Smales’ $\alpha$-theory for inexact Newton methods under the $\gamma$-condition

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\textbf{Article history:}
Received 13 September 2009
Available online 1 February 2010
Submitted by T. Ransford

\textbf{Keywords:}
Nonlinear equation
Inexact Newton method
$\gamma$-condition
Approximate zero

\textbf{A B S T R A C T}
The present paper is concerned with the convergence problem of inexact Newton methods. Assuming that the nonlinear operator satisfies the $\gamma$-condition, a convergence criterion for inexact Newton methods is established which includes Smale’s type convergence criterion. The concept of an approximate zero for inexact Newton methods is proposed in this paper and the criterion for judging an initial point being an approximate zero is established. Consequently, Smale’s $\alpha$-theory is generalized to inexact Newton methods. Furthermore, a numerical example is presented to illustrate the applicability of our main results.

\section{1. Introduction}
Let $X$ and $Y$ be (real or complex) Banach spaces, $\Omega \subseteq X$ be an open subset and let $f: \Omega \subseteq X \to Y$ be a nonlinear operator with the first and second continuous Fréchet derivatives denoted by $f'$ and $f''$, respectively. Finding solutions of the nonlinear operator equation

$$f(x) = 0$$

in Banach spaces is a very general subject which is widely used in both theoretical and applied areas of mathematics. The most practical method to find an approximation of a solution of (1.1) is Newton’s method which takes the following form:

$$x_{n+1} = x_n - f'(x_n)^{-1} f(x_n).$$

The convergence issue of Newton’s method has been studied extensively; see for example [8,9,12,13,25,26,29,31–34]. Usually these results can be distinguished into two classes: one is about local convergence that determines the convergence ball based on the information around the solution $x^*$ of (1.1) (cf. [32–34]), and the other is about semi-local convergence that provides the convergence criterion based on the information around the initial point $x_0$ (cf. [8,9,12,13,25,26,29,31]). Among the semi-local convergence results on Newton’s method, one of the famous results is the well-known Kantorovich’s theorem (cf. [13]) which guarantees convergence of Newton’s sequence to a solution under very mild conditions. Another important result is Smale’s $\alpha$-theory which was presented by Smale in his report written for the 20th International Conference of Mathematician (cf. [25]), where the concept of an approximate zero was proposed and the criteria to judge an initial point being an approximate zero were established for analytic functions, only depending on the information at the initial point.

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\textsuperscript{\ast} Supported in part by the National Natural Science Foundations of China (Grant No. 10731060) and Ministerio de Ciencia e Innovación, Grant MTM2009-110696-C02-01, Spain.
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0022-247X/$ – see front matter © 2010 Published by Elsevier Inc.
doi:10.1016/j.jmaa.2010.01.056
For recent progress on semi-local convergence of Newton’s method the reader is referred to [8,9,12,26,29–31]. In particular, by using the majorizing sequence, Wang and Han found the best α criterion in [30] improving the one due to Smale; and by introducing the notion of the ϴ-condition, they again discussed α criterion in [31] where Smale’s point estimate theory was generalized.

As expressed in (1.2), Newton’s method requires to exactly solve the following Newton equation at each step:

$$f'(x_n)s_n = -f(x_n). \quad (1.3)$$

This sometimes makes Newton’s method inefficient from the point of view of practical calculations especially when $f'(x_n)$ is large and dense. While using linear iterative methods to approximate the solution of (1.3) instead of solving it exactly can reduce some of the costs of Newton’s method which was studied extensively and applied in [1–4,7,10,16–21,27,35] (such a variant is the so-called inexact Newton method). In general, the inexact Newton method has the following general form:

**Algorithm 1.1.** For $n = 0$ and a given initial guess $x_0$ until convergence, do:

1. For the residual control $r_n$ and the iteration $x_n$, find the step $s_n$ satisfying

$$f'(x_n)s_n = -f(x_n) + r_n.$$

2. Set $x_{n+1} = x_n + s_n$.

3. Set $n = n + 1$ and turn to step 1.

Here $\{r_n\}$ is a sequence of elements in $Y$ (depending on $\{x_n\}$ in general).

As is well known, the convergence behavior of the inexact Newton method depends on the residual controls of $\{r_n\}$. Several authors (cf. [7,27]) have analyzed the local convergence behavior in some manner such that the stopping relative approximate zero are presented in the last section, where the concept of an approximate zero for the inexact Newton method was generalized.

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The above results concern mainly with the local convergence of the inexact Newton method. In the spirit of Kantorovich’s theorem, the semi-local convergence analysis of the inexact Newton method was studied recently; see [2,3,10] for example. As in the case of the local convergence analysis, different residual controls were used. For example, the residual controls $\|r_n\| \leq \eta_n \|f(x_n)\|$ were adopted in [3]; while in [10], Guo considered the residual controls $\|f'(x_0)^{-1}r_n\| \leq \eta_n \|f'(x_0)^{-1}f(x_n)\|$ and gave new convergence results under the Lipschitz continuity assumption on $f'$.

Motivated by the ideas of the inexact Newton-like method for the inverse eigenvalue problem (cf. [5]), the authors of the present paper presented in [15] the following residual controls:

$$\|P_n r_n\| \leq \eta_n \|P_n f(x_n)\|^{1+\kappa} \text{ for each } n = 0, 1, \ldots,$$

where $\{P_n\}$ is a sequence of invertible operators from $Y$ to $X$ and $0 \leq \kappa \leq 1$, and established the local convergence of order $1 + \kappa$ for the inexact methods. Moreover, by adopting the residual controls

$$\|f'(x_0)^{-1}r_n\| \leq \eta_n \|f'(x_0)^{-1}f(x_n)\|^{1+\kappa} \text{ for each } n = 0, 1, \ldots,$$  \hspace{1cm} (1.4)

and assuming the Lipschitz continuity on $f'(x_0)^{-1}f'$, Shen and Li presented a Kantorovich-type theorem in [23] for the inexact Newton method, which improves and extends some known results (cf. [10,22]).

