Constrained multi-degree reduction of triangular Bézier surfaces

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Abstract. This paper proposes and applies a method to sort two-dimensional control points of triangular Bézier surfaces in a row vector. Using the property of bivariate Jacobi basis functions, it further presents two algorithms for multi-degree reduction of triangular Bézier surfaces with constraints, providing explicit degree-reduced surfaces. The first algorithm can obtain the explicit representation of the optimal degree-reduced surfaces and the approximating error in both boundary curve constraints and corner constraints. But it has to solve the inversion of a matrix whose degree is related with the original surface. The second algorithm entails no matrix inversion to bring about computational instability, gives stable degree-reduced surfaces quickly, and presents the error bound. In the end, the paper proves the efficiency of the two algorithms through examples and error analysis.

§1 Introduction

Bézier curves(surfaces) are one of the most widely used modeling tools in CAD/CAM systems[1]. It is of great importance in geometry processing to achieve degree reduction, which is a process to find a lower degree curve(surface) to approximate another parameter curve(surface) with a given degree, while keeping the error within a given tolerance. Considerable methods have been developed to reduce degrees, which mainly fall into two categories. One adopts discretization methods[2-7], and the other, algebraic methods[8-12]. Some researchers have made certain achievements[13-17] in the degree reduction of tensor product Bézier surfaces. But few studies have been done on triangular Bézier surfaces[18-19]. Thus, what is needed now is to enable the multi-degree reduction of triangular Bézier surfaces at one time while keeping corner and boundary curve constraints, so as to create a more efficient and precise modeling system. The work of Lu[20] meets the stated requirement. Again it has to solve the inversion of a high degree matrix, which needs long computing time and is questionable in its stability.
With a fair view of former research achievements, we present the algorithms for multi-degree reduction of triangular Bézier surfaces under both corners and boundary curves constraints respectively. The basic idea is to achieve dimensional reduction of control points and basis functions by using the conversion between bivariate Jacobi basis functions and bivariate Bernstein basis functions as well as the orthogonality of Jacobi polynomials. The first algorithm derives the optimal degree-reduced surface and the best explicit presentation of approximating error under the two constraints in a most precise way. But it also needs to solve the inverse of a matrix whose degree is \( O(n) \), where \( n \) is the degree of the original surface. The second algorithm avoids the inverse matrix by using the property of bivariate Bernstein polynomials. It provides stable numerical calculation with minor computational error tolerance and presents the error bound.

§2 Preliminaries

Defined on triangular domain \( T := \{(s, t) : s, t \geq 0, 1 - s - t \geq 0 \} \), a triangular Bézier surface of degree \( n \) is defined as follows\(^1\):

\[
P(s, t) = \sum_{i,j \leq n} p_{i,j} B_{i,j}^n(s, t),
\]

where \( B_{i,j}^n(s, t) = \binom{n}{i,j} s^i t^j (1 - s - t)^{n-i-j} \) is the bivariate Bernstein basis function of degree \( n \). \( p_{i,j} \) are the control points of the surface.

2.1 Jacobi basis functions on triangular domain

Bivariate Jacobi basis functions \( J_{n,k}^{\alpha,\beta,\gamma}(s, t) \) on triangular domain are defined as follows\(^{21}\):

\[
J_{n,k}^{\alpha,\beta,\gamma}(s, t) = h_{n,k}^{-1} j_{n-k}^{2k+\beta+\gamma,\alpha-1/2}(s)(1 - s)^k j_k^{\gamma-1/2,\beta-1/2}(\frac{t}{1 - s}),
\]

where \( \alpha, \beta, \gamma > -1/2 \), and

\[
J_{n}^{(u,v)}(t) = \frac{(u + 1)n}{n!} F_1\left( \begin{array}{c} -n, n + u + v + 1 \\ u + 1 \end{array} | 1 - t \right)
\]

is univariate Jacobi polynomial\(^{22}\) of degree \( n \). Denote \( \lambda = \beta + \gamma, \sigma = \alpha + \beta + \gamma + 1/2 \), then

\[
b_{n,k}^2 \equiv [h_{n,k}^{\alpha,\beta,\gamma}]^2 = \frac{(\alpha + 1/2)_n(k + \lambda)_n}{k!(2k + \lambda)(2n/\sigma + 1)(\sigma)_n k!}.
\]

In this paper, generalized hypergeometric series\(^{23}\) is denoted as

\[
_1F_1\left( \begin{array}{c} a_1, \cdots, a_i \\ b_1, \cdots, b_j \end{array} | z \right) \equiv \sum_{k=0}^{\infty} \frac{\prod_{l=1}^{k} (a_l)_k}{\prod_{l=1}^{k} (b_l)_k} z^k,
\]

where \( i, j \in \mathbb{Z}_+, z, a_i, b_j, c \in \mathbb{C}, (c)_k = \prod_{l=0}^{k-1} (c + l) \).

