On increasing subsequences of minimal Erdős-Szekeres permutations

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Abstract Let \( \pi \) be minimal Erdős-Szekeres permutation of \( 1, 2, \ldots, n^2 \) and let \( l_{n,k} \) be the length of the longest increasing subsequence in the segment \( (\pi(1), \ldots, \pi(k)) \). Under uniform measure we establish an exponentially decaying bound of the upper tail probability for \( l_{n,k} \), and as a consequence we obtain a complete convergence, which is an improvement of Romik’s recent result. We also give a precise lower exponential tail for \( l_{n,k} \).

Keywords Large deviation; Minimal Erdős-Szekeres permutation; Robinson-Schensted-Knuth correspondence; Young tableaux

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1 Introduction and Main Results

In this paper, we consider a class of permutations of \( 1, 2, \ldots, n^2 \) which have a certain minimality property with respect to the length of their monotone subsequences. The celebrated Erdős-Szekeres theorem implies that a permutation \( \pi = (\pi(1), \pi(2), \ldots, \pi(n^2)) \) must contain a monotone (either increasing or decreasing) subsequence \( \pi(i_1), \pi(i_2), \ldots, \pi(i_n) \), \( i_1 < i_2 < \cdots < i_n \). Our interest of study is in those permutations whose longest increasing and longest decreasing subsequence have length exactly \( n \). Following Pittel and Romik [1], such permutations is called minimal Erdős-Szekeres permutations.

Denote by \( \mathcal{E}_n \) the set of all minimal Erdős-Szekeres permutations of \( 1, 2, \cdots, n^2 \). A remarkable result is

\[
|\mathcal{E}_n| = \left( \frac{(n^2)!}{1^2 \cdot 2^2 \cdot 3^2 \cdots n^2 \cdot (n + 1)^{n-1} \cdot (n + 2)^n \cdots (2n - 1)} \right)^2.
\]  

(1.1)

This can be seen from the well-known Robinson-Schensted-Knuth (RSK) correspondence and the hook formula (cf. Romik [2]).

Now equip a uniform probability measure \( P_n \) on \( \mathcal{E}_n \). It seems interesting to study the behavior of the typical minimal Erdős-Szekeres permutations. In contrast with the ordinary random permutations, there is no systematic study around Erdős-Szekeres permutations in the literature. An initial step in this direction was taken in [1, 2]. As an application of their limit shape result, Pittel and Romik [1] proved a law of large numbers for the length of the

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Theorem 2
Under the above notations we have
the length of the longest increasing subsequence in the segment π(1), · · · , π(k). Then for all η > 0,
\[ P_n \left( \pi \in \mathcal{E}_n : \left| \frac{l_{n,k}}{n} - 2\sqrt{\alpha(1-\alpha)} \right| > \eta \right) \to 0, \quad n \to \infty. \] (1.2)

See [1] for a slightly stronger statement including rate of convergence estimate.

The aim of this paper is to strengthen (1.2) to a complete convergence by giving an exponentially decaying bound for the upper tail probability. Our main result reads as follows.

**Theorem 1** Assume α < 1/2. Let \( \tau_n = n^{-2/3+\varepsilon} \) for some 0 < \( \varepsilon < 2/3 \). Then for any \( \theta > 0 \) and all sufficiently large \( n \geq n_0 \) (independent of \( \theta \)),
\[ P_n \left( \pi \in \mathcal{E}_n : l_{n,k} - 2\sqrt{\alpha_k(1-\alpha_k)} n \geq \theta n \sqrt{\tau_n} \right) \leq 2 \exp \left( -\frac{\theta^{3/2}}{n^{\varepsilon/6}} \right). \] (1.3)

