( \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \)  
   (d) \( (\mathbf{a} \cdot \mathbf{b}) \times \mathbf{c} \)  
   (e) \( (\mathbf{a} \cdot \mathbf{b}) \times (\mathbf{c} \cdot \mathbf{d}) \)  
   (f) \( (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) \)

2–3 Find \( |\mathbf{u} \times \mathbf{v}| \) and determine whether \( \mathbf{u} \times \mathbf{v} \) is directed into the page or out of the page.

4. The figure shows a vector \( \mathbf{a} \) in the \( xy \)-plane and a vector \( \mathbf{b} \) in the direction of \( \mathbf{k} \). Their lengths are \( |\mathbf{a}| = 3 \) and \( |\mathbf{b}| = 2 \).
   (a) Find \( |\mathbf{a} \times \mathbf{b}| \).
   (b) Use the right-hand rule to decide whether the components of \( \mathbf{a} \times \mathbf{b} \) are positive, negative, or 0.

5. A bicycle pedal is pushed by a foot with a 60-N force as shown. The shaft of the pedal is 18 cm long. Find the magnitude of the torque about \( P \).

6. Find the magnitude of the torque about \( P \) if a 36-lb force is applied as shown.

7–11 Find the cross product \( \mathbf{a} \times \mathbf{b} \) and verify that it is orthogonal to both \( \mathbf{a} \) and \( \mathbf{b} \).

7. \( \mathbf{a} = (1, -1, 0), \quad \mathbf{b} = (3, 2, 1) \)

8. \( \mathbf{a} = (-3, 2, 2), \quad \mathbf{b} = (6, 3, 1) \)

9. \( \mathbf{a} = (t, r^2, r^3), \quad \mathbf{b} = (1, 2t, 3t^2) \)

10. \( \mathbf{a} = \mathbf{i} + e^r \mathbf{j} + e^{-r} \mathbf{k}, \quad \mathbf{b} = 2\mathbf{i} + e^r \mathbf{j} - e^{-r} \mathbf{k} \)

11. \( \mathbf{a} = 3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}, \quad \mathbf{b} = \mathbf{i} - 2\mathbf{j} - 3\mathbf{k} \)

12. If \( \mathbf{a} = \mathbf{i} - 2\mathbf{k} \) and \( \mathbf{b} = \mathbf{j} + \mathbf{k} \), find \( \mathbf{a} \times \mathbf{b} \). Sketch \( \mathbf{a} \), \( \mathbf{b} \), and \( \mathbf{a} \times \mathbf{b} \) as vectors starting at the origin.

13. Find two unit vectors orthogonal to both \( \langle -1, 1, 1 \rangle \) and \( \langle 0, 4, 4 \rangle \).

14. Find two unit vectors orthogonal to both \( \mathbf{i} + \mathbf{j} \) and \( \mathbf{i} - \mathbf{j} + \mathbf{k} \).

15. Find the area of the parallelogram with vertices \( A(-2, 1), \ B(0, 4), \ C(4, 2), \) and \( D(2, -1) \).

16. Find the area of the parallelogram with vertices \( K(1, 2, 3), \ L(1, 3, 6), \ M(3, 8, 6), \) and \( N(3, 7, 3) \).

17–18 (a) Find a vector orthogonal to the plane through the points \( P, Q, \) and \( R \) and (b) find the area of triangle \( PQR \).

17. \( P(1, 0, 0), \quad Q(0, 2, 0), \quad R(0, 0, 3) \)

18. \( P(2, 0, -3), \quad Q(3, 1, 0), \quad R(5, 2, 2) \)

19. A wrench 30 cm long lies along the positive y-axis and grips a bolt at the origin. A force is applied in the direction \( \langle 0, 3, -4 \rangle \) at the end of the wrench. Find the magnitude of the force needed to supply 100 J of torque to the bolt.

20. Let \( \mathbf{v} = 5\mathbf{j} \) and let \( \mathbf{u} \) be a vector with length 3 that starts at the origin and rotates in the \( xy \)-plane. Find the maximum and minimum values of the length of the vector \( \mathbf{u} \times \mathbf{v} \). In what direction does \( \mathbf{u} \times \mathbf{v} \) point?

21–22 Find the volume of the parallelepiped determined by the vectors \( \mathbf{a}, \mathbf{b}, \) and \( \mathbf{c} \).

21. \( \mathbf{a} = (6, 3, -1), \quad \mathbf{b} = (0, 1, 2), \quad \mathbf{c} = (4, -2, 5) \)

22. \( \mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}, \quad \mathbf{b} = \mathbf{i} - \mathbf{j}, \quad \mathbf{c} = 2\mathbf{i} + 3\mathbf{k} \)

23–24 Find the volume of the parallelepiped with adjacent edges \( PQ, PR, \) and \( PS \).

23. \( P(1, 1, 1), \quad Q(2, 0, 3), \quad R(4, 1, 7), \quad S(3, -1, -2) \)

24. \( P(0, 1, 2), \quad Q(2, 4, 5), \quad R(-1, 0, 1), \quad S(6, -1, 4) \)
25. Use the scalar triple product to verify that the vectors 
\( \mathbf{a} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}, \mathbf{b} = \mathbf{i} - \mathbf{j}, \) and \( \mathbf{c} = 7\mathbf{i} + 3\mathbf{j} + 2\mathbf{k} \) are coplanar.

26. Use the scalar triple product to determine whether the points \( P(1, 0, 1), Q(2, 4, 6), R(3, -1, 2), \) and \( S(6, 2, 8) \) lie in the same plane.