Bivariate Jacobi basis functions have the following properties:

**Property 1.** Bivariate Jacobi basis functions of degree \( n \) on triangular domain \( T \) are orthogonal with weight function

\[
\omega_{\alpha,\beta,\gamma}(s, t) = A^{\alpha,\beta,\gamma} s^{\alpha-1/2} t^{\beta-1/2} (1 - s - t)^{\gamma-1/2},
\]

where

\[
A^{\alpha,\beta,\gamma} = \frac{(\alpha + \beta + \gamma + 1/2)!}{(\alpha - 1/2)! (\beta - 1/2)! (\gamma - 1/2)!}.
\]
That is,

\[ \int_{T} \omega^{(\alpha,\beta,\gamma)}(s,t) f_{m,l}^{(\alpha,\beta,\gamma)}(s,t) \omega_{n,k}^{(\alpha,\beta,\gamma)}(s,t) \, ds \, dt = \delta_{m,n}\delta_{l,k}. \]

**Lemma 1.** For any \(i, j, l, k, n \in \mathbb{N}_0\) such that \(i + j \leq n\) and \(l \leq k \leq n\), the bivariate Bernstein polynomials \(B_{i,j}^n(s,t)\) and bivariate Jacobi polynomials \(J_{l,k}^{(\alpha,\beta,\gamma)}(s,t)\) can be expressed each other as follows \([23]\):

\[
B_{i,j}^n(s,t) = \sum_{0 \leq k \leq l \leq n} e_{l,k}^{(\alpha,\beta,\gamma)}(n,i,j) J_{l,k}^{(\alpha,\beta,\gamma)}(s,t), \\
J_{l,k}^{(\alpha,\beta,\gamma)}(s,t) = \sum_{i+j \leq n} d_{i,j}^{(\alpha,\beta,\gamma)}(n,l,k) B_{i,j}^n(s,t),
\]

where

\[
e_{l,k}^{(\alpha,\beta,\gamma)}(n,i,j) = (-1)^{(l-k)} \left( \begin{array}{c} n \\ l \end{array} \right) \left( \begin{array}{c} n-i \\ j \end{array} \right) \left( \begin{array}{c} 1 \\ k \end{array} \right) \frac{(\alpha + 1/2)(\beta + 1/2)(\gamma + 1/2)n-i-j}{h_{l,k} l!(\sigma + 1)n+l} \\
\cdot H_{l,k}(j,n-i-j;\alpha-1/2,\beta-1/2,\gamma-1/2,n),
\]

\[
H_{l,k}(s,t;a,b,c,N) := (N-l+1)_{l-k}(a+1)_{n-k}(b+c+1)_k(-s-t)_k
\]

\[
\cdot Q_{l-k}(N-s-t;a,2k+b+c+1,N-k)Q_k(s,b,c,s+t),
\]

\[
Q_t(t;\mu,\nu,M) := F_2(-l,l+\mu+\nu+1,-t,M)_{\mu+1,-M} \\
\]

\[
d_{i,j}^{(\alpha,\beta,\gamma)}(n,l,k) = \left( \begin{array}{c} 1 \\ k \end{array} \right) \frac{(l-n)!H_{l,k}(j,n-i-j;\alpha-1/2,\beta-1/2,\gamma-1/2,n)}{|l(-n)|h_{l,k}}.
\]

### 2.2 One dimensional sorting of control points of surface and basis functions

In 1959 when de Casteljau invented Bézier curves, he realized the need for the extension of curve ideas to surfaces. Interestingly enough, the first surface type he considered was what we now call Bézier triangular. This historical ‘first’ of triangular patches is reflected by the mathematical statement that they are a more ‘natural’ generalization of Bézier curves than tensor product patches \([1]\). Based on this ‘natural’ generalization, this paper first considers the re-sorting of the control points. Then using the conversion between Bernstein basis and Jacobi basis, and the orthogonality of Jacobi polynomials, we derive the optimal degree-reduced surface and the corresponding error.

**Definition 1.** If a binary array \(p_{i,j}(i+j \leq n)\) arranged in a row vector conforms to the order of the subscript \((i,j)\) in the index set \(\Omega^n = \{(n,0),(n-1,1),\ldots,(0,n),(0,n-1),\ldots,(0,0),(1,0),\ldots,(n-1,0),(n-2,1),\ldots,(1,1),\ldots,(n-3,1),\ldots\}\), we say the array satisfies \((i,j) \in \Omega^n\). This sorting method is called helical sorting. When \(n = 5\), the sorting is presented in Fig. 1(a).

**Definition 2.** If a binary array \(J_{k,l}(s,t)(0 \leq l \leq k \leq n)\) arranged in a row vector conforms to the order of the subscript \((i,j)\) in the index set \(\Lambda^n = \{(n,n),(n,n-1),\ldots,(0,n),(n-1,n-1),(n-1,n-2),\ldots,(1,0),(n-2,0),\ldots,(0,0)\}\), we say the array satisfies \((k,l) \in \Lambda^n\). When \(n = 5\), the sorting is presented in Fig. 1(b).