We also establish a precise lower exponential tail for \( l_{n,k} \). It is expected that \( l_{n,k} \) exhibit quite different behaviors in their upper and lower tail. To state the result, we introduce some additional notations. For 0 < \( \alpha < 1 \), let \( \mathcal{F}_\alpha \) be the class of weakly decreasing functions \( f : [0,1] \to [0,1] \) such that \( \int_0^1 f(x) dx = \alpha \). Set
\[ I(f) = \int_0^1 \int_0^1 \log |f(x) - y + f^{-1}(y) - x| dx dy. \]

Let, for 0 < \( c < 2\sqrt{\alpha(1-\alpha)} \)
\[ H_\alpha(c) = \inf \left\{ I(f) : f \in \mathcal{F}_\alpha, f(0) = 2\sqrt{\alpha(1-\alpha)} - c \right\} + H(\alpha) + C, \]
where \( H(\alpha) = -\alpha \log \alpha - (1-\alpha) \log(1-\alpha), C = \frac{3}{2} - 2 \log 2 \).

**Theorem 2** Under the above notations we have
\[ \lim_{n \to \infty} \frac{1}{n^2} \log P_n \left( \pi \in \mathcal{E}_n : l_{n,k} \leq \left( 2\sqrt{\alpha(1-\alpha)} - c \right) n \right) = -H_\alpha(c). \] (1.4)

As a consequence of Theorems 1 and 2, we immediately have

**Corollary** Assume 0 < \( \alpha < 1/2 \). Then for all \( \eta > 0 \),
\[ \sum_{n=1}^{\infty} P_n \left( \pi \in \mathcal{E}_n : \left| \frac{l_{n,k}}{n} - 2\sqrt{\alpha(1-\alpha)} \right| > \eta \right) < \infty. \] (1.5)

The proofs of Theorems 1 and 2 will be given in the next section. The basic idea is to use the Schensted algorithm [3] to express the probability distribution of \( l_{n,k} \) in terms of random square Young tableaux. From the RSK correspondence it follows that to each \( \pi \in \mathcal{E}_n \) there correspond a pair \( (T, T') \) of square \( n \times n \) tableaux. And moreover, if we let \( \lambda_T^k \) be the sub-diagram of the square comprised of those boxes where the value of the entry \( T' \) is \( \leq k \), then by the Schensted algorithm the length \( l_{n,k} \) is equal to the length \( \lambda_T^k(1) \) of the first row of \( \lambda_T^k \). Let \( T_n \) be the set of \( n \times n \) square Young tableaux, and let \( \bar{P}_n \) be the uniform probability measure on \( T_n \). So the distribution of \( l_{n,k} \) under \((\mathcal{E}_n, P_n)\) is equal to the distribution of \( \lambda_T^k(1) \) under \((T_n, \bar{P}_n)\), namely,
\[ l_{n,k} \overset{d}{=} \lambda_T^k(1). \] (1.6)
Based on such an elegant identity, we can, and do in the sequel, reformulate our main results (1.3) and (1.4) in terms of $\lambda_T^k(1)$. Throughout Section 2, we will carry out our calculation only with respect to probability $\bar{P}_n$ and expectation $\bar{E}_n$.

The technique used in the proof of Theorem 1 is to estimate the moment generating function $\exp(x\lambda_T^k(1))$ using the conditioning devices. This technique was already used in Kim [4] on increasing subsequences of ordinary random permutations. See also Pittel and Romik [1] for the upper bound of the expected value of $\lambda_T^k(1)$. Interestingly, we need a nice control for the lower bound of $\lambda_T^k(1)$ in the computation of upper bound.

The proof of Theorem 2 is a direct application of Lemma 1 of Pittel and Romik [1], which estimates the probability that the sub-diagram $\lambda_T^k(1)$ has a given shape.

2 Proofs

Let us start with the proof of Theorem 1. To this end, we need a lemma to estimate the moment generating function of $\lambda_T^k(1)$.