27. (a) Let \( \mathbf{a} \) be a point not on the line that passes through the points \( Q \) and \( R. \) Show that the distance \( d \) from the point \( P \) to the line \( L \) is
\[
d = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}|}
\]
where \( \mathbf{a} = \overrightarrow{QR} \) and \( \mathbf{b} = \overrightarrow{QP}. \)

(b) Use the formula in part (a) to find the distance from the point \( P(1, 1, 1) \) to the line through \( Q(0, 6, 8) \) and \( R(-1, 4, 7). \)

28. (a) Let \( \mathbf{a} \) be a point not on the plane that passes through the points \( Q, R, \) and \( S. \) Show that the distance \( d \) from \( P \) to the plane is
\[
d = \frac{|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|}{|\mathbf{a} \times \mathbf{b}|}
\]
where \( \mathbf{a} = \overrightarrow{QS}, \mathbf{b} = \overrightarrow{QR}, \) and \( \mathbf{c} = \overrightarrow{QP}. \)

(b) Use the formula in part (a) to find the distance from the point \( P(2, 1, 4) \) to the plane through the points \( Q(1, 0, 0), R(0, 2, 0), \) and \( S(0, 0, 3). \)

29. Prove that \( (\mathbf{a} - \mathbf{b}) \times (\mathbf{a} + \mathbf{b}) = 2(\mathbf{a} \times \mathbf{b}). \)

30. Prove the following formula for the vector triple product:
\[
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}
\]

31. Use Exercise 30 to prove that
\[
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = 0
\]

32. Prove that
\[
(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = \begin{vmatrix} \mathbf{a} \cdot \mathbf{c} & \mathbf{b} \cdot \mathbf{c} \\ \mathbf{a} \cdot \mathbf{d} & \mathbf{b} \cdot \mathbf{d} \end{vmatrix}
\]

33. Suppose that \( \mathbf{a} \neq 0. \)

(a) If \( \mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}, \) does it follow that \( \mathbf{b} = \mathbf{c}? \)

(b) If \( \mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}, \) does it follow that \( \mathbf{b} = \mathbf{c}? \)

(c) If \( \mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c} \) and \( \mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}, \) does it follow that \( \mathbf{b} = \mathbf{c}? \)

34. If \( \mathbf{v}_1, \mathbf{v}_2, \) and \( \mathbf{v}_3 \) are noncoplanar vectors, let
\[
\mathbf{k}_1 = \frac{\mathbf{v}_2 \times \mathbf{v}_3}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)}, \quad \mathbf{k}_2 = \frac{\mathbf{v}_1 \times \mathbf{v}_3}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)}, \quad \mathbf{k}_3 = \frac{\mathbf{v}_1 \times \mathbf{v}_2}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)}.
\]

(These vectors occur in the study of crystallography. Vectors of the form \( n_1 \mathbf{v}_1 + n_2 \mathbf{v}_2 + n_3 \mathbf{v}_3 \), where each \( n_i \) is an integer, form a lattice for a crystal. Vectors written similarly in terms of \( \mathbf{k}_1, \mathbf{k}_2, \) and \( \mathbf{k}_3 \) form the reciprocal lattice.)

(a) Show that \( \mathbf{k}_1 \) is perpendicular to \( \mathbf{v}_i \) if \( i \neq j. \)

(b) Show that \( \mathbf{k}_i \cdot \mathbf{v}_j = 1 \) for \( i = 1, 2, 3. \)

(c) Show that \( \mathbf{k}_1 \cdot (\mathbf{k}_2 \times \mathbf{k}_3) = \frac{1}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)}. \)

### Discovery Project

#### The Geometry of a Tetrahedron

A tetrahedron is a solid with four vertices, \( P, Q, R, \) and \( S, \) and four triangular faces, as shown in the figure.

1. Let \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \) and \( \mathbf{v}_4 \) be vectors with lengths equal to the areas of the faces opposite the vertices \( P, Q, R, \) and \( S, \) respectively, and directions perpendicular to the respective faces and pointing outward. Show that
\[
\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 = 0
\]

2. The volume \( V \) of a tetrahedron is one-third the distance from a vertex to the opposite face, times the area of that face.

(a) Find a formula for the volume of a tetrahedron in terms of the coordinates of its vertices \( P, Q, R, \) and \( S. \)

(b) Find the volume of the tetrahedron whose vertices are \( P(1, 1, 1), Q(1, 2, 3), \) \( R(1, 1, 2), \) and \( S(3, -1, 2). \)
SOLUTION Since the two lines $L_1$ and $L_2$ are skew, they can be viewed as lying on two parallel planes $P_1$ and $P_2$. The distance between $L_1$ and $L_2$ is the same as the distance between $P_1$ and $P_2$, which can be computed as in Example 9. The common normal vector to both planes must be orthogonal to both $v_1 = (1, 3, -1)$ (the direction of $L_1$) and $v_2 = (2, 1, 4)$ (the direction of $L_2$). So a normal vector is

$$
\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
1 & 3 & -1 \\
2 & 1 & 4
\end{vmatrix} = 13 \mathbf{i} - 6 \mathbf{j} - 5 \mathbf{k}
$$

If we put $s = 0$ in the equations of $L_2$, we get the point $(0, 3, -3)$ on $L_2$ and so an equation for $P_2$ is

$$13(x - 0) - 6(y - 3) - 5(z + 3) = 0 \quad \text{or} \quad 13x - 6y - 5z + 3 = 0$$

If we now set $t = 0$ in the equations for $L_1$, we get the point $(1, -2, 4)$ on $P_1$. So the distance between $L_1$ and $L_2$ is the same as the distance from $(1, -2, 4)$ to $13x - 6y + 5z + 3 = 0$. By Formula 8, this distance is

$$D = \frac{|13(1) - 6(-2) - 5(4) + 3|}{\sqrt{13^2 + (-6)^2 + (-5)^2}} = \frac{8}{\sqrt{230}} \approx 0.53$$

---

**Exercises**

1. Determine whether each statement is true or false.
   a) Two lines parallel to a third line are parallel.
   b) Two lines perpendicular to a third line are parallel.
   c) Two planes parallel to a third plane are parallel.
   d) Two lines parallel to a third plane are parallel.
   e) Two planes parallel to a line are parallel.
   f) Two lines perpendicular to a plane are parallel.
   g) Two planes parallel to a line are parallel.
   h) Two planes parallel to a line are parallel.
   i) Two planes either intersect or are parallel.
   j) Two lines either intersect or are parallel.
   k) A plane and a line either intersect or are parallel.

2–5 Find a vector equation and parametric equations for the line.