However, to our knowledge, Smale’s $\alpha$-theory for the inexact Newton method has not been found to explore. The purpose of the present paper, by considering residual controls (1.4) with $\kappa = 1$, i.e.,

$$\|f'(x_0)^{-1}r_n\| \leq \eta_n \|f'(x_0)^{-1}f(x_n)\|^2 \text{ for each } n = 0, 1, \ldots,$$  \hspace{1cm} (1.5)

we try to use the $\gamma$-condition, which was introduced in [31] and has been extensively applied in [11,14,28,29], to study the convergence issue of the inexact Newton method. Under the assumption that $f$ satisfies the $\gamma$-condition at the initial point $x_0$, we establish Smale’s $\alpha$-theory for the inexact Newton method. In particular, in the special case when $\eta_n \equiv 0$, Algorithm 1.1 reduces to Newton’s method and our result extends the corresponding one in [29]. Applications to Smale’s approximate zero are presented in the last section, where the concept of an approximate zero for the inexact Newton method is extended and a criterion to judge an initial point being an approximate zero is provided. Furthermore, a numerical example is also presented to illustrate the applicability of our results.

2. Preliminaries

Let $X$ and $Y$ be Banach spaces. Throughout the whole paper, we use $B(x, r)$ to stand for the open ball in $X$ with center $x$ and radius $r > 0$. Let $\gamma$, $\lambda$, and $c$ be positive constants. We define an important majorizing function $\varphi$ which was used by
Wang in his work [29] on approximate zeros of Smale (cf. [25]):
\[
\varphi(t) = \lambda - t + \frac{c\gamma t^2}{1 - \gamma t} \quad \text{for each } 0 \leq t < \frac{1}{\gamma}.
\] (2.1)

Note that the derivative of \( \varphi \) is
\[
\varphi'(t) = \frac{c}{(1 - \gamma t)^2} - c - 1 \quad \text{for each } 0 \leq t < \frac{1}{\gamma}.
\]

Then the derivative \( \varphi' \) is strictly increasing on \([0, \frac{1}{\gamma})\) and has the values \( \varphi'(0) < 0 \) and \( \varphi'(\frac{1}{\gamma} - 0) = +\infty \). It follows that the equation \( \varphi'(t) = 0 \) has a unique positive solution in \((0, \frac{1}{\gamma})\). In the remainder, we denote the solution by \( r_\gamma \), that is,
\[
r_\gamma = \left( 1 - \sqrt{\frac{c}{c+1}} \right) \gamma.
\] (2.2)

We first list some known lemmas (cf. [29]), which are crucial for the convergence analysis of the inexact Newton method. Let \( \{t_n\} \) denote the sequence generated by Newton’s method with initial point \( t_0 = 0 \), which is defined by
\[
t_{n+1} = t_n - \frac{\varphi(t_n)}{\varphi'(t_n)} \quad \text{for each } n = 0, 1, \ldots.
\] (2.3)

**Lemma 2.1.** Let \( \varphi \) be defined by (2.1). If
\[
\lambda \gamma \leq 1 + 2c - 2\sqrt{c(c+1)},
\] (2.4)
then the function \( \varphi \) has two zeros
\[
\left\{ \begin{array}{c}
t^* \\
t^{**}
\end{array} \right\} = \frac{1 + \lambda \gamma \mp \sqrt{(1 + \lambda \gamma)^2 - 4(1+c)\lambda \gamma}}{2(1+c)\gamma}
\] (2.5)
satisfying
\[
\lambda < t^* \leq r_\gamma \leq t^{**}.
\]

**Lemma 2.2.** Let \( t^* \) be defined by (2.5) and \( \{t_n\} \) be Newton’s sequence generated by (2.3). Suppose that (2.4) holds. Then
\[
t_n < t_{n+1} < t^* \quad \text{for each } n = 0, 1, \ldots.
\]
Consequently, \( \{t_n\} \) converges increasingly to \( t^* \).

**Lemma 2.3.** Let \( t^* \) be defined by (2.5) and \( \{t_n\} \) be Newton’s sequence generated by (2.3). Suppose that (2.4) holds. Then for each \( n \geq 1 \), the following estimates hold:
\[
\frac{t^* - t_n}{t^* - t_{n-1}} \leq q^{n-1}, \quad \frac{t_{n+1} - t_n}{t_n - t_{n-1}} \leq q^{n-1} \quad \text{and} \quad \frac{\varphi(t_n)}{\varphi(t_{n-1})} \leq q^{n-1},
\]
where
\[
q = \frac{1 - \lambda \gamma - \sqrt{(1 + \lambda \gamma)^2 - 4(1+c)\lambda \gamma}}{1 - \lambda \gamma + \sqrt{(1 + \lambda \gamma)^2 - 4(1+c)\lambda \gamma}}.
\]

3. Convergence analysis

Recall that \( f : \Omega \subseteq X \rightarrow Y \) is an operator with the first and second continuous Fréchet derivatives denoted by \( f' \) and \( f'' \), respectively. Let \( x_0 \in \Omega \) be such that the inverse \( f'(x_0)^{-1} \) exists. Definition 3.1 about the \( \gamma \)-condition and the related Lemma 3.1 are taken from [31].

**Definition 3.1.** Let \( 0 < r \leq \frac{1}{\gamma} \) be such that \( B(x_0, r) \subseteq \Omega \). The function \( f \) is said to satisfy the \( \gamma \)-condition at \( x_0 \) on \( B(x_0, r) \) if
\[
\|f'(x_0)^{-1}f''(x)\| \leq \frac{2\gamma}{(1 - \gamma \|x - x_0\|)^3} \quad \text{for each } x \in B(x_0, r).
\] (3.1)
Lemma 3.1. Let $0 < r \leq \frac{1}{2}$ be such that $B(x_0, r) \subseteq \Omega$. Suppose that $f$ satisfies the γ-condition (3.1) at $x_0$ on $B(x_0, r)$. If $\|x - x_0\| \leq (1 - \frac{1}{\sqrt{2}}) \frac{1}{r}$, then $f'(x)$ is invertible and satisfies that

$$\|f'(x)^{-1} f'(x_0)\| \leq \left(2 - \frac{1}{(1 - \gamma \|x - x_0\|)^2}\right)^{-1}.$$