Let $m = \lceil \tau_n n^2 \rceil$ where $\tau_n$ is as in Theorem 1 and $\lceil \cdot \rceil$ stands for the integer part, and let $\delta = n^{-\frac{1}{2} (1 - \varepsilon)}$. Define

$$a_{n,j} = \begin{cases} \frac{2 \sinh \frac{x}{n}}{n^{1/2}} \frac{1}{(1 - \frac{j}{n})} & 1 \leq j \leq m, \\ \frac{2 \sinh \frac{x}{n}}{n^{1/2}} \left(1 - \left(2 \sqrt{\frac{j}{\pi n}} - \frac{1}{\pi} \right)^2 \right)^{1/2} & m + 1 \leq j \leq k, \end{cases}$$

and for $\hat{c} > 0, \kappa > 1$

$$b_n = \exp(-\hat{c} n^\kappa).$$

Lemma 1 For $x > 0$ such that $\max_{j \geq 1} a_{n,j} < 1$, we have

$$\bar{E}_n e^{x\lambda_T^j(1)} \leq \begin{cases} \frac{1}{1 - a_{n,j}} \bar{E}_n e^{x\lambda_T^{j-1}(1)}, & 1 \leq j \leq m, \\ \frac{1 + b_n}{1 - a_{n,j}} \bar{E}_n e^{x\lambda_T^{j-1}(1)}, & m + 1 \leq j \leq n. \end{cases}$$

Proof Let, for $j \geq 1$

$$I_{n,j} = \lambda_T^j(1) - \lambda_T^{j-1}(1).$$

Trivially,

$$I_{n,j} = \begin{cases} 1, & \lambda_T^j(1) \text{ is obtained from } \lambda_T^{j-1}(1) \text{ by adding a box to the first row}, \\ 0, & \text{or else}. \end{cases}$$

Then using the total expectation formula, we have for any $x$,

$$\bar{E}_n e^{x\lambda_T^j(1)} = \bar{E}_n \left[ \bar{E}_n \left( e^{x\lambda_T^j(1)} | \lambda_T^{j-1} \right) \right]$$

$$= \bar{E}_n \left[ e^{x\lambda_T^{j-1}(1)} \bar{E}_n \left( e^{xI_{n,j}} | \lambda_T^{j-1} \right) \right].$$

(2.4)
Conditioning on $\lambda_T^{j-1}$, it obviously follows
\[
\bar{E}_n \left( e^{x|I_{n,j}|} \lambda_T^{j-1} \right) = 1 + (e^x - 1) \bar{P}_n \left( I_{n,j} = 1 | \lambda_T^{j-1} \right). \tag{2.5}
\]
Inserting (2.5) into (2.4) yields
\[
\bar{E}_n e^{x\lambda_T^{j-1}(1)} = \bar{E}_n e^{x\lambda_T^{j-1}(1)} + (e^x - 1) \bar{E}_n \left[ e^{x\lambda_T^{j-1}(1)} \bar{P}_n \left( I_{n,j} = 1 | \lambda_T^{j-1} \right) \right]. \tag{2.6}
\]
In turn, it easily follows via the Cauchy-Schwarz inequality
\[
\bar{E}_n \left[ e^{x\lambda_T^{j-1}(1)} \bar{P}_n \left( I_{n,j} = 1 | \lambda_T^{j-1} \right) \right] \leq \left( \bar{E}_n e^{x\lambda_T^{j-1}(1)} \right)^{1/2} \left( \bar{E}_n \bar{P}_n^2 (I_{n,j} = 1 | \lambda_T^{j-1}) \right)^{1/2}. \tag{2.7}
\]
Next we shall estimate the second expectation of the right hand side of (2.7) using a similar argument to Lemma 10 of Pittel and Romik [1]. Let $\mathcal{Y}_{n,j}$ be the set of Young diagrams of area $j$ contained in the $n \times n$ square. For a diagram $\lambda_0 \in \mathcal{Y}_{n,j-1}$, denote by $\lambda_0^*$ the diagram obtained from $\lambda_0$ by adding a box to the first row. For a Young diagram $\lambda$, denote by $d(\lambda)$ the number of Young tableaux of shape. Then we have
\[
\bar{P}_n^2 \left( I_{n,j} = 1 | \lambda_T^{j-1} = \lambda_0 \right) \bar{P}_n \left( \lambda_T^{j-1} = \lambda_0 \right) = \frac{d^2(\lambda_0) d^2(\square_n \setminus \lambda_0^*)}{d^2(\square_n)} \times \frac{d(\square_n \setminus \lambda_0)}{d(\square_n \setminus \lambda_0^*)},
\]
where $\square_n$ stands for the $n \times n$ square and $\square_n \setminus \lambda_0$ is thought of as an ordinary diagram from the opposite corner of the square.