2. The line through the point $(1, 0, -3)$ and parallel to the vector $2\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$

3. The line through the point $(-2, 4, 10)$ and parallel to the vector $(3, 1, -8)$

4. The line through the origin and parallel to the line $x = 2t$, $y = 1 - t$, $z = 4 + 3t$

5. The line through the point $(1, 0, 6)$ and perpendicular to the plane $x + 3y + z = 5$

6–10 Find parametric equations and symmetric equations for the line.

6. The line through the origin and the point $(1, 2, 3)$

7. The line through the points $(3, 1, -1)$ and $(3, 2, -6)$

8. The line through the points $(-1, 0, 5)$ and $(4, -3, 3)$

9. The line through the points $(0, \frac{1}{2}, 1)$ and $(2, 1, -3)$

10. The line of intersection of the planes $x + y + z = 1$ and $x + z = 0$

11. Show that the line through the points $(2, -1, -5)$ and $(8, 8, 7)$ is parallel to the line through the points $(4, 2, -6)$ and $(8, 8, 2)$.

12. Show that the line through the points $(0, 1, 1)$ and $(1, -1, 6)$ is perpendicular to the line through the points $(-4, 2, 1)$ and $(-1, 6, 2)$.

13. a) Find symmetric equations for the line that passes through the point $(0, 2, -1)$ and is parallel to the line with parametric equations $x = 1 + 2t$, $y = 3t$, $z = 5 - 7t$.
   b) Find the points in which the required line in part (a) intersects the coordinate planes.
14. (a) Find parametric equations for the line through (5, 1, 0) that is perpendicular to the plane \(2x - y + z = 1\).
(b) In what points does this line intersect the coordinate planes?

15–18 Determine whether the lines \(L_1\) and \(L_2\) are parallel, skew, or intersecting. If they intersect, find the point of intersection.

15. \(L_1: \frac{x - 4}{2} = \frac{y + 5}{4} = \frac{z - 1}{-3}\)
\(L_2: \frac{x - 2}{1} = \frac{y + 1}{3} = \frac{z}{2}\)

16. \(L_1: \frac{x - 1}{2} = \frac{y - 1}{4} = \frac{z - 1}{4}\)
\(L_2: \frac{x}{1} = \frac{y + 2}{2} = \frac{z + 2}{3}\)

17. \(L_1: x = -6t, \quad y = 1 + 9t, \quad z = -3t\)
\(L_2: x = 1 + 2s, \quad y = 4 - 3s, \quad z = s\)

18. \(L_1: x = 1 + t, \quad y = 2 - t, \quad z = 3t\)
\(L_2: x = 2 - s, \quad y = 1 + 2s, \quad z = 4 + s\)

19–28 Find an equation of the plane.

19. The plane through the point (6, 3, 2) and perpendicular to the vector \((-2, 1, 5)\)

20. The plane through the point (4, 0, -3) and with normal vector \(1 + 2k\)

21. The plane through the origin and parallel to the plane \(2x - y + 3z = 1\)

22. The plane that contains the line \(x = 3 + 2t, y = t, \quad z = 8 - t\) and is parallel to the plane \(2x + 4y + 8z = 17\)

23. The plane through the points (0, 1, 1), (1, 0, 1), and (1, 1, 0)

24. The plane through the origin and the points (2, -4, 6) and (5, 1, 3)

25. The plane that passes through the point (6, 0, -2) and contains the line \(x = 4 - 2t, y = 3 + 5t, z = 7 + 4t\)

26. The plane that passes through the point (1, -1, 1) and contains the line with symmetric equations \(x = 2y = 3z\)

27. The plane that passes through the point (-1, 2, 1) and contains the line of intersection of the planes \(x + y - z = 2\) and \(2x - y + 3z = 1\)

28. The plane that passes through the line of intersection of the planes \(x - z = 1\) and \(y + 2z = 3\) and is perpendicular to the plane \(x + y - 2z = 1\)

29–30 Find the point at which the line intersects the given plane.

29. \(x = 1 + 2t, \quad y = -1, \quad z = t; \quad 2x + y - z + 5 = 0\)

30. \(x = 1 - t, \quad y = t, \quad z = 1 + t; \quad z = 1 - 2x + y\)

31–34 Determine whether the planes are parallel, perpendicular, or neither. If neither, find the angle between them.

31. \(x + z = 1, \quad y + z = 1\)

32. \(-8x - 6y + 2z = 1, \quad z = 4x + 3y\)

33. \(x + 4y - 3z = 1, \quad -3x + 6y + 7z = 0\)

34. \(2x + 2y - z = 4, \quad 6x - 3y + 2z = 5\)

35. (a) Find symmetric equations for the line of intersection of the planes \(x + y - z = 2\) and \(3x - 4y + 5z = 6\).
(b) Find the angle between these planes.

36. Find an equation for the plane consisting of all points that are equidistant from the points \((-4, 2, 1)\) and \((2, -4, 3)\).

37. Find an equation of the plane with x-intercept \(a\), y-intercept \(b\), and z-intercept \(c\).

38. (a) Find the point at which the given lines intersect:
\[ \mathbf{r} = (1, 1, 0) + t(1, -1, 2) \]
\[ \mathbf{r} = (2, 0, 2) + s(-1, 1, 0) \]
(b) Find an equation of the plane that contains these lines.

39. Find parametric equations for the line through the point \((0, 1, 2)\) that is parallel to the plane \(x + y + z = 2\) and perpendicular to the line \(x = 1 + t, y = 1 - t, z = 2t\).

40. Find parametric equations for the line through the point \((0, 1, 2)\) that is perpendicular to the line \(x = 1 + t, y = 1 - t, z = 2t\) and intersects this line.

41. Which of the following four planes are parallel? Are any of them identical?
\[ P_1: \quad 4x - 2y + 6z = 3 \quad P_2: \quad 4x - 2y - 2z = 6 \]
\[ P_3: \quad -6x + 3y - 9z = 5 \quad P_4: \quad z = 2x - y - 3 \]

42. Which of the following four lines are parallel? Are any of them identical?
\[ L_1: \quad x = 1 + t, \quad y = t, \quad z = 2 - 5t \]
\[ L_2: \quad x + 1 = y - 2 = 1 - z \]
\[ L_3: \quad x = 1 + t, \quad y = 4 + t, \quad z = 1 - t \]
\[ L_4: \quad \mathbf{r} = (2, 1, -3) + t(2, 2, -10) \]

43–44 Use the formula in Exercise 27 in Section 9.4 to find the distance from the point to the given line.