In the present paper, we adopt the residuals $\{r_n\}$ satisfying (1.5) and assume that $\eta = \sup_{n \geq 0} \eta_n < 1$. Thus, if $n \geq 0$ and $x_n$ is well defined, then

$$\|f'(x_0)^{-1} r_n\| \leq \eta_n \|f'(x_0)^{-1} f(x_n)\|^2 \leq \eta \|f'(x_0)^{-1} f(x_n)\|^2.$$  \hfill (3.2)

Let

$$\beta = \|f'(x_0)^{-1} f(x_0)\|.$$  \hfill (3.3)

Without loss of generality, we may assume throughout the whole paper that $x_0$ is not a zero of $f$. This means that $\beta > 0$. Write

$$\lambda = (1 + \sqrt{\gamma}) \beta \quad \text{and} \quad c = \frac{\sqrt{2} \eta (1 + \sqrt{\eta})}{\gamma (1 - \sqrt{\gamma})^2} + 1 + \sqrt{\eta}. \hfill (3.4)$$

Recall that $r_0$ and $t^*$ are respectively determined by (2.2) and (2.5), and that $\{r_n\}$ is Newton's sequence generated by (2.3) with $\lambda$ and $c$ given by (3.4). We now first verify the following key lemma.

Lemma 3.2. Let $\{x_n\}$ be a sequence generated by Algorithm 1.1. Suppose that $f$ satisfies the γ-condition (3.1) with $r = t^*$ and that

$$(1 + \sqrt{\eta}) \beta \gamma \leq 1 + 2c - 2\sqrt{c(c + 1)}. \hfill (3.5)$$

Then the following assertions hold.

(i) $t^* \leq (1 - \frac{1}{\sqrt{2}}) \frac{1}{r}$ or equivalently $\frac{1}{r - t^*} \leq \sqrt{2}$.

(ii) If $n \geq 1$ is such that $\|x_n - x_0\| \leq t_n$, then $\|f'(x_n)^{-1} f'(x_0)\| \leq -\psi'(t_n)^{-1}$.

(iii) Let $m = 1, 2, \ldots, n$. If

$$\sqrt{\eta} \|f'(x_0)^{-1} f(x_{n-1})\| \leq 1 \hfill (3.6)$$

and

$$\|x_n - x_{n-1}\| \leq t_n - t_{n-1} \hfill (3.7)$$

hold for each $1 \leq n \leq m$, then the following inequalities hold:

$$\|f'(x_0)^{-1} f(x_m)\| \leq \psi(t_m) \hfill (3.8)$$

$$\|f'(x_0)^{-1} f(x_m)\| \leq 1; \hfill (3.9)$$

$$\frac{\|f'(x_0)^{-1} f(x_m)\|}{\|f'(x_0)^{-1} f(x_{m-1})\|} \leq \frac{\psi(t_m)}{\psi(t_{m-1})}; \hfill (3.10)$$

$$\|x_{m+1} - x_m\| \leq \frac{t_m + 1 - t_m}{t_m - t_{m-1}} \|x_m - x_{m-1}\|. \hfill (3.11)$$

Proof. Recalling that $\lambda$ is defined by (3.4), the condition (3.5) is equivalent to (2.4). Thus, Lemmas 2.1 and 2.2 are applicable. Hence, $\{r_n\}$ is strictly increasing and the following estimate holds for each $n \geq 0$:

$$t_n < t^* \leq r_0. \hfill (3.12)$$

Noting by (3.4) that $c \geq 1$, one has $1 - \frac{1}{\sqrt{2}} \leq 1 - \frac{1}{\sqrt{2}}$. Thus assertion (i) follows from (2.2) and (3.12). To show assertion (ii), suppose that $\|x_n - x_0\| \leq t_n$. Then, by (3.12) and assertion (i),

$$\|x_n - x_0\| \leq t_n < t^* \leq \left(1 - \frac{1}{\sqrt{2}}\right) \frac{1}{\gamma}.$$
Below we estimate
and assertion (ii) is seen to hold. It remains to prove assertion (iii). For this end, let \( m = 1, 2, \ldots \) and suppose that (3.6) and (3.7) hold for each \( 1 \leq n \leq m \). Write
\[
x_{m-1}^{rs} = x_{m-1} + \tau s(x_m - x_{m-1}) \quad \text{for any } 0 \leq \tau, s \leq 1.
\]
Then, applying Algorithm 1.1, we have that
\[
\begin{align*}
\int (x_m) &= f(x_m) - f(x_{m-1}) - f'(x_{m-1})(x_m - x_{m-1}) + r_{m-1} \\
&= \int \int f''(x_{m-1}^rs) \tau \, ds \, d\tau \, (x_m - x_{m-1})^2 + r_{m-1}.
\end{align*}
\]
Hence
\[
\begin{align*}
\| f'(x_0)^{-1} f(x_m) \| &\leq \| f'(x_0)^{-1} \int \int f''(x_{m-1}^rs) \tau \, ds \, d\tau \, (x_m - x_{m-1})^2 \| + \| f'(x_0)^{-1} r_{m-1} \| \\
&= I_1 + I_2. 
\end{align*}
\]
We first estimate \( I_1 \). To do this, we note by (3.7) and (3.12) that
\[
\| x_{m-1}^{rs} - x_0 \| \leq \sum_{n=1}^{m-1} \| x_n - x_{n-1} \| + \tau s \| x_m - x_{m-1} \| \leq \tau s t_m + (1 - \tau s) t_{m-1} < t^*.
\]
In particular,\[
\| x_{m-1} - x_0 \| \leq t_{m-1} < t^* \quad \text{and} \quad \| x_m - x_0 \| \leq t_m < t^*. 
\]
Thus, by the \( \gamma \)-condition, we get
\[
\begin{align*}
I_1 &\leq \int \int \frac{2\gamma}{(1 - \gamma \| x_{m-1}^{rs} - x_0 \|^2)^3} \tau \, ds \, d\tau \| x_m - x_{m-1} \|^2 \\
&\leq \int \int \frac{2\gamma}{(1 - \gamma \| x_{m-1} - x_0 \| - \gamma \| x_m - x_{m-1} \|)^2} \tau \, ds \, d\tau \| x_m - x_{m-1} \|^2 \\
&= \frac{\gamma \| x_m - x_{m-1} \|^2}{(1 - \gamma \| x_{m-1} - x_0 \| - \gamma \| x_m - x_{m-1} \|)(1 - \gamma \| x_m - x_{m-1} \|^2)}.
\end{align*}
\]
Combining this with (3.7) (with \( n = m \)) and (3.15) implies that
\[
I_1 \leq \frac{\gamma (t_m - t_{m-1})^2}{(1 - \gamma t_m)(1 - \gamma t_{m-1})^2} \left( \frac{\| x_m - x_{m-1} \|}{t_m - t_{m-1}} \right)^2.
\]
Below we estimate \( I_2 \). Since
\[
\begin{align*}
f'(x_0)^{-1} f'(x_{m-1}) &= I + f'(x_0)^{-1} \left( f'(x_{m-1}) - f'(x_0) \right) \\
&= I + f'(x_0)^{-1} \int_0^1 f''(x_0 + \tau(x_{m-1} - x_0)) \, d\tau \, (x_{m-1} - x_0).
\end{align*}
\]
it follows from the $\gamma$-condition that