Using the above two sorting methods and Lemma 1, the conversion of the two base is described as follows:

**Theorem 1.** Bivariate Bernstein basis \(B_{i,j}^n(s,t)\) and Jacobi basis \(J_{l,k}^{(\alpha,\beta,\gamma)}(s,t)\) can represent
each other with
\[ B_n = J_n^{(\alpha,\beta,\gamma)} E_n^{(\alpha,\beta,\gamma)}, \quad J_n^{(\alpha,\beta,\gamma)} = B_n D_n^{(\alpha,\beta,\gamma)}. \]

Here,
\[ B_n = (B_{i,j}^{n}(s,t))_{1 \times [(n+2)(n+1)/2]}, \quad J_n^{(\alpha,\beta,\gamma)} = (J_{l,k}^{(\alpha,\beta,\gamma)}(s,t))_{1 \times [(n+2)(n+1)/2]}, \]
\[ E_n^{(\alpha,\beta,\gamma)} = (e_{i,k}^{(\alpha,\beta,\gamma)}(n,i,j)), \quad D_n^{(\alpha,\beta,\gamma)} = (d_{i,j}^{(\alpha,\beta,\gamma)}(n,l,k)), \quad (i,j) \in \Omega^n, \quad (l,k) \in \Lambda^n, \]
where the elements of each row of matrix \( E_n^{(\alpha,\beta,\gamma)} \) satisfy \( (i,j) \in \Omega^n \) and the elements of each column of matrix \( D_n^{(\alpha,\beta,\gamma)} \) satisfy \( (l,k) \in \Lambda^n \). The elements of each row of matrix \( D_n^{(\alpha,\beta,\gamma)} \) satisfy \( (l,k) \in \Lambda^n \) and the elements of each column of matrix \( D_n^{(\alpha,\beta,\gamma)} \) satisfy \( (i,j) \in \Omega^n \).

As this paper only discusses Jacobi polynomials in two occasions, when \( \alpha = \beta = \gamma = 0 \) and \( \alpha = \beta = \gamma = 5/2 \), the following abbreviated notations are used for convenience:
\[ J_0 := J_0^{(0,0,0)}, E_0 := E_0^{(0,0,0)}, D_0 := D_0^{(0,0,0)}, \]
\[ J_{5/2} := J_{5/2}^{(5/2,5/2,5/2)}, E_{5/2} := E_{5/2}^{(5/2,5/2,5/2)}, D_{5/2} := D_{5/2}^{(5/2,5/2,5/2)}. \]

### §3 The problem of degree reduction and preconditioning of boundary curves

\((n-m)\)-degree reduction of triangular Bézier surface \( P_n(s,t) \) in (2.1) is to find a triangular Bézier surface of degree \( m(m < n) \),
\[ Q^m(s,t) = \sum_{i+j \leq m} q_{i,j} B^m_{i,j}(s,t), \tag{3.1} \]
whose control points are \( \{q_{i,j}\}_{i+j \leq m} \), so that the distance function between two surfaces in \( L_2 \)-norm is minimized. That is,
\[ d(P^n(s,t), Q^m(s,t)) = \sqrt{\int_T \int_T \|P^n(s,t) - Q^m(s,t)\|^2 ds dt} = \min. \]

As the information of corners of surfaces in a CAD/CAM system usually needs to be preserved, we need to consider the continuities at the corners of degree-reduced surfaces. Moreover, most of the products consist of complex surfaces which are composed of several patches. In
order to preserve the continuities globally, we should make constraints on the boundary curves of the degree-reduced surfaces. So this section presents the preconditioning of boundary curves under these two constraints.

3.1 Boundary constraints

To a single triangular Bézier surface, its three boundary curves are respectively as follows:

\[ s + t = 1 : \quad \mathbf{P}_1(s) = \sum_{i=0}^{n} p_{i,n} B_n^i(s), \quad 0 \leq s \leq 1, \]  
\[ s = 0 : \quad \mathbf{P}_2(t) = \sum_{j=0}^{n} p_{0,j} B_n^j(t), \quad 0 \leq t \leq 1, \]  
\[ t = 0 : \quad \mathbf{P}_3(s) = \sum_{i=0}^{n} p_{i,0} B_n^i(s), \quad 0 \leq s \leq 1, \]  

where \( B_n^i(s) = \binom{n}{i}(1-s)^{n-i}s^i \) is the Bernstein basis function of degree \( n \). The three boundary curves are Bézier curves of degree \( n \).

To make constraints on these boundary curves, we need to keep high-order interpolation at the endpoints. In order to get the degree-reduced curves of (3.2)-(3.4) with constraints of endpoints high-order interpolation. The basic idea is summarized as follows.

Given a Bézier curve of degree \( n \),

\[ \mathbf{S}_n(t) = \sum_{k=0}^{n} b_k B_n^k(t), \quad t \in [0, 1]. \]  

The optimal \((n - n_1)\)-degree-reduced approximating curve having equal derivatives up to \((r - 1)\)-th and \((s - 1)\)-th orders at the endpoints, respectively i.e.,

\[ \mathbf{S}_n^{(k)}(0) = \mathbf{T}_{n_1}^{(k)}(0), \quad k = 0, 1, \cdots, r - 1; \quad \mathbf{S}_n^{(l)}(1) = \mathbf{T}_{n_1}^{(l)}(1), \quad l = 0, 1, \cdots, s - 1, \]  

is denoted as

\[ \mathbf{T}_{n_1}(t) = \sum_{k=0}^{n_1} \tilde{b}_k B_{n_1}^k(t). \]