43. \((1, 2, 3); \quad x = 2 + t, \quad y = 2 - 3t, \quad z = 5t\)

44. \((1, 0, -1); \quad x = 5 - t, \quad y = 3t, \quad z = 1 + 2t\)

45–46 Find the distance from the point to the given plane.

45. \((2, 8, 5); \quad x - 2y - 2z = 1\)

46. \((3, -2, 7); \quad 4x - 6y + z = 5\)
47–48 Find the distance between the given parallel planes.
47. \( z = x + 2y + 1, \quad 3x + 6y - 3z = 4 \)
48. \( 3x + 6y - 9z = 4, \quad x + 2y - 3z = 1 \)
49. Show that the distance between the parallel planes
\[ ax + by + cz + d_1 = 0 \text{ and } ax + by + cz + d_2 = 0 \]
\[ D = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}} \]
50. Find equations of the planes that are parallel to the plane
\( x + 2y - 2z = 1 \) and two units away from it.
51. Show that the lines with symmetric equations \( x = y = z \)
and \( x + 1 = y/2 = z/3 \) are skew, and find the distance
between these lines.

52. Find the distance between the skew lines with parametric
equations \( x = 1 + t, \quad y = 1 + 6t, \quad z = 2t, \) and \( x = 1 + 2t, \)
\( y = 5 + 15t, \quad z = -2 + 6t. \)
53. If \( a, b, \) and \( c \) are not all 0, show that the equation
\( ax + by + cz + d = 0 \) represents a plane and \( (a, b, c) \) is a
normal vector to the plane.
Hint: Suppose \( a \neq 0 \) and rewrite the equation in the form
\[ a(x + \frac{d}{a}) + b(y - 0) + c(z - 0) = 0 \]
54. Give a geometric description of each family of planes.
(a) \( x + y + z = c \)
(b) \( x + y + cz = 1 \)
(c) \( y \cos \theta + z \sin \theta = 1 \)

9.6 Functions and Surfaces

In this section we take a first look at functions of two variables and their graphs, which
are surfaces in three-dimensional space. We will give a much more thorough treatment
of such functions in Chapter 11.

Functions of Two Variables

The temperature \( T \) at a point on the surface of the earth at any given time depends on
the longitude \( x \) and latitude \( y \) of the point. We can think of \( T \) as being a function of
the two variables \( x \) and \( y \), or as a function of the pair \( (x, y) \). We indicate this functional
dependence by writing \( T = f(x, y) \).

The volume \( V \) of a circular cylinder depends on its radius \( r \) and its height \( h \). In
fact, we know that \( V = \pi r^2 h \). We say that \( V \) is a function of \( r \) and \( h \), and we write
\( V(r, h) = \pi r^2 h \).

**Definition** A function \( f \) of two variables is a rule that assigns to each ordered
pair of real numbers \( (x, y) \) in a set \( D \) a unique real number denoted by \( f(x, y) \).
The set \( D \) is the domain of \( f \) and its range is the set of values that \( f \) takes on,
that is, \( \{ f(x, y) \mid (x, y) \in D \} \).

We often write \( z = f(x, y) \) to make explicit the value taken on by \( f \) at the general
point \( (x, y) \). The variables \( x \) and \( y \) are independent variables and \( z \) is the dependent
variable. [Compare this with the notation \( y = f(x) \) for functions of a single variable.]
The domain is a subset of \( \mathbb{R}^2 \), the \( xy \)-plane. We can think of the domain as the set
of all possible inputs and the range as the set of all possible outputs. If a function \( f \) is
given by a formula and no domain is specified, then the domain of \( f \) is understood
to be the set of all pairs \( (x, y) \) for which the given expression is a well-defined real number.
EXAMPLE 9 Classify the quadric surface \( x^2 + 2z^2 - 6x - y + 10 = 0 \).

**Solution** By completing the square we rewrite the equation as
\[
y - 1 = (x - 3)^2 + 2z^2
\]
Comparing this equation with Table 2, we see that it represents an elliptic paraboloid. Here, however, the axis of the paraboloid is parallel to the \( y \)-axis, and it has been shifted so that its vertex is the point \((3, 1, 0)\). The traces in the plane \( y = k \) \((k > 1)\) are the ellipses
\[
(x - 3)^2 + 2z^2 = k - 1 \quad y = k
\]
The trace in the \( xy \)-plane is the parabola with equation \( y = 1 + (x - 3)^2 \), \( z = 0 \). The paraboloid is sketched in Figure 13.

### Exercises

1. In Example 3 we considered the function \( h = f(v, t) \), where \( h \) is the height of waves produced by wind at speed \( v \) for a time \( t \). Use Table 1 to answer the following questions.
   (a) What is the value of \( f(40, 15) \)? What is its meaning?
   (b) What is the meaning of the function \( h = f(30, t) \)? Describe the behavior of this function.
   (c) What is the meaning of the function \( h = f(v, 30) \)? Describe the behavior of this function.

2. The figure shows vertical traces for a function \( z = f(x, y) \). Which one of the graphs I–IV has these traces? Explain.

3. Let \( f(x, y) = x^2 e^{by} \).
   (a) Evaluate \( f(2, 0) \).
   (b) Find the domain of \( f \).
   (c) Find the range of \( f \).

4. Let \( f(x, y) = \ln(x + y - 1) \).
   (a) Evaluate \( f(1, 1) \).
   (b) Evaluate \( f(e, 1) \).
   (c) Find and sketch the domain of \( f \).
   (d) Find the range of \( f \).

5–8 Find and sketch the domain of the function.

5. \( f(x, y) = \sqrt{x + y} \)

6. \( f(x, y) = \sqrt{x} + \sqrt{y} \)

7. \( f(x, y) = \sqrt{y - x^2} \)

8. \( f(x, y) = \sqrt{x^2 + y^2 - 1} + \ln(4 - x^2 - y^2) \)

9–13 Sketch the graph of the function.