$$\|f'(x_0)^{-1} f'(x_{m-1})\| \leq 1 + \int_0^1 \frac{2\gamma}{(1 - \gamma t \|x_{m-1} - x_0\|)^2} \, dt \|x_{m-1} - x_0\| = \frac{1}{(1 - \gamma \|x_{m-1} - x_0\|)^2}. \quad (3.18)$$

Furthermore, by Algorithm 1.1, we have that

$$\|f'(x_0)^{-1} f'(x_{m-1})(x_m - x_{m-1})\| \geq \|f'(x_0)^{-1} f(x_{m-1})\| - \|f'(x_0)^{-1} r_{m-1}\|,$$

Therefore, thanks to (3.2) and (3.6) (with $n = m$),

$$\|f'(x_0)^{-1} f'(x_{m-1})(x_m - x_{m-1})\| \geq \|f'(x_0)^{-1} f(x_{m-1})\| - \eta \|f'(x_0)^{-1} f(x_{m-1})\|^2 \geq (1 - \sqrt{\eta}) \|f'(x_0)^{-1} f(x_{m-1})\|,$$

and so

$$\|f'(x_0)^{-1} f(x_{m-1})\| \leq \frac{\|f'(x_0)^{-1} f'(x_{m-1})\| \cdot \|x_m - x_{m-1}\|}{1 - \sqrt{\eta}} \leq \frac{\|x_m - x_{m-1}\|}{(1 - \sqrt{\eta})(1 - \gamma \|x_{m-1} - x_0\|)^2}, \quad (3.19)$$

where the last inequality holds because of (3.18). Thus by (3.2) and (3.15) together with (3.19),

$$I_2 \leq \eta \|f'(x_0)^{-1} f(x_{m-1})\|^2 \leq \frac{\eta \|x_m - x_{m-1}\|^2}{(1 - \sqrt{\eta})^2(1 - \gamma \|x_{m-1} - x_0\|)^4} \leq \frac{\eta (tm - t_{m-1})^2}{(1 - \sqrt{\eta})^2(1 - \gamma t_{m-1})^4} \left( \frac{\|x_m - x_{m-1}\|}{tm - t_{m-1}} \right)^2. \quad (3.20)$$

Noting that $tm-1 < tm < t^*$, (3.20) and assertion (i) together entail that

$$I_2 \leq \frac{\eta (tm - t_{m-1})^2}{(1 - \sqrt{\eta})^2(1 - \gamma t^*)(1 - \gamma t_{m})(1 - \gamma t_{m-1})^2} \left( \frac{\|x_m - x_{m-1}\|}{tm - t_{m-1}} \right)^2 \leq \frac{\sqrt{2}\eta (tm - t_{m-1})^2}{(1 - \sqrt{\eta})^2(1 - \gamma t_{m})(1 - \gamma t_{m-1})^2} \left( \frac{\|x_m - x_{m-1}\|}{tm - t_{m-1}} \right)^2. \quad (3.21)$$

Consequently, combining (3.14), (3.17) and (3.21) gives

$$\|f'(x_0)^{-1} f(x_m)\| \leq I_1 + I_2 \leq \left(1 + \frac{\sqrt{2}\eta}{\gamma(1 - \sqrt{\eta})^2} \right) \frac{\gamma (tm - t_{m-1})^2}{(1 - \gamma t_{m})(1 - \gamma t_{m-1})^2} \left( \frac{\|x_m - x_{m-1}\|}{tm - t_{m-1}} \right)^2.$$

Recall that $c = \frac{\sqrt{2}\eta}{\gamma(1 - \sqrt{\eta})^2} + 1 + \sqrt{\eta}$. This yields that

$$(1 + \sqrt{\eta}) \|f'(x_0)^{-1} f(x_m)\| \leq \frac{c\gamma (tm - t_{m-1})^2}{(1 - \gamma t_{m})(1 - \gamma t_{m-1})^2} \left( \frac{\|x_m - x_{m-1}\|}{tm - t_{m-1}} \right)^2. \quad (3.22)$$

Since, by (2.1) and (2.3),

$$\frac{c\gamma (tm - t_{m-1})^2}{(1 - \gamma t_{m})(1 - \gamma t_{m-1})^2} = \varphi(t_m) - \varphi(t_{m-1}) - \varphi'(t_{m-1})(t_m - t_{m-1}) = \varphi(t_m),$$

one has by (3.22) that

$$I_2 = I_1 + I_2 \leq \varphi(t_m) \left( \frac{\|x_m - x_{m-1}\|}{tm - t_{m-1}} \right)^2 \leq \varphi(t_{m-1}) \left( \frac{\|x_m - x_{m-1}\|}{tm - t_{m-1}} \right)^2 \leq \varphi(t_0) = \lambda.$$
This implies that
\[ \sqrt{\eta} \left\| f'(x_0)^{-1} f(x_m) \right\| \leq \frac{\sqrt{\eta}}{1 + \sqrt{\eta}} \lambda = \sqrt{\eta} \beta = \sqrt{\eta} \left\| f'(x_0)^{-1} f(x_0) \right\| \]

thanks to the definitions of \( \lambda \) and \( \beta \). Hence (3.9) holds by (3.6) (with \( n = 1 \)). Thus to complete the proof of the lemma, we have to show that (3.10) and (3.11) hold. For this end, we note by Algorithm 1.1 that

\[ x_{n+1} - x_n = f'(x_0)^{-1} f'(x_0) \left( - f'(x_0)^{-1} f(x_n) + f'(x_0)^{-1} r_n \right). \]