The control points of the curve \( \mathbf{T}_{n_1}(t) \) of degree \( n_1 \) can be expressed in matrix form

\[
\begin{pmatrix}
\tilde{b}_0 \\
\vdots \\
\tilde{b}_{n_1}
\end{pmatrix}
= \begin{pmatrix}
\mathbf{E}_{n_1} \mathbf{B}^a - \mathbf{E}^a \mathbf{A}^{a-1} \mathbf{E}^b \\
\vdots
\end{pmatrix}
\begin{pmatrix}
b_0 \\
\vdots \\
b_{n_1}
\end{pmatrix}.
\]

Here,

\[ \mathbf{B}_n = (b_{i,j})_{(n+1)\times(n+1)}, \quad b_{i,j} = \begin{cases} 0, & i < j, \\ (-1)^{i+j} \binom{n}{j}, & i \geq j, \end{cases} \quad i, j = 0, 1, \cdots, n, \]

\[ \mathbf{E}_{n_1} = (f_{i,j})_{(n_1+1)\times(n_1+1)}, \quad f_{i,j} = \begin{cases} 0, & i < j, \\ \binom{n_1-j}{i-j}, & i \geq j, \end{cases} \quad i, j = 0, 1, \cdots, n_1, \]
The corresponding approximating error is denoted as \( \varepsilon \). The matrix \( F \) is a conversion matrix from degree \( n \) Bernstein basis to power basis of the same degree. The matrix \( F_{n,1} \) is a conversion matrix from degree \( n_1 \) power basis to Bernstein basis at the same degree.

Denote the error function between original curve \( S_n(t) \) and the degree-reduced curve \( T_{n_1}(t) \) as
\[
\varepsilon(t) = S_n(t) - T_{n_1}(t).
\]
The corresponding approximating error is denoted as \( \varepsilon \). Its explicit expression is
\[
\varepsilon = \sqrt{\varepsilon_{m-r-s}^2 + \varepsilon_{m-r-s+2}^2 + \cdots + \varepsilon_{m-r-s+2s}^2},
\]
where
\[
\begin{align*}
\varepsilon_{m-r-s} & = \sum_{i=0}^{r+s+1} \frac{\binom{2s+2r}{r+s+1} \binom{2s+2r}{i} \binom{2s+2r}{i}}{2^{2s+2r+1}} \\
\delta_{m-r-s} & = \frac{2^{r+s+1}}{2^{n+1}(n+1)\cdots(n+s+1)}.
\end{align*}
\]
Using the above principle of degree reduction of curves, we can derive three degree-reduced matrices \( K_1, K_2, K_3 \), whose degrees are all \((m+1) \times (n+1)\), such that
\[
\begin{pmatrix}
q_{0,0} \\
q_{0,m} \\
q_{0,m-1} \\
\vdots \\
q_{0,n}
\end{pmatrix} = K_1 
\begin{pmatrix}
p_{0,0} \\
p_{0,n} \\
p_{0,m-1} \\
\vdots \\
p_{0,r}
\end{pmatrix}, 
\begin{pmatrix}
q_{0,0} \\
q_{0,m} \\
q_{0,m-1} \\
\vdots \\
q_{0,n}
\end{pmatrix} = K_2 
\begin{pmatrix}
p_{1,0} \\
p_{1,n-1} \\
p_{1,m-1} \\
\vdots \\
p_{1,0}
\end{pmatrix}, 
\begin{pmatrix}
q_{0,0} \\
q_{0,m} \\
q_{0,m-1} \\
\vdots \\
q_{0,n}
\end{pmatrix} = K_3 
\begin{pmatrix}
p_{0,0} \\
p_{0,n} \\
p_{0,m-1} \\
\vdots \\
p_{0,0}
\end{pmatrix}.
\]
The three degree-reduced curves are denoted respectively as

\[ Q_1(s) = \sum_{i=0}^{m} q_{i,n-i} B_i^m(s), \quad Q_2(t) = \sum_{j=0}^{m} q_{0,j} B_j^m(t), \quad Q_3(s) = \sum_{i=0}^{m} q_{i,0} B_i^m(s). \]  

They have equal derivatives \((r_1, s_1), (r_2, s_2), (r_3, s_3)\) at the endpoints with the original curve respectively (see Fig.2). Then, denote the error function and approximating error of (3.7) as \(\varepsilon_1(s), \varepsilon_2(t), \varepsilon_3(s)\) and \(\varepsilon_1, \varepsilon_2, \varepsilon_3\) respectively.

Write triangular Bézier surface of degree \(m\) as

\[ \tilde{Q}_m(s, t) = \sum_{i+j \leq m} \tilde{q}_{i,j} B_{i,j}^m(s, t), \quad \tilde{q}_{i,j} = \begin{cases} q_{0,j}, & j = 0, \ldots, m, \\ q_{i,0}, & i = 0, \ldots, m, \\ q_{m-j,j}, & j = 0, \ldots, m, \\ 0, & \text{else}. \end{cases} \]  

Its control points \(\tilde{q}_{i,j}\) sorted in helix order, that is, \((i, j) \in \Omega^m\), construct a row vector, which is denoted as \(\tilde{Q}_m\). Also, the control points \(p_{i,j}\) of the surface in (1) satisfying \((i, j) \in \Omega^m\), make a row vector which is denoted as \(P_n\). The control points \(q_{l,k}\) of degree-reduced surface satisfying \((l, k) \in \Omega^m\), constitute a row vector which is denoted as \(Q_m\).