9. \( f(x, y) = 3 \)

10. \( f(x, y) = x \)

11. \( f(x, y) = 1 - x - y \)

12. \( f(x, y) = \sin y \)

13. \( f(x, y) = 1 - x^2 \)

14. (a) Find the traces of the function \( f(x, y) = x^2 + y^2 \) in the planes \( x = k, \ y = k, \) and \( z = k \). Use these traces to sketch the graph.
   (b) Sketch the graph of \( g(x, y) = -x^2 - y^2 \). How is it related to the graph of \( f \)?
   (c) Sketch the graph of \( h(x, y) = 3 - x^2 - y^2 \). How is it related to the graph of \( g \)?

15. Match the function with its graph (labeled I–VI). Give reasons for your choices.
   (a) \( f(x, y) = |x| + |y| \)
   (b) \( f(x, y) = |xy| \)
(c) $f(x, y) = \frac{1}{1 + x^2 + y^2}$

(d) $f(x, y) = (x^2 - y^2)^2$

(e) $f(x, y) = (x - y)^2$

(f) $f(x, y) = \sin(|x| + |y|)$

16. $f(x, y) = \sqrt{16 - x^2 - 16y^2}$

17. $f(x, y) = x^2 + 9y^2$

18. $f(x, y) = x^2 - y^2$

19. $y = z^2 - x^2$

20. $y = x^2 + z^2$

21–22 Use traces to sketch the graph of the function.

21. $x = 4y^2 + z^2 - 4z + 4$

22. $x^2 + 4y^2 + z^2 - 2x = 0$

23. (a) What does the equation $x^2 + y^2 = 1$ represent as a curve in $\mathbb{R}^2$?

(b) What does it represent as a surface in $\mathbb{R}^3$?

(c) What does the equation $x^2 + z^2 = 1$ represent?

24. (a) Identify the traces of the surface $z^2 = x^2 + y^2$.

(b) Sketch the surface.

(c) Sketch the graphs of the functions $f(x, y) = \sqrt{x^2 + y^2}$

25. (a) Find and identify the traces of the quadric surface $x^2 + y^2 - z^2 = 1$ and explain why the graph looks like

the graph of the hyperboloid of one sheet in Table 2.

(b) If we change the equation in part (a) to $x^2 - y^2 + z^2 = 1$, how is the graph affected?

(c) What if we change the equation in part (a) to $x^2 + y^2 + 2y - z^2 = 0$?

26. (a) Find and identify the traces of the quadric surface $-x^2 - y^2 + z^2 = 1$ and explain why the graph looks like the graph of the hyperboloid of two sheets in Table 2.

(b) If the equation in part (a) is changed to $x^2 - y^2 - z^2 = 1$, what happens to the graph? Sketch the new graph.

27–28 Use a computer to graph the function using various domains and viewpoints. Get a printout that gives a good view of the “peaks and valleys.” Would you say the function has a maximum value? Can you identify any points on the graph that you might consider to be “local maximum points”? What about “local minimum points”?

27. $f(x, y) = 3x - x^3 - 4y^2 - 10xy$

28. $f(x, y) = xye^{-x^2-y^2}$

29–30 Use a computer to graph the function using various domains and viewpoints. Comment on the limiting behavior of the function. What happens as both $x$ and $y$ become large? What happens as $(x, y)$ approaches the origin?

29. $f(x, y) = \frac{x + y}{x^2 + y^2}$

30. $f(x, y) = \frac{xy}{x^2 + y^2}$

31. Graph the surfaces $z = x^2 + y^2$ and $z = 1 - y^2$ on a common screen using the domain $|x| \leq 1.2, |y| \leq 1.2$ and observe the curve of intersection of these surfaces. Show that the projection of this curve onto the $xy$-plane is an ellipse.

32. Show that the curve of intersection of the surfaces $x^2 + 2y^2 - z^2 + 3x = 1$ and $2x^2 + 4y^2 - 2z^2 - 5y = 0$ lies in a plane.

33. Show that if the point $(a, b, c)$ lies on the hyperbolic paraboloid $z = y^2 - x^2$, then the lines with parametric equations $x = a + t, y = b + t, z = c + 2(b - a)t$ and $x = a + t, y = b - t, z = c - 2(b - a)t$ both lie entirely on this paraboloid. (This shows that the hyperbolic paraboloid is what is called a \textit{ruled surface}; that is, it can be generated by the motion of a straight line. In fact, this exercise shows that through each point on the hyperbolic paraboloid there are two generating lines. The only other quadric surfaces that are ruled surfaces are cylinders, cones, and hyperboloids of one sheet.)

34. Find an equation for the surface consisting of all points $P$ for which the distance from $P$ to the $x$-axis is twice the distance from $P$ to the $yz$-plane. Identify the surface.
**EXAMPLE 8** Use a computer to draw a picture of the solid that remains when a hole of radius 3 is drilled through the center of a sphere of radius 4.

**SOLUTION** To keep the equations simple, let’s choose the coordinate system so that the center of the sphere is at the origin and the axis of the cylinder that forms the hole is the z-axis. We could use either cylindrical or spherical coordinates to describe the solid, but the description is somewhat simpler if we use cylindrical coordinates. Then the equation of the cylinder is \( r = 3 \) and the equation of the sphere is \( x^2 + y^2 + z^2 = 16, \) or \( r^2 + z^2 = 16. \) The points in the solid lie outside the cylinder and inside the sphere, so they satisfy the inequalities

\[
3 \leq r \leq \sqrt{16 - z^2}
\]

To ensure that the computer graphs only the appropriate parts of these surfaces, we find where they intersect by solving the equations \( r = 3 \) and \( r = \sqrt{16 - z^2}: \)

\[
\sqrt{16 - z^2} = 3 \quad \Rightarrow \quad 16 - z^2 = 9 \quad \Rightarrow \quad z^2 = 7 \quad \Rightarrow \quad z = \pm \sqrt{7}
\]

The solid lies between \( z = -\sqrt{7} \) and \( z = \sqrt{7}, \) so we ask the computer to graph the surfaces with the following equations and domains:

\[
\begin{align*}
& r = 3 \quad \quad \quad \quad \quad 0 \leq \theta \leq 2\pi \quad -\sqrt{7} \leq z \leq \sqrt{7} \\
& r = \sqrt{16 - z^2} \quad 0 \leq \theta \leq 2\pi \quad -\sqrt{7} \leq z \leq \sqrt{7}
\end{align*}
\]

The resulting picture, shown in Figure 11, is exactly what we want.