Since \( \sqrt{\eta} \| f'(x_0)^{-1} f(x_m) \| \leq 1 \) by (3.6) (with \( n = m \)), it follows from (3.2) that

\[ \left\| f'(x_0)^{-1} r_{m-1} \right\| \leq \eta \left\| f'(x_0)^{-1} f(x_{m-1}) \right\|^2 \leq \sqrt{\eta} \left\| f'(x_0)^{-1} f(x_{m-1}) \right\|. \]

Consequently we have

\[ \| x_m - x_{m-1} \| \leq \left\| f'(x_{m-1})^{-1} f'(x_0) \right\| \left( \left\| f'(x_{m-1})^{-1} f(x_{m-1}) \right\| + \left\| f'(x_0)^{-1} r_{m-1} \right\| \right) \]

\[ \leq (1 + \sqrt{\eta}) \left\| f'(x_{m-1})^{-1} f'(x_0) \right\| \cdot \left\| f'(x_0)^{-1} f(x_{m-1}) \right\|. \quad (3.24) \]

Similarly, we also have that (noting that \( \sqrt{\eta} \| f'(x_0)^{-1} f(x_m) \| \leq 1 \) by just proved (3.9))

\[ \| x_{m+1} - x_m \| \leq (1 + \sqrt{\eta}) \left\| f'(x_m)^{-1} f'(x_0) \right\| \cdot \left\| f'(x_0)^{-1} f(x_m) \right\|. \quad (3.25) \]

By (3.23) and (3.24), we get that

\[ \frac{\| f'(x_0)^{-1} f(x_m) \|}{\| f'(x_0)^{-1} f(x_m) \|} = \frac{(1 + \sqrt{\eta}) \left\| f'(x_0)^{-1} f(x_m) \right\|}{(1 + \sqrt{\eta}) \left\| f'(x_0)^{-1} f(x_m) \right\|} \]

\[ \leq \varphi(t_m) \left( \frac{\| x_m - x_{m-1} \|}{t_m - t_{m-1}} \right)^2 \]

\[ \leq \varphi(t_m) \left\| f'(x_{m-1})^{-1} f'(x_0) \right\| \left\| x_m - x_{m-1} \right\| \frac{t_{m-1} - t_{m}}{(t_m - t_{m-1})} \]

\[ \leq \frac{\varphi(t_m)}{t_m - t_{m-1}} \left\| f'(x_{m-1})^{-1} f'(x_0) \right\|. \quad (3.26) \]

where (3.7) (with \( n = m \)) has been used for the last inequality. Similarly, by (3.23) and (3.25) together with (3.7) (with \( n = m \)), one can verify the following assertion:

\[ \| x_{m+1} - x_m \| \leq \varphi(t_m) \left\| f'(x_m)^{-1} f'(x_0) \right\| \left\| x_m - x_{m-1} \right\| \frac{t_{m-1} - t_{m}}{t_m - t_{m-1}} . \quad (3.27) \]

Furthermore, thanks to (3.15), assertion (ii) entails that

\[ \left\| f'(x_{m-1})^{-1} f'(x_0) \right\| \leq -\varphi(t_{m-1})^{-1} \text{ and } \left\| f'(x_m)^{-1} f'(x_0) \right\| \leq -\varphi'(t_m)^{-1} . \]

This together with (3.26) and (3.27) implies that

\[ \frac{\| f'(x_0)^{-1} f(x_m) \|}{\| f'(x_0)^{-1} f(x_m) \|} \leq \frac{\varphi(t_m)}{-\varphi(t_m) \left( t_m - t_{m-1} \right)} \]

\[ \text{and } \quad \frac{\| x_{m+1} - x_m \|}{t_{m-1} - t_{m}} \leq \frac{\varphi(t_m)}{t_{m-1} - t_m} \left\| x_m - x_{m-1} \right\|. \]

That is, (3.10) and (3.11) hold and the proof is complete. \( \square \)

Now we are ready to prove the main theorem of the present paper. Recall that \( \lambda \) and \( c \) are given by (3.4).

**Theorem 3.1.** Suppose that \( f \) satisfies the \( \gamma \)-condition (3.1) with \( r = t^* \). Suppose also that

\[ \beta \leq \min \left\{ \frac{1}{\sqrt{\eta}}, \frac{1 + 2c - 2\sqrt{c(c + 1)}}{\gamma(1 + \sqrt{\eta})} \right\}. \quad (3.28) \]

Let \( \{ x_n \} \) be a sequence generated by Algorithm 1.1. Then \( \{ x_n \} \) converges to a solution \( x^* \) of (1.1) and the following assertions hold for each \( n \geq 1 \):
\[
\|x_n - x^*\| \leq q^{n-1} \|x_{n-1} - x^*\|; \\
\|x_{n+1} - x_0\| \leq q^{n-1} \|x_n - x_{n-1}\|; \\
\|f'(x_0)^{-1} f(x_0)\| \leq q^{n-1} \|f'(x_0)^{-1} f(x_{n-1})\|,
\]
where
\[
q = \frac{1 - \lambda \gamma - \sqrt{(1 + \lambda \gamma)^2 - 4(1 + c)\lambda \gamma}}{1 - \lambda \gamma + \sqrt{(1 + \lambda \gamma)^2 - 4(1 + c)\lambda \gamma}}.
\]