Obviously, the relation between the control points vector \(\tilde{Q}_m\) and \(P_n\) can be shown as follows:

\[ (\tilde{Q}_m)^T = C(P_n)^T, \]  

where the matrix \(C\) can be described in Matlab language as:

\[
\begin{align*}
C &= \text{zeros}((m+2)(m+1)/2, (n+2)(n+1)/2), \\
C(1 : m + 1, 1 : n + 1) &= K_1, \\
C(m + 2 : 2m + 1, n + 1 : 2n + 1) &= K_2(2 : m + 1, n + 1 : 2n + 1), \\
C(2m + 2 : 3m, 2n + 1 : 3n) &= K_3(2 : m, 1 : n), \\
C(2m + 2 : 3m, 1) &= K_3(2m + 2 : 3m, n + 1).
\end{align*}
\]

Here, \(\text{zeros}(n, m)\) denote a zero matrix of degree \(n \times m\). The notation \(C(\cdots, \cdots)\) means the submatrix of the matrix \(C\) obtained by extracting the specific rows and columns.

Finally, write vector

\[ \overline{Q}_m = Q_m - \tilde{Q}_m. \]
When \( m = 4 \), the sorting of vectors \( \overline{Q}_m, \tilde{Q}_m, \check{Q}_m \) is shown in Fig. 3.

### 3.2 Interpolation at corners

In (3.8), we have got the degree-reduced matrices \( K_1, K_2, K_3 \) of the three boundary curves. Based on the interpolation at the corners and the three matrices, we also construct three corner constraints matrices. Denote

\[
H_i = \begin{pmatrix}
K_1^i \\
0 \\
K_2^i
\end{pmatrix}, \quad i = 1, 2, 3,
\]

where the submatrices \( K_1^i \) and \( K_2^i \) are formed respectively by the first \((r_i + 1)\) rows and last \((s_i + 1)\) columns of the matrix \( K_i \). Matrix \( 0_i \) is a zero matrix of degree \((m - r_i - s_i - 1) \times (n + 1)\).

Using the three corner constraints matrices, we can derive the constrained control points of the degree-reduced surface at the corners. Just the same as (3.10), denote the triangular Bézier surface of degree \( m \) as

\[
\check{Q}_m^c(s, t) = \sum_{i+j \leq m} \check{q}_{i,j}^c B_{i,j}^m(s, t), \quad \check{q}_{i,j}^c = \begin{cases}
q_{0,j}, & j = 0, \ldots, s_2, m - r_2, \ldots, m - 1; \\
q_{i,0}, & j = 0, \ldots, r_2, m - s_2, \ldots, m - 1; \\
q_{m-i,j}, & j = 0, \ldots, r_1, m - s_1, \ldots, m; \\
0, & \text{else.}
\end{cases}
\]

The control points \( \check{q}_{i,j}^c \) satisfying \((i, j) \in \Omega^m \) make a row vector \( \check{Q}_m^c \). Similarly to the construction of matrix \( C \) in (3.11), we construct a matrix \( H \) such that

\[
(\check{Q}_m^c)^T = H(P_n)^T.
\]

Finally, denote vector

\[
\overline{Q}_m = Q_m - \check{Q}_m^c.
\]

When \( m = 4 \) and the continuities at the three corners are all 0, the sorting of the vectors \( \overline{Q}_m \) and \( \check{Q}_m \) is shown in Fig. 4.

### §4 Algorithm I of degree reduction of triangular Bézier surfaces

The paper here first presents the algorithm for the optimal explicit multi-degree reduction of triangular Bézier surfaces with constraints of three boundary curves. As to corners interpolation in Section 3.2, we can obtain the degree-reduced surface by a simple alteration of the
To obtain the inner control points, we should subtract the surface \( Q_m(s, t) \) from the surface \( P_n(s, t) \). The identity

\[
P_n(s, t) - Q_m(s, t) = B_n(P_n)^T - B_m(Q_m + \overline{Q}_m)^T
\]

\[
= J_0^0_\hat{E}_0^0(P_n)^T - J_0^0_\hat{E}_0^0(C(P_n)^T + (\overline{Q}_m)^T)
\]

\[
= \begin{bmatrix} J_{n-m}^0 & J_0^0 \end{bmatrix} \begin{bmatrix} \hat{E}_0^0_{n-m} - (\hat{E}_0^0_{m})^T \end{bmatrix} \begin{bmatrix} (P_n)^T - J_0^0_\hat{E}_0^0(Q_m)^T \end{bmatrix}
\]

\[
= J_{n-m}^0 \hat{E}_0^0_{n-m}(P_n)^T - J_0^0_\hat{E}_0^0[(\hat{E}_0^0_{m} - \hat{E}_0^0_{m} C)(P_n)^T - \hat{E}_0^0_{m}(\overline{Q}_m)^T],
\]

where \( J_0^0 = \begin{bmatrix} J_{n-m}^0 & J_0^0 \end{bmatrix} \), \( \hat{E}_0^0 = \begin{bmatrix} \hat{E}_0^0_{n-m} \\ \hat{E}_0^0_{m} \end{bmatrix} \), the dimensions of the four block matrices are shown as follows.