▲ Most three-dimensional graphing programs can graph surfaces whose equations are given in cylindrical or spherical coordinates. As Example 8 demonstrates, this is often the most convenient way of drawing a solid.

**FIGURE 11**

---

**Exercises**

1. What are cylindrical coordinates? For what types of surfaces do they provide convenient descriptions?

2. What are spherical coordinates? For what types of surfaces do they provide convenient descriptions?

3–4 Plot the point whose cylindrical coordinates are given. Then find the rectangular coordinates of the point.

3. (a) \((3, \pi/2, 1)\) (b) \((4, -\pi/3, 5)\)

4. (a) \((1, \pi, e)\) (b) \((5, \pi/6, 6)\)

5–6 Change from rectangular to cylindrical coordinates.

5. (a) \((1, -1, 4)\) (b) \((-1, -\sqrt{3}, 2)\)

6. (a) \((3, 3, -2)\) (b) \((3, 4, 5)\)

7–8 Plot the point whose spherical coordinates are given. Then find the rectangular coordinates of the point.

7. (a) \((1, 0, 0)\) (b) \((2, \pi/3, \pi/4)\)

8. (a) \((5, \pi, \pi/2)\) (b) \((2, \pi/4, \pi/3)\)
Families of Surfaces

In this project you will discover the interesting shapes that members of families of surfaces can take. You will also see how the shape of the surface evolves as you vary the constants.

1. Use a computer to investigate the family of functions

\[ f(x, y) = (ax^2 + by^2)e^{-x^2-y^2} \]

How does the shape of the graph depend on the numbers \( a \) and \( b \)?

2. Use a computer to investigate the family of surfaces \( z = x^2 + y^2 + cxy \). In particular, you should determine the transitional values of \( c \) for which the surface changes from one type of quadric surface to another.

3. Members of the family of surfaces given in spherical coordinates by the equation

\[ \rho = 1 + 0.2\sin m\theta \sin n\phi \]

have been suggested as models for tumors and have been called *bumpy spheres* and *wrinkled spheres*. Use a computer to investigate this family of surfaces, assuming that \( m \) and \( n \) are positive integers. What roles do the values of \( m \) and \( n \) play in the shape of the surface?
CHAPTER 9 VECTORS AND THE GEOMETRY OF SPACE

Review

1. What is the difference between a vector and a scalar?
2. How do you add two vectors geometrically? How do you add them algebraically?
3. If \( \mathbf{a} \) is a vector and \( c \) is a scalar, how is \( c \mathbf{a} \) related to \( \mathbf{a} \) geometrically? How do you find \( c \mathbf{a} \) algebraically?
4. How do you find the vector from one point to another?
5. How do you find the dot product \( \mathbf{a} \cdot \mathbf{b} \) of two vectors if you know their components?
6. How are dot products useful?
7. Write expressions for the scalar and vector projections of \( \mathbf{b} \) onto \( \mathbf{a} \). Illustrate with diagrams.
8. How do you find the cross product \( \mathbf{a} \times \mathbf{b} \) of two vectors if you know their lengths and the angle between them? What if you know their components?
9. How are cross products useful?
10. (a) How do you find the area of the parallelogram determined by \( \mathbf{a} \) and \( \mathbf{b} \)?
    (b) How do you find the volume of the parallelepiped determined by \( \mathbf{a} \), \( \mathbf{b} \), and \( \mathbf{c} \)?
11. How do you find a vector perpendicular to a plane?
12. How do you find the angle between two intersecting planes?
13. Write a vector equation, parametric equations, and symmetric equations for a line.
14. Write a vector equation and a scalar equation for a plane.
15. (a) How do you tell if two vectors are parallel?
    (b) How do you tell if two vectors are perpendicular?
    (c) How do you tell if two planes are parallel?
16. (a) Describe a method for determining whether three points \( P, Q, \) and \( R \) lie on the same line.
    (b) Describe a method for determining whether four points \( P, Q, R, \) and \( S \) lie in the same plane.
17. (a) How do you find the distance from a point to a line?
    (b) How do you find the distance from a point to a plane?
    (c) How do you find the distance between two lines?
18. How do you sketch the graph of a function of two variables?
19. Write equations in standard form of the six types of quadric surfaces.
20. (a) Write the equations for converting from cylindrical to rectangular coordinates. In what situation would you use cylindrical coordinates?
    (b) Write the equations for converting from spherical to rectangular coordinates. In what situation would you use spherical coordinates?

True–False Quiz

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. For any vectors \( \mathbf{u} \) and \( \mathbf{v} \) in \( \mathbb{V}_3 \), \( \mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u} \).
2. For any vectors \( \mathbf{u} \) and \( \mathbf{v} \) in \( \mathbb{V}_3 \), \( \mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{u} \).
3. For any vectors \( \mathbf{u} \) and \( \mathbf{v} \) in \( \mathbb{V}_3 \), \( |\mathbf{u} \times \mathbf{v}| = |\mathbf{v} \times \mathbf{u}| \).
4. For any vectors \( \mathbf{u} \) and \( \mathbf{v} \) in \( \mathbb{V}_3 \) and any scalar \( k \), \( k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v} \).
5. For any vectors \( \mathbf{u} \) and \( \mathbf{v} \) in \( \mathbb{V}_3 \) and any scalar \( k \), \( k(\mathbf{u} \times \mathbf{v}) = (k\mathbf{u}) \times \mathbf{v} \).
6. For any vectors \( \mathbf{u}, \mathbf{v}, \) and \( \mathbf{w} \) in \( \mathbb{V}_3 \), \( (\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w} \).
7. For any vectors \( \mathbf{u}, \mathbf{v}, \) and \( \mathbf{w} \) in \( \mathbb{V}_3 \), \( \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} \).
8. For any vectors \( \mathbf{u}, \mathbf{v}, \) and \( \mathbf{w} \) in \( \mathbb{V}_3 \), \( \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \times \mathbf{w} \).
9. For any vectors \( \mathbf{u} \) and \( \mathbf{v} \) in \( \mathbb{V}_3 \), \( (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = 0 \).
10. For any vectors \( \mathbf{u} \) and \( \mathbf{v} \) in \( \mathbb{V}_3 \), \( (\mathbf{u} + \mathbf{v}) \times \mathbf{v} = \mathbf{u} \times \mathbf{v} \).
11. The cross product of two unit vectors is a unit vector.
12. A linear equation \( Ax + By + Cz + D = 0 \) represents a line in space.
13. The set of points \( \{ (x, y, z) | x^2 + y^2 = 1 \} \) is a circle.
14. If \( \mathbf{u} = \langle u_1, u_2 \rangle \) and \( \mathbf{v} = \langle v_1, v_2 \rangle \), then \( \mathbf{u} \cdot \mathbf{v} = \langle u_1v_1, u_2v_2 \rangle \).
1. (a) Find an equation of the sphere that passes through the point \( (6, -2, 3) \) and has center \((-1, 2, 1)\).
(b) Find the curve in which this sphere intersects the \(yz\)-plane.
(c) Find the center and radius of the sphere \(x^2 + y^2 + z^2 - 8x + 2y + 6z + 1 = 0\).