**Proof.** Clearly the condition (3.28) implies the condition (3.5) which is equivalent to (2.4). We first verify that (3.6) and (3.7) hold for each \( n \geq 1 \). We will proceed by mathematical induction. Note by (3.3) and (3.28), (3.6) is clear for \( n = 1 \). To show (3.7) holds for \( n = 1 \), we have by Algorithm 1.1 that
\[
x_1 - x_0 = -f'(x_0)^{-1} f(x_0) + f'(x_0)^{-1} r_0.
\]
Using (3.2)–(3.3), one has that
\[
\|x_1 - x_0\| \leq \beta + \eta \beta^2.
\]
This together with (3.28) and (3.4) gives that
\[
\|x_1 - x_0\| \leq \beta + \sqrt{\eta} \beta = \lambda = t_1 - t_0,
\]
that is, (3.7) holds for \( n = 1 \). Assume now that (3.6) and (3.7) hold for all \( n \leq m \). Then, Lemma 3.2(iii) is applicable to concluding that
\[
\sqrt{\eta} \|f'(x_0)^{-1} f(x_m)\| \leq 1
\]
and
\[
\|x_{m+1} - x_m\| \leq \frac{t_{m+1} - t_m}{t_m - t_{m-1}} \|x_m - x_{m-1}\| \leq t_{m+1} - t_m.
\]
Hence (3.6) and (3.7) hold for \( n = m + 1 \) and so for each \( n \geq 1 \). Consequently, for any \( n \geq 0 \) and \( k \geq 0 \),
\[
\|x_{k+n} - x_n\| \leq \sum_{i=1}^{k} \|x_{i+n} - x_{i+n-1}\| \leq \sum_{i=1}^{k} (t_{i+n} - t_{i+n-1}) = t_{k+n} - t_n.
\]
Since (2.4) is satisfied as noted earlier, one sees that \( \{t_n\} \) is convergent. This together with (3.32) means \( \{x_n\} \) is a Cauchy sequence and so converges to some \( x^* \). Then letting \( n = 0 \) in (3.32), we have that
\[
\|x_k - x_0\| \leq \sum_{i=1}^{k} \|x_i - x_{i-1}\| \leq t_k \quad \text{for each } k \geq 0;
\]
while taking \( k \to \infty \) in (3.32), we get
\[
\|x_n - x^*\| \leq t^* - t_n \quad \text{for each } n \geq 0.
\]
Moreover, since (3.5) is satisfied, Lemma 3.2 is applicable. In particular,
\[
(1 + \sqrt{\eta}) \|f'(x_0)^{-1} f(x_0)\| \leq \psi(t_n),
\]
\[
\frac{\|f'(x_0)^{-1} f(x_0)\|}{\|f'(x_0)^{-1} f(x_{n-1})\|} \leq \frac{\psi(t_n)}{\psi(t_{n-1})} \quad \text{and} \quad \|x_{n+1} - x_n\| \leq \frac{t_{n+1} - t_0}{t_n - t_{n-1}} \|x_n - x_{n-1}\|
\]
hold for each \( n \geq 1 \) (noting that (3.6) and (3.7) hold for each \( n \geq 1 \)). Letting \( n \to \infty \) in (3.35) shows that the limit \( x^* \) is a solution of (1.1); while applying Lemma 2.3 to (3.36) shows that (3.30) and (3.31) hold for each \( n \geq 1 \). Thus, to complete the proof, it remains to prove (3.29) for each \( n \geq 1 \). By Lemma 2.3, it suffices to prove that
\[
\|x_n - x^*\| \leq \frac{t^* - t_0}{t^* - t_{n-1}} \|x_{n-1} - x^*\| \quad \text{for each } n \geq 1.
\]
To do this, we let \( n = 1, 2, \ldots \) and write
\[
x_{n-1, \tau} = x_{n-1} + \tau(x^* - x_{n-1}) \quad \text{for each } 0 \leq \tau \leq 1.
\]
Then, by (3.33)–(3.34) and (3.12), one has for each $0 \leq \tau \leq 1$,
\[
\|x_{n-1}^\tau_{*,*} - x_0\| \leq \|x_{n-1} - x_0\| + \tau \|x^* - x_{n-1}\| \leq t_{n-1} + \tau (t^* - t_{n-1}) = \tau t^* + (1 - \tau)t_{n-1} < t^*.
\] (3.38)

Further, by Algorithm 1.1, we have
\[
x_n - x^* = f'(x_{n-1})^{-1}(f(x_n) - f(x_{n-1}) - f'(x_{n-1})(x^* - x_{n-1}) + r_{n-1})
\]
\[
= f'(x_{n-1})^{-1}f'(x_0)\left(\int_0^1 \int_0^1 f'(x_0)^{-1}f''(x_{n-1}^\tau_{*,*}) \tau ds d\tau (x^* - x_{n-1})^2 + f'(x_0)^{-1}r_{n-1}\right).
\]
Hence
\[
\|x_n - x^*\| \leq \|f'(x_{n-1})^{-1}f'(x_0)\| \left(\int_0^1 \int_0^1 \|f'(x_0)^{-1}f''(x_{n-1}^\tau_{*,*})\| ds d\tau \|x^* - x_{n-1}\|^2 + \|f'(x_0)^{-1}r_{n-1}\|\right). \tag{3.39}
\]

Thanks to (3.33) and using Lemma 3.2(ii), we conclude that
\[
\|f'(x_{n-1})^{-1}f'(x_0)\| \leq -\varphi'(t_{n-1}^{-1}). \tag{3.40}
\]

Moreover, by (3.38), $\|x_{n-1}^\tau_{*,*} - x_0\| \leq t_{n-1} + \tau s(t^* - t_{n-1})$ and the $\gamma$-condition is applicable. Hence we get
\[
\int_0^1 \int_0^1 \|f'(x_0)^{-1}f''(x_{n-1}^\tau_{*,*})\| ds d\tau \leq \int_0^1 \int_0^1 \frac{2\gamma}{(1 - \gamma \|x_{n-1}^\tau_{*,*} - x_0\|)^3} \tau ds d\tau
\]
\[
\leq \int_0^1 \int_0^1 \frac{2\gamma}{(1 - \gamma t_{n-1} - \gamma \tau s(t^* - t_{n-1}))^3} \tau ds d\tau
\]
\[
= \frac{\gamma}{(1 - \gamma t^*)(1 - \gamma t_{n-1})^2}. \tag{3.41}
\]