Now note the amusing identity due to Pittel and Romik [1] (see (71) therein):
\[
d(\lambda_0) d(\square_n \setminus \lambda_0^*) = \frac{n^2 - \lambda_0^*(1)}{j(n^2 - j + 1)}.
\]
Then we have
\[
\bar{E}_n e^{x\lambda_T^{j-1}(1)} \bar{P}_n^2 \left( I_{n,j} = 1 | \lambda_T^{j-1} = \lambda_0 \right) = \sum_{\lambda_0 \in \mathcal{Y}_{n,j-1}} e^{x\lambda_0(1)} \bar{P}_n^2 \left( I_{n,j} = 1 | \lambda_T^{j-1} = \lambda_0 \right) \bar{P}_n \left( \lambda_T^{j-1} = \lambda_0 \right) = e^{-x} \sum_{\lambda_0 \in \mathcal{Y}_{n,j-1}} e^{x\lambda_0^*(1)} \frac{d(\lambda_0^*) d(\square_n \setminus \lambda_0^*)}{d(\square_n)} \times \frac{n^2 - \lambda_0^*(1)}{j(n^2 - j + 1)}.
\]
This is nearly an average over $\mathcal{Y}_{n,j}$; in fact, slightly less, since not any $\lambda \in \mathcal{Y}_{n,j}$ is of the form $\lambda_0^*$ for some $\lambda_0 \in \mathcal{Y}_{n,j-1}$. So we must have
\[
\bar{E}_n e^{x\lambda_T^{j-1}(1)} \bar{P}_n^2 \left( I_{n,j} = 1 | \lambda_T^{j-1} \right) \leq \frac{e^{-x}}{j(n^2 - j)} \bar{E}_n e^{x\lambda_T^{j}(1)} \left( n^2 - (\lambda_T^j(1) - 1)^2 \right). \tag{2.8}
\]
For $1 \leq j \leq m$, we use the trivial upper bound to get
\[
\bar{E}_n e^{x\lambda_T^{j-1}(1)} \bar{P}_n^2 \left( I_{n,j} = 1 | \lambda_T^{j-1} \right) \leq \frac{n^2}{j(n^2 - j)} e^{-x} \bar{E}_n e^{x\lambda_T^{j}(1)}. \tag{2.9}
\]
Combining (2.6), (2.7) and (2.9) yields
\[ E_n e^{\lambda^j I(1)} = E_n e^{\lambda^j I - 1(1)} + \frac{2 \sinh \frac{\theta}{2}}{n \sqrt{\frac{j}{n^2}(1 - \frac{j}{n^2})}} E_n e^{\lambda^j I(1)} \]  
(2.10)
from which we obtain (2.3) for 1 \leq j \leq m.

For j \geq m + 1, we adapt a non-trivial lower bound for \lambda^j I(1) given by Lemma 9 of Pittel and Romik [1]. Specifically speaking,  
\[ \mathbb{P}_n \left( T \in T_n : \min_{\tau_n \leq j \leq n^2/2} \left( \lambda^j I(1) - 2n \sqrt{\frac{j}{n^2}(1 - \frac{j}{n^2})} \right) \leq -\delta n \right) \leq \exp(-\hat{c} \kappa n), \]  
(2.11)
where \kappa > 1 and where and in the sequel \hat{c} > 0 is numerical constant, whose value may change from line to line.