2. Copy the vectors in the figure and use them to draw each of the following vectors.
(a) \( \mathbf{a} + \mathbf{b} \)  
(b) \( \mathbf{a} - \mathbf{b} \)  
(c) \(-\frac{1}{2}\mathbf{a}\)  
(d) \(2\mathbf{a} + \mathbf{b}\)

3. If \(\mathbf{u}\) and \(\mathbf{v}\) are the vectors shown in the figure, find \(\mathbf{u} \cdot \mathbf{v}\) and \(|\mathbf{u} \times \mathbf{v}|\). Is \(\mathbf{u} \times \mathbf{v}\) directed into the page or out of it?

4. Calculate the given quantity if \(\mathbf{a} = i + j - 2k\), \(\mathbf{b} = 3i - 2j + k\), and \(\mathbf{c} = j - 5k\).
(a) \(2\mathbf{a} + 3\mathbf{b}\)  
(b) \(|\mathbf{b}|\)  
(c) \(\mathbf{a} \cdot \mathbf{b}\)  
(d) \(\mathbf{a} \times \mathbf{b}\)  
(e) \(|\mathbf{b} \times \mathbf{c}|\)  
(f) \(\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})\)  
(g) \(\mathbf{c} \times \mathbf{c}\)  
(h) \(\mathbf{a} \times (\mathbf{b} \times \mathbf{c})\)  
(i) \(\text{comp}_\mathbf{a} \mathbf{b}\)  
(j) \(\text{proj}_\mathbf{a} \mathbf{b}\)  
(k) The angle between \(\mathbf{a}\) and \(\mathbf{b}\) (correct to the nearest degree)

5. Find the values of \(x\) such that the vectors \((3, 2, x)\) and \((2x, 4, x)\) are orthogonal.

6. Find two unit vectors that are orthogonal to both \(\mathbf{j} + 2\mathbf{k}\) and \(\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}\).

7. Suppose that \(\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 2\). Find
(a) \((\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}\)  
(b) \(\mathbf{u} \cdot (\mathbf{w} \times \mathbf{v})\)  
(c) \(\mathbf{v} \cdot (\mathbf{u} \times \mathbf{w})\)  
(d) \((\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v}\)

8. Show that if \(\mathbf{a}\), \(\mathbf{b}\), and \(\mathbf{c}\) are in \(V_3\), then
\[(\mathbf{a} \times \mathbf{b}) \cdot [(\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a})] = [\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})]^2\]

9. Find the acute angle between two diagonals of a cube.

10. Given the points \(A(1, 0, 1), B(2, 3, 0), C(-1, -1, 4),\) and \(D(0, 3, 2)\), find the volume of the parallelepiped with adjacent edges \(AB, AC,\) and \(AD\).

11. (a) Find a vector perpendicular to the plane through the points \(A(1, 0, 0), B(2, 0, -1),\) and \(C(1, 4, 3)\).
(b) Find the area of triangle \(ABC\).

12. A constant force \(\mathbf{F} = 3\mathbf{i} + 5\mathbf{j} + 10\mathbf{k}\) moves an object along the line segment from \((1, 0, 2)\) to \((5, 3, 8)\). Find the work done if the distance is measured in meters and the force in newtons.

13. A boat is pulled onto shore using two ropes, as shown in the diagram. If a force of 255 N is needed, find the magnitude of the force in each rope.

14. Find the magnitude of the torque about \(P\) if a 50-N force is applied as shown.

15–21 Find parametric equations for the line that satisfies the given conditions.
15. Passing through \((1, 2, 4)\) and in the direction of \(\mathbf{v} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}\)
16. Passing through \((-6, -1, 0)\) and \((2, -3, 5)\)
17. Passing through \((1, 0, 1)\) and parallel to the line with parametric equations \(x = 4t, y = 1 - 3t, z = 2 + 5t\)

18–21 Find an equation of the plane that satisfies the given conditions.
18. Passing through \((4, -1, -1)\) and with normal vector \((2, 6, -3)\)
19. Passing through \((-4, 1, 2)\) and parallel to the plane \(x + 2y + 5z = 3\)
20. Passing through \((-1, 2, 0), (2, 0, 1),\) and \((-5, 3, 1)\)
21. Passing through the line of intersection of the planes \(x - z = 1\) and \(y + 2z = 3\) and perpendicular to the plane \(x + y - 2z = 1\)
22. Find the point in which the line with parametric equations
\( x = 2 - t, \ y = 1 + 3t, \ z = 4t \) intersects the plane
\( 2x - y + z = 2 \).

23. Determine whether the lines given by the symmetric equations
\[
\frac{x - 1}{2} = \frac{y - 2}{3} = \frac{z - 3}{4}
\]
and
\[
\frac{x + 1}{6} = \frac{y - 3}{-1} = \frac{z + 5}{2}
\]
are parallel, skew, or intersecting.