To estimate the term $\|f'(x_0)^{-1}r_{n-1}\|$, recall that $x_{n-1}^\tau_{*,*} = x^* + \tau (x_{n-1} - x^*)$ for each $0 \leq \tau \leq 1$. Then,
\[
f(x_{n-1}) = f(x_{n-1}) - f(x^*) - f'(x_0)(x_{n-1} - x^*) + f'(x_0)(x_{n-1} - x^*)
\]
\[
= \int_0^1 \int_0^1 f'(x_0 + s(x_{n-1}^1 - x_0) - x_0) (x_{n-1}^1 - x_0) ds d\tau (x_{n-1}^1 - x_0) + f'(x_0)(x_{n-1} - x^*)
\]
hence,
\[
\|f'(x_0)^{-1}f(x_{n-1})\| \leq \left(\int_0^1 \int_0^1 \|f'(x_0)^{-1}f''(x_0 + s(x_{n-1}^1 - x_0))\| \cdot \|x_{n-1}^1 - x_0\| ds d\tau + 1\right) \|x_{n-1} - x^*\|. \tag{3.42}
\]

Since by (3.38)
\[
\|x_{n-1}^\tau_{*,*} - x_0\| \leq (1 - \tau)t^* + \tau t_{n-1}
\]
and since by Lemma 2.2
\[
\|x_0 + s(x_{n-1}^1 - x_0) - x_0\| \leq \|x_{n-1}^1 - x_0\| \leq (1 - \tau)t^* + \tau t_{n-1} < t^*,
\]
it follows from the $\gamma$-condition that
\[
\int_0^1 \int_0^1 \|f'(x_0)^{-1}f''(x_0 + s(x_{n-1}^1 - x_0))\| \cdot \|x_{n-1}^1 - x_0\| ds d\tau
\]
\[
\leq \int_0^1 \int_0^1 \frac{2\gamma \|x_{n-1}^1 - x_0\|}{(1 - \gamma s\|x_{n-1}^1 - x_0\|)^3} ds d\tau
\]
we define $\{x_n\}$ be the sequence generated by Newton's method. Then $\{x_n\}$ converges to a solution $x^\ast$ of (1.1) and the assertions (3.29)–(3.31) hold for each $n \geq 1$ with $q$ defined by

$$q = \frac{1 - \beta \gamma - \sqrt{(1 + \beta \gamma)^2 - 8 \beta \gamma}}{1 - \beta \gamma + \sqrt{(1 + \beta \gamma)^2 - 8 \beta \gamma}}.$$ 

One typical and important class of examples satisfying the $\gamma$-conditions is the one of analytic functions. Following [25], we define

$$\gamma := \sup_{n \geq 2} \left\| f'(x_0)^{-1} \frac{f^{(n)}(x_0)}{n!} \right\|^{\frac{1}{n-1}}. \quad (3.45)$$

Then $0 \leq \gamma < +\infty$ (recalling that $f'(x_0)^{-1}$ exists as assumed at the beginning of this section). Then the following lemma is known in [29].
Lemma 3.3. Let $0 < r \leq \frac{1}{2}$ be such that $B(x_0, r) \subseteq \Omega$. Then the analytic operator $f$ satisfies the $\gamma$-condition (3.1) at $x_0$ on $B(x_0, r)$ with $\gamma$ defined by (3.45).

Thus the following is immediate.

Corollary 3.2. Suppose that $f$ is an analytic operator and that (3.28) holds. Let $\{x_n\}$ be a sequence generated by Algorithm 1.1. Then $\{x_n\}$ converges to a solution $x^*$ of (1.1) and assertions (3.29)–(3.31) hold for each $n \geq 1$.

4. Applications to approximate zeros

As in the previous section, throughout this section, we assume that the nonlinear operator $f : \Omega \subseteq X \to Y$ is of the second continuous Fréchet derivative. Let $x_0 \in \Omega$ be such that the inverse $f'(x_0)^{-1}$ exists.

To study the computational complexity of Newton’s method for nonlinear operators in Banach spaces, Smale introduced in [24] the notion of an approximate zero for Newton’s method. However, it was found that it did not describe completely the property of quadratic convergence of Newton’s method and was inconvenient for the application in the study of the computational complexity. Hence, Smale proposed in [25] two kinds of the modifications of the notion: the first kind (in sense of $\|x_n - x_{n-1}\|$) and the second kind (in sense of $\|x_n - x^*\|$) of an approximate zero, and used the criterion

$$\alpha := \beta \gamma$$

to judge $x_0$ is an approximate zero of $f$, where $\beta := \|f'(x_0)^{-1} f(x_0)\|$. The notion of an approximate zero in the sense of $f'(x_0)^{-1} f(x_n)$ was also defined and studied in [6]; while in [29], Wang introduced the following unified definition of an approximate zero for Newton’s method. Let $e(x_n)$ denote some measurement of the approximation degree between $x_n$ and $x^*$.

Definition 4.1. Let $x_0 \in \Omega$ be such that the sequence $\{x_n\}$ generated by Newton’s method is well defined and satisfies

$$e(x_n) \leq \left( \frac{1}{2} \right)^{n-1} e(x_{n-1}) \quad \text{for each } n \geq 1. \tag{4.1}$$

Then $x_0$ is called an approximate zero of $f$ in sense of $e(x_n)$.

Note that if $x_0$ is an approximate zero of $f$, then Newton’s sequence $\{x_n\}$ converges to a solution $x^*$ of $f$. We now extend the notion of the approximate zero to the inexact Newton method.

Definition 4.2. Let $x_0 \in \Omega$ be such that the sequence $\{x_n\}$ generated by the inexact Newton method is well defined and satisfies (4.1). Then $x_0$ is called an approximate zero of $f$ in sense of $e(x_n)$.

By Theorem 3.1, we have the following theorem which gives a criterion for the approximate zero related to the inexact Newton method. Recall that $c = \frac{\sqrt{2n(1+\eta)}}{\gamma(1+\sqrt{\eta})^2} + 1 + \sqrt{\eta}$.