For simplicity, we write \( A_n \) for the random event of the left hand side of (2.11). Then 
\[ E_n e^{\lambda^j I(1)} \left( n^2 - (\lambda^j I(1) - 1)^2 \right) \leq n^2 \left( 1 - \left( 2 \sqrt{\frac{j}{n^2}(1 - \frac{j}{n^2}) - \frac{1}{\kappa} \right) \right) E_n e^{\lambda^j I(1)} + E_n \left[ e^{\lambda^j I(1)} \left( n^2 - (\lambda^j I(1) - 1)^2 \right) I_{A_n} \right] \]
\[ \leq n^2 \left( 1 - \left( 2 \sqrt{\frac{j}{n^2}(1 - \frac{j}{n^2}) - \frac{1}{\kappa} \right) \right) E_n e^{\lambda^j I(1)} + \exp(-\hat{c} \kappa n), \]
from which it easily follows 
\[ \left( E_n e^{\lambda^j I(1)}(n^2 - (\lambda^j I(1) - 1)^2) \right)^{1/2} \leq n \left( 1 - \left( 2 \sqrt{\frac{j}{n^2}(1 - \frac{j}{n^2}) - \frac{1}{\kappa} \right) \right)^{1/2} \left( E_n e^{\lambda^j I(1)} \right)^{1/2} + \exp(-\hat{c} \kappa n). \]  
(2.12)

Analogously to (2.10), combining (2.6), (2.7) and (2.12) yields 
\[ E_n e^{\lambda^j I(1)} \leq E_n e^{\lambda^j I - 1(1)} + \exp(-\hat{c} \kappa n)E_n e^{\lambda^j I - 1(1)} \]
\[ + \frac{2 \sinh \frac{\theta}{2}}{n \sqrt{\frac{j}{n^2}(1 - \frac{j}{n^2})}} \left( 1 - \left( 2 \sqrt{\frac{j}{n^2}(1 - \frac{j}{n^2}) - \frac{1}{\kappa} \right) \right) \right)^{1/2} E_n e^{\lambda^j I(1)}. \]

Hence we have for j \geq m + 1 
\[ E_n e^{\lambda^j I(1)} \leq \frac{1 + b_n}{1 - a_{n,j}} E_n e^{\lambda^j I - 1(1)}. \]
This completes the proof of Lemma 1. \( \square \)