24. (a) Show that the planes \( x + y - z = 1 \) and
\( 2x - 3y + 4z = 5 \) are neither parallel nor perpendicular.
(b) Find, correct to the nearest degree, the angle between these planes.

25. Find the distance between the planes \( 3x + y - 4z = 2 \)
and \( 3x + y - 4z = 24 \).

26. Find the distance from the origin to the line \( x = 1 + t, \ y = 2 - t, \ z = -1 + 2t \).

27–28 ■ Find and sketch the domain of the function.
27. \( f(x, y) = x \ln(x - y^2) \)
28. \( f(x, y) = \sqrt{\sin \pi(x^2 + y^2)} \)

29–32 ■ Sketch the graph of the function.
29. \( f(x, y) = 6 - 2x - 3y \)
30. \( f(x, y) = \cos x \)
31. \( f(x, y) = 4 - x^2 - 4y^2 \)
32. \( f(x, y) = \sqrt{4 - x^2 - 4y^2} \)

33–36 ■ Identify and sketch the graph of each surface.
33. \( y^2 + z^2 = 1 - 4x^2 \)
34. \( y^2 + z^2 = x \)
35. \( y^2 + z^2 = 1 \)
36. \( y^2 + z^2 = 1 + x^2 \)

37. The cylindrical coordinates of a point are \((2, \pi/6, 2)\). Find the rectangular and spherical coordinates of the point.

38. The rectangular coordinates of a point are \((2, 2, -1)\). Find the cylindrical and spherical coordinates of the point.

39. The spherical coordinates of a point are \((4, \pi/3, \pi/6)\). Find the rectangular and cylindrical coordinates of the point.

40. Identify the surfaces whose equations are given.
(a) \( \theta = \pi/4 \)
(b) \( \phi = \pi/4 \)

41–42 ■ Write the equation in cylindrical coordinates and in spherical coordinates.
41. \( x^2 + y^2 + z^2 = 4 \)
42. \( x^2 + y^2 = 4 \)

43. The parabola \( z = 4y^2, \ x = 0 \) is rotated about the \( z \)-axis. Write an equation of the resulting surface in cylindrical coordinates.

44. Sketch the solid consisting of all points with spherical coordinates \((\rho, \theta, \phi)\) such that \(0 \leq \theta \leq \pi/2, 0 \leq \phi \leq \pi/6, \) and \(0 \leq \rho \leq 2 \cos \phi\).
1. Each edge of a cubical box has length 1 m. The box contains nine spherical balls with the same radius \( r \). The center of one ball is at the center of the cube and it touches the other eight balls. Each of the other eight balls touches three sides of the box. Thus, the balls are tightly packed in the box. (See the figure.) Find \( r \). (If you have trouble with this problem, read about the problem-solving strategy entitled Use analogy on page 88.)

2. Let \( B \) be a solid box with length \( L \), width \( W \), and height \( H \). Let \( S \) be the set of all points that are a distance at most 1 from some point of \( B \). Express the volume of \( S \) in terms of \( L, W, \) and \( H \).

3. Let \( L \) be the line of intersection of the planes \( cx + y + z = c \) and \( x - cy + cz = -1 \), where \( c \) is a real number.
   (a) Find symmetric equations for \( L \).
   (b) As the number \( c \) varies, the line \( L \) sweeps out a surface \( S \). Find an equation for the curve of intersection of \( S \) with the horizontal plane \( z = t \) (the trace of \( S \) in the plane \( z = t \)).
   (c) Find the volume of the solid bounded by \( S \) and the planes \( z = 0 \) and \( z = 1 \).

4. A plane is capable of flying at a speed of 180 km/h in still air. The pilot takes off from an airfield and heads due north according to the plane’s compass. After 30 minutes of flight time, the pilot notices that, due to the wind, the plane has actually traveled 80 km at an angle 5° east of north.
   (a) What is the wind velocity?
   (b) In what direction should the pilot have headed to reach the intended destination?

5. Suppose a block of mass \( m \) is placed on an inclined plane, as shown in the figure. The block’s descent down the plane is slowed by friction; if \( \theta \) is not too large, friction will prevent the block from moving at all. The forces acting on the block are the weight \( W \), where \( |W| = mg \) (\( g \) is the acceleration due to gravity); the normal force \( N \) (the normal component of the reactionary force of the plane on the block), where \( |N| = n \); and the force \( F \) due to friction, which acts parallel to the inclined plane, opposing the direction of motion. If the block is at rest and \( \theta \) is increased, \( |F| \) must also increase until ultimately \( |F| \) reaches its maximum, beyond which the block begins to slide. At this angle \( \theta_c \), it has been observed that \( |F| \) is proportional to \( n \). Thus, when \( |F| \) is maximal, we can say that \( |F| = \mu_n n \), where \( \mu_n \) is called the coefficient of static friction and depends on the materials that are in contact.
   (a) Observe that \( N + F + W = 0 \) and deduce that \( \mu_n = \tan \theta_c \).
   (b) Suppose that, for \( \theta > \theta_c \), an additional outside force \( H \) is applied to the block, horizontally from the left, and let \( |H| = h \). If \( h \) is small, the block may still slide down the plane; if \( h \) is large enough, the block will move up the plane. Let \( h_{\text{min}} \) be the smallest value of \( h \) that allows the block to remain motionless (so that \( |F| \) is maximal).

   By choosing the coordinate axes so that \( F \) lies along the \( x \)-axis, resolve each force into components parallel and perpendicular to the inclined plane and show that

   \[
   h_{\text{min}} \sin \theta + mg \cos \theta = n \quad \text{and} \quad h_{\text{min}} \cos \theta + \mu_n n = mg \sin \theta
   \]

   (c) Show that

   \[
   h_{\text{min}} = mg \tan(\theta - \theta_c)
   \]

   Does this equation seem reasonable? Does it make sense for \( \theta = \theta_c \)? As \( \theta \to 90^\circ \)? Explain.

   (d) Let \( h_{\text{max}} \) be the largest value of \( h \) that allows the block to remain motionless. (In which direction is \( F \) heading?) Show that

   \[
   h_{\text{max}} = mg \tan(\theta + \theta_c)
   \]

   Does this equation seem reasonable? Explain.