<table>
<thead>
<tr>
<th>Matrix</th>
<th>Dimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>( J_{n-m}^0 )</td>
<td>([n-m(n+m+3)/2] \times [(n+2)(n+1)/2] )</td>
</tr>
<tr>
<td>( J_0^0 )</td>
<td>([(n+2)(n+1)/2] \times [(n+2)(n+1)/2] )</td>
</tr>
<tr>
<td>( \hat{E}<em>0^0</em>{n-m} )</td>
<td>([n-m(n+m+3)/2] \times [(n+2)(n+1)/2] )</td>
</tr>
<tr>
<td>( \hat{E}<em>0^0</em>{m} )</td>
<td>([(n+2)(n+1)/2] \times [(n+2)(n+1)/2] )</td>
</tr>
</tbody>
</table>

Because the elements of the first \( 3m \) columns of the vector \( \overline{Q}_m \) are all 0, we take out the first \( 3m \) columns of vector \( \overline{Q}_m \) and the matrix \( \hat{E}_0^0 \). Denote the new vector and matrix as \( Q_m \) and \( M \) respectively, then

\[
M(Q_m)^T = \hat{E}_0^0(Q_m)^T.
\]

Based on the orthogonality of bivariate Jacobi basis functions, we can see that the approximating error \( \|P_n(s, t) - Q_m(s, t)\|_2 \) reaches minimum if and only if

\[
\iint_{T} J_0^0 \hat{E}_0^0_{m} C(P_n)^T - \hat{E}_0^0_{m}(\overline{Q}_m)^T][[\hat{E}_0^0_{m} - \hat{E}_0^0_{m} C)(P_n)^T - \hat{E}_0^0_{m}(\overline{Q}_m)^T]^T(J_0^0)^T dsdt = \min.
\]

That is,

\[
\epsilon_1 = [[\hat{E}_0^0_{m} - \hat{E}_0^0_{m} C)(P_n)^T - M(Q_m)^T][[\hat{E}_0^0_{m} - \hat{E}_0^0_{m} C)(P_n)^T - M(Q_m)^T] = \min,
\]

In Section 3.1, we have pretreated the three boundary curves of degree-reduced surface. The vectors \( \overline{Q}_4 \) and \( \hat{Q}_4^c \) sorted in helix order are presented above-presented algorithm.

### 4.1 Degree reduction with boundary constraints

Fig. 4 The vectors \( \overline{Q}_4 \) and \( \hat{Q}_4^c \) sorted in helix order are presented above-presented algorithm.
so

\[(Q_m)^T = \left[(M^T M)^{-1} M^T (E_0^m - E_0^n C)\right](P_n)^T. \quad (4.3)\]

According to (3.13), (4.2)-(4.3), the control points of the degree-reduced surface \(Q_m(s, t)\) are

\[(Q_m)^T = [C + \left(M^T M)^{-1} M^T (E_0^m - E_0^n C)\right)](P_n)^T. \]

Its approximating error in matrix form is

\[d_l = \sqrt{(P_n (E_0^m)^T E_0^m (P_n)^T + \epsilon_1)/2. \quad (4.4)\]

### 4.2 Degree reduction with boundary constraints

As the same in Section 4.1, we also obtain the equation (4.1). When \(s_3 > 0, \) the \(1, \ldots, r_1 + 1, \ldots, n - s_1 + 1, \ldots, n + r_2 + 1, 1, \ldots, n - s_1 + 1, \ldots, n + r_2 + 1\) elements of vector \(Q_m\) are all zero. When \(s_3 = 0, \) the \(1, \ldots, r_1 + 1, \ldots, n + r_2 + 1, 2n - 1 - s_2, \ldots, 2n + 1 + r_3\) elements of vector \(Q_m\) are all zero. So we subtract the corresponding column of vector \(Q_m\) and matrix \(E_0^m, \) and denote the new vector and matrix as \(\tilde{Q}_m\) and \(\textbf{N}\) respectively. Then the unconstrained control points can be expressed as follows:

\[(\tilde{Q}_m)^T = (\textbf{N}^T \textbf{N})^{-1} \textbf{N}^T (E_0^m - E_0^n \textbf{H}) (P_n)^T. \]

Based on (3.13), the control points of degree-reduced surface are given by

\[Q_m = \tilde{Q}_m + \tilde{Q}_m. \]

The approximating error is similar to (4.4).

### §5 Algorithm II of degree reduction of triangular Bézier surfaces

Using the degree elevation property of Bézier triangle

\[
\tilde{p}_{i,j} = \frac{i}{j + 1} \tilde{p}_{i-1,j} + \frac{j}{j + 1} \tilde{p}_{i,j-1} + \frac{n + 1 - i - j}{n + 1} \tilde{p}_{i,j}, \quad (5.1)
\]

we can raise the degree of a triangular Bézier surface from \(m\) to \(n\) and write it as

\[Q_n^1(s, t) = \sum_{i+j \leq n} q_{i,j} B_{i,j}^n(s, t). \quad (5.2)\]

Denote \(Q_n^1 = (q_{i,j}), (i, j) \in \Omega^n.\)