**Proof of Theorem 1.** Without loss of generality, we may and do assume 0 < \theta \leq n^{1/3-\varepsilon/2}, since \lambda^j I(1) \leq n. Let x = \theta^{1/2}/(n^{1/2}). Obviously, the hypothesis in Lemma 1 is satisfied for all sufficiently large n.
Repeatedly using (2.3), we have
\[
\bar{E}_n e^{x\lambda_n^k(1)} \leq \prod_{j=m+1}^{k} \frac{1 + b_n}{1 - a_{n,j}} \bar{E}_n e^{x\lambda_n^m(1)} \\
\leq \exp \left( k b_n - \sum_{j=1}^{k} \log(1 - a_{n,j}) \right). \tag{2.13}
\]
It remains to estimating the logarithmic sum in the exponent of the right hand side of (2.13). Note \( \max_{1 \leq j \leq k} a_{n,j} \leq 1/2 \) and use \( \log(1 - z) \geq -z - z^2, \quad 0 < z \leq 1/2 \). We have
\[
-\sum_{j=1}^{k} \log(1 - a_{n,j}) \leq \sum_{j=1}^{k} a_{n,j} + \sum_{j=1}^{k} a_{n,j}^2. \tag{2.14}
\]
By the Euler sum formula,
\[
\sum_{j=1}^{m} \frac{1}{\sqrt{j/n^2}(1 - \frac{j}{n^2})} = n^2 \int_{0}^{\tau_n} \frac{1}{\sqrt{t(1-t)}} dt + O(n) \\
= 2n^2 \arcsin \sqrt{\tau_n} + O(n).
\]
So, it follows from the definition of \( a_{n,j} \) over \( 1 \leq j \leq m \)
\[
\sum_{j=1}^{m} a_{n,j} = 2 \sinh \frac{x}{2} \left( 2n \arcsin \sqrt{\tau_n} + O(1) \right). 
\]
For the remaining sum of \( a_{n,j} \) over \( m+1 \leq j \leq k \), recall \( \alpha < 1/2 \) and
\[
a_{n,j} = \frac{2 \sinh \frac{x}{2}}{n \sqrt{\frac{1}{n^2}}(1 - \frac{j}{n^2})} \left( 1 - \left( 2 \sqrt{\frac{j}{n^2}(1 - \frac{j}{n^2})} - \delta - \frac{1}{n} \right)^2 \right)^{1/2}, \quad m+1 \leq j \leq k.
\]
Then using \((1 + x)^{1/2} \leq 1 + \frac{x}{2}\),
\[
a_{n,j} \leq \frac{2 \sinh \frac{x}{2}}{n \sqrt{\frac{1}{n^2}}(1 - \frac{j}{n^2})} \left( 1 - \left( \frac{2j}{n^2} \right)^2 + 4 \left( \delta + \frac{1}{n} \right) \sqrt{\frac{j}{n^2}(1 - \frac{j}{n^2})} \right)^{1/2} \\
\leq \frac{2 \sinh \frac{x}{2}}{n \sqrt{\frac{1}{n^2}}(1 - \frac{j}{n^2})} \left[ \frac{1 - \frac{2j}{n^2}}{n \sqrt{\frac{1}{n^2}}(1 - \frac{j}{n^2})} + \frac{2(\delta + \frac{1}{n})}{n(1 - \frac{j}{n^2})} \right].
\]
Again, by the Euler sum formula,
\[
\sum_{j=m+1}^{k} \frac{1 - \frac{2j}{n^2}}{\sqrt{\frac{1}{n^2}}(1 - \frac{j}{n^2})} = n^2 \int_{\tau_n}^{\alpha_k} \frac{1 - 2t}{\sqrt{t(1-t)}} dt + O \left( n^{2/3 - \epsilon/2} \right) \\
= 2n^2 \left[ \sqrt{\alpha_k(1 - \alpha_k)} - \sqrt{\tau_n(1 - \tau_n)} \right] + O \left( n^{2/3 - \epsilon/2} \right) \tag{2.15}
\]
and
\[
\sum_{j=m+1}^{k} \frac{1}{(1 - \frac{2j}{n^2})} = n^2 \int_{\tau_n}^{\alpha_k} \frac{1}{1 - 2t} dt + O(1) \\
= -\frac{n^2}{2} \log(1 - 2\alpha_k) + O(m). \tag{2.16}
\]
Combining (2.15) and (2.16) gives
\[
\sum_{j=m+1}^{k} a_{n,j} = 2 \sinh \frac{x}{2} \left[ 2n \left( \sqrt{\alpha_k (1 - \alpha_k)} - \sqrt{\tau_n (1 - \tau_n)} \right) - 2n\delta \log (1 - \alpha_k) + O(n^{\epsilon/3}) \right].
\]

By noting \( \tau_n = n^{-2/3+\epsilon} \to 0 \) and using a simple analysis, we have
\[
\sum_{j=1}^{k} a_{n,j} = \sum_{j=1}^{m} a_{n,j} + \sum_{j=m+1}^{k} a_{n,j}
= 2 \sinh \frac{x}{2} \left[ 2n \sqrt{\alpha_k (1 - \alpha_k)} - 2n\delta \log (1 - \alpha_k) + O(n^{3\epsilon/2}) \right].
\] (2.17)