Theorem 4.1. Suppose that $f$ satisfies the $\gamma$-condition (3.1) at $x_0$ with $r = t^*$. Suppose also that

$$\beta \leq \min \left\{ \frac{1}{\sqrt{\eta}}, \frac{4 + 9c - 3\sqrt{c(9c + 8)}}{4\gamma(1 + \sqrt{\eta})^2} \right\}. \tag{4.2}$$

Let $\{x_n\}$ be a sequence generated by Algorithm 1.1. Then $\{x_n\}$ converges to a solution $x^*$ of (1.1) and $x_0$ is an approximate zero of $f$ in sense of $\|x_n - x^*\|$, $\|x_{n+1} - x_n\|$ and $\|f'(x_0)^{-1} f(x_n)\|$.

Proof. Since $(18c + 17)^2 - 36(c + 1)(9c + 8) = 1 > 0$, we obtain

$$(3\sqrt{9c + 8} - 8\sqrt{c + 1})^2 - c = 8(18c + 17 - 6\sqrt{(c + 1)(9c + 8)}) > 0.$$  

Thus, it follows that

$$1 + 2c - 2\sqrt{c(c + 1)} = \frac{4 + 9c - 3\sqrt{c(9c + 8)}}{4} = \frac{c}{4} (3\sqrt{9c + 8} - 8\sqrt{c + 1} - \sqrt{c}) > 0.$$  

This and (4.2) imply (3.28) holds. Thus, Theorem 3.1 is applicable; and hence for each $n \geq 1$ assertions (3.29)–(3.31) hold. Consequently, to show $x_0$ is an approximate zero of $f$, it suffices to prove

$$q = \frac{1 - \lambda \gamma - \sqrt{(1 + \lambda \gamma)^2 - 4(1 + c)\lambda \gamma}}{1 - \lambda \gamma + \sqrt{(1 + \lambda \gamma)^2 - 4(1 + c)\lambda \gamma}} \leq \frac{1}{2}.$$
Corollary 4.1. Suppose that $f$ satisfies $\gamma$-condition (3.1) with $r = \frac{1 + \beta \gamma - \sqrt{(1 + \beta \gamma)^2 - 8 \beta \gamma}}{4 \gamma}$. Suppose that
\[ \beta \gamma \leq \frac{13 - 3 \sqrt{17}}{4}. \]

Let $\{x_n\}$ be the sequence generated by Algorithm 1.1. Then $\{x_n\}$ converges to a solution $x^*$ of (1.1) and $x_0$ is an approximate zero of $f$ in sense of $\|x_n - x^*\|$, $\|x_{n+1} - x_n\|$ and $\|f'(x_0)^{-1} f(x_0)\|$.

Thanks to Theorem 4.1 and Lemma 3.3, the following corollary is also immediate.

Corollary 4.2. Suppose that $f$ is an analytic operator and that (4.2) holds. Let $\{x_n\}$ be a sequence generated by Algorithm 1.1. Then $\{x_n\}$ converges to a solution $x^*$ of (1.1) and $x_0$ is an approximate zero of $f$ in sense of $\|x_n - x^*\|$, $\|x_{n+1} - x_n\|$ and $\|f'(x_0)^{-1} f(x_0)\|$.

We end this paper with an example to illustrate the applicability of our results.

Example 4.1. Let $X = Y = \mathbb{R}^2$. Let $X$ and $Y$ be endowed with the $l_1$-norm and $l_\infty$-norm respectively. Define the analytic function $f : X \to Y$ by
\[ f(x) := \begin{pmatrix} g(x) \\ h(x) \end{pmatrix} = \begin{pmatrix} 10se^t + 5st \\ 5s^2 + \sin s + 10t \end{pmatrix} \text{ for each } x = (s, t)^T \in X. \]

Let $x_0 = (u, v)^T \in X$. Then
\[ f'(x_0) = \begin{pmatrix} 10e^u + 5v \\ 10u + \cos u \end{pmatrix}, \]
and
\[ f'(x_0)^{-1} = \frac{1}{d} \begin{pmatrix} 10 & -(10ue^u + 5u) \\ -(10u + \cos u) & 10e^u + 5v \end{pmatrix}, \]

where
\[ d = \det \begin{pmatrix} 10e^u + 5v & 10ue^u + 5u \\ 10u + \cos u & 10 \end{pmatrix} = 100e^u + 50v - (10ue^u + 5u)(10u + \cos u). \]

Thus
\[ f'(x_0)^{-1} f(x_0) = \frac{1}{d} \begin{pmatrix} 10g(x_0) - (10ue^u + 5u)h(x_0) \\ (10e^u + 5v)h(x_0) - (10u + \cos u)g(x_0) \end{pmatrix}. \]

Consequently,
\[ \beta := \frac{1}{|d|} \left( |10g(x_0) - (10ue^u + 5u)h(x_0)| + |(10e^u + 5v)h(x_0) - (10u + \cos u)g(x_0)| \right). \]
whether the convergence results are applicable depends upon the different choices of \( \eta \) in the case when \( \eta = 0,0.01,0.05,0.1,0.2,0.4,0.6,0.8,0.9 \) and use \( "T" \) and \( "F" \) represent that a criterion \( (3.28) \) or \( (4.2) \) holds and fails, respectively. Then the TF values of \( (3.28) \) and \( (4.2) \) corresponding to the above different points \( x_0 = (0.002,0.002)^T \), \( (0.002,-0.002)^T \), \( (0.01,0.035)^T \), \( (0.05,0.05)^T \), for which the estimates of the values of \( \gamma \) and \( \beta \) are given in Table 1.
Table 2
TF values of (3.28) for different $x_0$ and $\eta$.

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<th>$x_0$</th>
<th>$\eta = 0$</th>
<th>$\eta = 0.01$</th>
<th>$\eta = 0.05$</th>
<th>$\eta = 0.1$</th>
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<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
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<tr>
<td>(0.002, -0.002)$^T$</td>
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<td>T</td>
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<td>T</td>
<td>F</td>
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<tr>
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<td>T</td>
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<tr>
<td>(0.05, 0.05)$^T$</td>
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<td>F</td>
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</table>

Table 3
TF values of (4.2) for different $x_0$ and $\eta$.

<table>
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References