Subtract the degree-reduced curves of the three boundary curves in (3.2)-(3.4) from surface \(P_n(s, t)\) in (2.1). Based on (5.1)-(5.2), we obtain a new surface

\[\tilde{P}_n(s, t) = \sum_{i+j \leq n} p_{i,j} B_{i,j}^m(s, t) - \sum_{i=0}^m q_{0,i} B_{0,i}^m(0, t) - \sum_{i=0}^m q_{i,0} B_{i,0}^m(s, 0) - \sum_{i=0}^m q_{i,m} B_{i,m}^m(s, 1 - s) - q_{0,0}(1 - t)^m + q_{0,m} s^m + q_{0,m} t^m \quad (5.3)\]

where

\[\tilde{P}_n = P_n - Q_n^1 = (\tilde{p}_{i,j}), (i, j) \in \Omega^n.\]

According to the degree elevation in (5.1), it is clear that the error functions of the three boundary curves and the control points of surface \(P_n(s, t)\) have the following relationship:

\[\tilde{p}_{n,0} = 0, \quad \tilde{p}_{0,n} = 0, \quad \tilde{p}_{0,0} = 0, \quad (5.4)\]

\[\varepsilon_1(s) = \sum_{i=0}^m \tilde{p}_{i,n-i} B_{i,n-i}^m(s), \quad \varepsilon_2(t) = \sum_{j=0}^m \tilde{p}_{0,j} B_{0,j}^m(t), \quad \varepsilon_3(s) = \sum_{i=0}^m \tilde{p}_{i,0} B_{i,0}^m(s). \]
The above equations illuminate that the three boundary curves of surface \( \tilde{\mathbf{P}}_n(s, t) \) are just the error functions of the boundary curves of surface \( \mathbf{P}_n(s, t) \). So we remove the three boundary curves of \( \tilde{\mathbf{P}}_n(s, t) \) although they create minor error in this progressing. This is done to give the surface
\[
\mathbf{P}_n(s, t) = \sum_{ij(n-i-j) \neq 0} \tilde{p}_{i,j} B_{n}^{i,j}(s, t)
\]
\[
= st(1-s-t) \sum_{ij(n-i-j) \neq 0} \tilde{p}_{i,j} \frac{n(n-1)(n-2)}{i(n-i-j)} B_{n-1,i,j-1}(s, t)
\]
\[
= st(1-s-t) \sum_{i+j \leq n-3} \mathbf{P}_{i,j} B_{n-3}^{i,j}(s, t)
\]
\[
\mathbf{P}_{i,j} = \tilde{p}_{i+j+1} \frac{n(n-1)(n-2)}{(i+1)(j+1)(n-2-i-j)}, \quad 0 \leq i+j \leq n-3, \quad (i, j) \in \Omega^{n-3} \tag{5.5}
\]
a common factor \( st(1-s-t) \) so that we can use the orthogonality of bivariate Jacobi basis function to derive the optimal degree-reduced surface of \( \mathbf{P}_n(s, t) \).

Write the surface \( \mathbf{P}_n(s, t) \) in matrix form, and then make basis transformation,
\[
\mathbf{P}_n(s, t) = \mathbf{B}_{n-3}(\mathbf{P}_{n-3})^T
\]
\[
= st(1-s-t)J_{n-3}^{5/2}E_{n-3}^{5/2}(\mathbf{P}_{n-3})^T
\]
where
\[
\mathbf{P}_{n-3} = (\mathbf{p}_{i,j}), \quad (i, j) \in \Omega^{n-3},
\]
\[
\tilde{\mathbf{P}}_{n-3} = \mathbf{E}_{n-3}^{5/2}(\mathbf{P}_{n-3})^T = (\mathbf{p}_{i,j}), \quad (i, j) \in \Lambda^{n-3} \tag{5.6}
\]
Based on property 1, we can see that the optimal \((n-m)\) degree-reduced surface of \( \mathbf{P}_n(s, t) \) is
\[
\mathbf{R}(s, t) = st(1-s-t)J_{m-3}^{5/2} \tilde{\mathbf{P}}^{II} = st(1-s-t)\mathbf{B}_{m-3}D_{m-3}^{5/2} \tilde{\mathbf{P}}^{II} = st(1-s-t)\mathbf{B}_{m-3} \mathbf{R},
\]
where
\[
\mathbf{R} = D_{m-3}^{5/2} \tilde{\mathbf{P}}^{II} = (\mathbf{r}_{i,j}), \quad (i, j) \in \Omega^{m-3},
\]
\[
\tilde{\mathbf{P}}_{n-3} = (\tilde{\mathbf{p}}_k)^{\frac{(n-i-j)(n-i-j-2)}{2}} \mathbf{J}^5_{m-3} = (\mathbf{J}^5_{m-3})^{\frac{(n-i-j)(n-i-j-2)}{2}} = \left( \mathbf{J}^5 \mathbf{J}^{II} \right)
\]
Column vector \( \tilde{\mathbf{P}}^{II} \) and row vector \( \mathbf{J}^{II} \) are formed by the last \((m-1)(m-2)/2\) elements of column vector \( \tilde{\mathbf{P}}_{n-3} \) and row vector \( \mathbf{J}^{5/2} \) respectively. Using the orthogonality of Jacobi basis function, the approximating error of \( \mathbf{P}_n(s, t) \) is
\[
\varepsilon_{II} = \left\| \mathbf{P}_n(s, t) - \mathbf{R}(s, t) \right\|_{L_2}
\]
\[
= \left( \int_0^1 \int_0^{1-t} s t^2 (1-s-t)^2 (\mathbf{J}^5 \tilde{\mathbf{P}}^{II})^2 ds dt \right)^{1/2}
\]
\[
= \sqrt{\left\| (\mathbf{J}^5) (\mathbf{P}^{II}) \right\|_{L_1}} / 5040. \tag{5.9}
\]
Then we transform the surface \( \mathbf{R}(s, t) \) in bivariate Bernstein basis,
\[
\mathbf{R}(s, t) = \mathbf{B}_{m-3} \sum_{i+j \leq m-3} \mathbf{r}_{i,j} B_{m-3}^{i,j}(s, t)
\]
\[
= \sum_{i+j \leq m-3} \mathbf{r}_{i,j} \frac{(n-i-j)(n-i-j-2)}{\min(m-1)(m-2)} B_{m-1,i,j+1}(s, t)
\]
\[
= \sum_{i+j \leq m-3} \mathbf{q}_{i,j} B_{m-1,i,j}(s, t) \tag{5.10}
\]
Fig. 5 (a) a triangular Bézier surface of degree 6; (b)-(c) optimal 2-degree-reduced surfaces with constraints (1) and (2) by algorithm I respectively; (d) 2-degree-reduced surfaces with constraint (2) by algorithm II; (e)-(g) the corresponding error functions of (b)-(d), respectively
where
\[ q_{i,j} = r_{i-1,j-1} \frac{ij(m-i-j)}{m(m-1)(m-2)}, \quad 2 \leq i+j \leq m-1 \quad \text{or} \quad ij(m-i-j) \neq 0. \] (5.11)