Turn to the square sum. It easily follows
\[
\sum_{j=1}^{k} a_{n,j}^2 \leq \left( 2 \sinh \frac{x}{2} \right)^2 \sum_{j=1}^{k} \frac{1}{j(1 - \frac{\alpha}{n^2})} \leq 4 \left( 2 \sinh \frac{x}{2} \right)^2 \log n.
\] (2.18)

Substituting (2.17) and (2.18) into (2.13), and then into (2.14) yields a key control of moment generating function:
\[
\bar{E}_n e^{x \lambda^k_T(1)} \leq \exp \left[ n^2 b_n + 2 \sinh \frac{x}{2} \left( 2n \sqrt{\alpha_k (1 - \alpha_k)} - 2n\delta \log (1 - \alpha_k) + O(n^{3\epsilon/2}) \right) + 4 \left( 2 \sinh \frac{x}{2} \right)^2 \log n \right].
\]

Applying the Markov inequality, we have
\[
\bar{P}_n \left( \lambda^k_T(1) - 2n \sqrt{\alpha_k (1 - \alpha_k)} \geq \theta n \sqrt{\tau_n} \right)
\leq \exp \left( -x^2 n \sqrt{\alpha_k (1 - \alpha_k)} - x\theta n \sqrt{\tau_n} \right) \bar{E}_n e^{x \lambda^k_T(1)}.
\]

Since \( x \to 0 \) as \( n \to \infty \), \( 2 \sinh \frac{x}{2} = x + O(x^3) \). So, a simple algebra leads to
\[
\bar{P}_n \left( \lambda^k_T(1) - 2n \sqrt{\alpha_k (1 - \alpha_k)} \geq \theta n \sqrt{\tau_n} \right) \leq 2 \exp \left( -\frac{\theta^2}{n^2} + \frac{\theta}{n^{3/2}} \right)
\]
for all sufficiently large \( n \), as desired. \(\square\)

**Proof of Theorem 2.** Let us start with an estimate of the probability that the sub-diagram \( \lambda^k_T \) has a given shape. For any Young diagram \( \lambda = (\lambda(1), \lambda(2), \cdots, \lambda(n)) \) (some of them may be 0) whose graph lies within the \( n \times n \) square, define \( f_\lambda : [0, 1] \to [0, 1] \) by

\[
f_\lambda(x) = \frac{1}{n} \lambda([nx]).
\]

Then for any given diagram \( \lambda_0 \subseteq \square_n \) of size \( k \), we have
\[
\bar{P}_n \left( T \in T_n : \lambda^k_T = \lambda \right) = \frac{d(\lambda) d(\square_n \setminus \lambda)}{d(\square_n)}
= \exp \left( -\frac{1}{n^2} \right) \left( I(f_\lambda) + H(\alpha_k) + C \right),
\] (2.19)

where \( o(1) \) is uniform over all \( \lambda \) and all \( 1 \leq k \leq n^2 \).

(2.19) was proved by Pittel and Romik [1] through a careful computation with the help of the classic hook formula. Our proof will be a direct application of this asymptotic estimate. See Deuschel and Zeitouni [5] for similar arguments.
First, by virtue of (2.19), it follows
\[
\bar{P}_n \left( T \in \mathcal{T}_n : \lambda^k_\mathcal{T}(1) \leq \left( 2\sqrt{\alpha_k(1-\alpha_k) - c} \right) n \right) = \sum_{\lambda \in \mathcal{P}_k(\alpha_k, c)} \bar{P}_n \left( T \in \mathcal{T}_n : \lambda^k_\mathcal{T} = \lambda \right) = \sum_{\lambda \in \mathcal{P}_k(\alpha_k, c)} \exp \left( -(1 + o(1))n^2 (I(f_\lambda) + H(\alpha_k) + C) \right),
\]
(2.20)
where \( \mathcal{P}_k(\alpha_k, c) \) denotes the set of all partitions of \( k \) with the largest summand at most \( (2\sqrt{\alpha_k(1-\alpha_k) - c})n \).

Let \( p(k) \) be the number of partitions of \( k \). It is well known \( p(k) \leq \exp \left( \pi \sqrt{2k/3} \right) \). So, (2.20) is bounded by
\[
p(k) \exp \left( -(1 + o(1))n^2 \min_{\lambda \in \mathcal{P}_k(\alpha_k, c)} (I(f_\lambda) + H(\alpha_k) + C) \right) \leq p(k) \exp \left( -(1 + o(1))n^2 H(\alpha_k(c)) \right).
\]
Taking the logarithm and then letting \( n \to \infty \) gives
\[
\limsup_{n \to \infty} \frac{1}{n^2} \log \bar{P}_n \left( T \in \mathcal{T}_n : \lambda^k_\mathcal{T}(1) \leq \left( 2\sqrt{\alpha(1-\alpha) - c} \right) n \right) \leq -H(\alpha(c)).
\]
Let us turn to proving the lower bound:
\[
\liminf_{n \to \infty} \frac{1}{n^2} \log \bar{P}_n \left( T \in \mathcal{T}_n : \lambda^k_\mathcal{T}(1) \leq \left( 2\sqrt{\alpha(1-\alpha) - c} \right) n \right) \geq -H(\alpha(c)).
\]
(2.21)
Let \( f_0^c \in \mathcal{F}_\alpha \) be such that
\[
I(f_0^c) = \inf \left\{ I(f) : f \in \mathcal{F}_\alpha, f(0) = 2\sqrt{\alpha(1-\alpha) - c} \right\}.
\]
Assume it has the support \( b_0(c) \). We construct a particular Young diagram out of \( f_0^c \). For \( 1 \leq i \leq |b_0(c)| =: i_{\text{max}}, \) set \( j(i) = \lfloor f_0^c(i/n) \rfloor n \). Note that \( j(i) \) is a decreasing sequence, and
\[
n^2\alpha - cn \leq m_n = \sum_{i=1}^{i_{\text{max}}} j(i) \leq n^2\alpha.
\]
The sequence \( \{(i, j(i)), i = 1, \cdots, i_{\text{max}}\} \) defines a Young diagram \( \lambda_n \) of size \( m_n \). Applying (2.19) to the \( \lambda_n \) yields
\[
\bar{P}_n \left( T \in \mathcal{T}_n : \lambda^m_n(1) \leq \left( 2\sqrt{\alpha(1-\alpha) - c} \right)n \right) \geq \bar{P}_n \left( T \in \mathcal{T}_n : \lambda^m_n = \lambda_n \right) = \exp \left( -(1 + o(1))n^2 (I(f_{\lambda_n}) + H(\alpha) + C) \right).
\]
This immediately gives
\[
\liminf_{n \to \infty} \frac{1}{n^2} \log \bar{P}_n \left( T \in \mathcal{T}_n : \lambda^m_n(1) \leq \left( 2\sqrt{\alpha(1-\alpha) - c} \right)n \right) \geq -H(\alpha(c)).
\]
The lower bound (2.21) follows from the continuity of \( H(\alpha(c)) \) in \( c \). This concludes the proof. \( \square \)

A remarkable achievement in combinatorics probability in the end of the nineties is the work of Baik, Deift and Johansson [6] on the asymptotic distribution of the longest increasing subsequences of random uniform permutations. In virtue of the RSK correspondence, they equivalently obtained the asymptotic distribution for the first row of standard Young tableaux
under the Plancherel measure. The similar results hold for the first $m$ rows where $m$ is arbitrary fixed integer. Just recently did Bogachev and Su [7] show the central limit theorem in the bulk. It is natural to ask whether we could find a normalizing constant and limit distribution for $\lambda_k^T(m)$ as $n$ tends to infinity, where $m$ may be fixed or depend on $n$. See Pittel and Romik [1] for the limit shape and more details. Honestly, the present bounds for $\lambda_T^k(1)$ is only a bit effort toward this goal.

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References