Finally, according to (3.8)-(3.11), (5.1)-(5.3), (5.5)-(5.8), (5.11), the degree-reduced surface
\[ Q_m(s,t) = \sum_{i+j \leq m} q_{i,j} B_{i,j}^m(s,t). \]

Based on (3.7), (5.4), (5.9)-(5.10), the error bound is given
\[ d_2 = \left\| P_n(s,t) - Q_m(s,t) \right\|_{L_2} = \left\| \tilde{P}_n(s,t) - R(s,t) \right\|_{L_2} \]
\[ = \left\| \sum_{i=0}^{n} \tilde{p}_{i,n} B_{i}^n(s) + \sum_{j=0}^{m} \tilde{p}_{0,j} B_{0,j}^m(s) + \sum_{i=0}^{n} \tilde{p}_{i,0} B_{i}^0(s) \right\|_{L_2} \]
\[ + \sum_{i,j(n-i-j) \neq 0} \tilde{p}_{i,j} B_{i,j}^n(s,t) - R(s,t) \right\|_{L_2} \]
\[ = \| \epsilon_1(s) + \epsilon_2(t) + \epsilon_3(s) + \tilde{P}_n(s,t) - R(s,t) \|_{L_2} \]
\[ \leq \epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_{II}. \] (5.12)

§6 Examples and Analysis

Example 1. Given a triangular Bézier surface of degree 6, its control points in helix order are
\{(6, 0, 2), (5.2, 1, 3), (4.5, 2, 5), (4, 3, 6), (3.5, 3.5, 4), (3, 4.2, 2), (2, 5, 1), (2, 4, 2), (1.5, 3.5, 3), (1, 2, 3), (0.8, 1.5, 1.5), (0.4, 0.8, 0.6), (0, 0, 0), (0.4, 0.4, 1), (1.5, 0.5, 2), (3, 0.2, 4), (4, 0.5, 3.5), (5, 0, 3), (4.5, 1, 3), (4, 2.2, 4), (2.5, 2.5, 5), (2.5, 3, 5), (1.5, 2.8, 4), (2, 1.2, 2), (1.2, 1, 2), (2, 0.9, 3), (3, 0.6, 4), (3, 2, 3) \}

We consider two constraints, that is,
(1) preserving $C^1$ continuities at three corners.
(2) preserving boundary constraints and the continuities at the endpoints of the boundary curves are all 1.

First, using algorithm I, we present the 2-degree-reduced surface with constraints (1)-(2) respectively. Then, using algorithm II, we present the 2-degree-reduced surface with constraint (2) (see Fig. 5). The approximating errors of Fig. 5(b)-(c) derived by (4.4) are 0.0399, 0.0482 respectively. The error bound of Fig. 5(d) is 0.2125, which is obtained by (5.12). The absolute error of Figure 5(d) is 0.0498, which is obtained by numerical method.

The errors of Fig. 5(c) are greater than those of Fig. 5(b). This is because the former has more constraints than the latter.

Compare Fig. 5(c) with 5(d), we can find that with boundary constraints, the best error 0.0482 is near to the absolute error 0.0498. It suggests that algorithm II has high precision and avoids inverse matrix. And algorithm II has many advantages such as boundary constraints, multi-degree reduction at one time, explicit expression, error forecast, time saving and high precision and so on. It has practical value in CAD design.

References

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