Limiting subdifferentials of perturbed distance functions in Banach spaces

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\section{1. Introduction}

Let $X$ be a real Banach space endowed with norm $\| \cdot \|$ and $S$ be a nonempty closed subset of $X$. Let $J : S \to \mathbb{R}$ be a lower semicontinuous (lsc) function. Let $x \in X$ and consider the following perturbed optimization problem, which is denoted by

$$
\min_{w \in S} \{ \| x - w \| + J(w) \}.
$$

(1.1)

This kind of perturbed problem, which was first presented and investigated by Baranger in [1], has been studied extensively, and was applied to optimal control problems governed by partial differential equations; see for example [2–5]. Generic results on the existence and/or well posedness of perturbed optimization problems have been established in [6–13]. The perturbed optimization problem (1.1) is associated with the optimal value function, which is called here the perturbed distance function $d^J_S : X \to \mathbb{R}$, defined by

$$
d^J_S(x) := \inf_{s \in S} \{ \| x - s \| + J(s) \}
$$

(1.2)
In particular, in the case when $J \equiv 0$, the perturbed distance function $d^\varepsilon_\varepsilon(x)$ is reduced to the (classic) distance function defined by
\[ d_\varepsilon(x) := \inf_{s \in S} \|x - s\| \quad \text{for each } x \in X. \]

As is well known, the distance function plays an important role in optimization and variational analysis and has been studied extensively. In particular, its various subdifferentials, such as the Clarke subdifferentials, the proximal subdifferentials, the Fréchet subdifferentials as well as the limiting subdifferentials, have been explored in [14–22] and the references therein.

Among them, we pay special interest to the work on the limiting subdifferentials of the distance function developed by Mordukhovich and Nam in [22] (see also [21]), where the limiting subdifferentials of distance functions are estimated in terms of the corresponding limiting normal cones of the associated subsets. Extensions of these results to the setting of a minimal time function determined by a closed convex set and a closed set have been done recently, see for example [23,24] and the references therein. For the general perturbed distance function $d^\varepsilon_\varepsilon$, the proximal subdifferentials and the Fréchet subdifferentials at some points in the target set were explored recently in [25], which extends the corresponding results due to [14,15,21].

In the present paper, we develop and compute the Fréchet type $\varepsilon$-subdifferential and the limiting subdifferential for the perturbed distance function $d^\varepsilon_\varepsilon$ at some points in the target set and out of the target set. Our main results extend the corresponding ones in [14,26,21,22] from distance functions to general perturbed distance functions. In particular, almost all the results obtained in the present paper are new.

2. Preliminaries

Let $X$ be a real Banach space with norm $\| \cdot \|$ and $X^*$ the dual of $X$, respectively. We use $\langle \cdot, \cdot \rangle$ to denote the inner product between $X^* \times X$, that is $\langle x^*, x \rangle := x^*(x)$ for each pair $(x^*, x) \in X^* \times X$. The closed ball in $X$ (resp. $X^*$) with radius $r$ and center $x$ (resp. $x^*$) is denoted by $B(x, r)$ (resp. $B^*(x^*, r)$); while the corresponding open ball is denoted by $U(x, r)$ (resp. $U^*(x^*, r)$). In particular, we use $B_r$ for the closed balls with radius $r$ at the origin and omit the subscript $r$ when $r = 1$, that is, $B_1 = B(0, r)$ and $B_1 = B(0, 1)$; while $B^*_r$, $B^*_1$, $U_r$, $U_1$, $U^*_r$ and $U^*_1$ are understood similarly. Let $C$ be a nonempty closed subset of $X$. We use $\delta_C$ to denote the indicator function of $C$, i.e.,
\[ \delta_C(x) := \begin{cases} 0 & x \in C \\ +\infty & \text{otherwise}. \end{cases} \]

For a function $h$ defined on $C$, we define the extended function $h + \delta_C : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ by
\[ (h + \delta_C)(x) := \begin{cases} h(x) & x \in C \\ +\infty & \text{otherwise}. \end{cases} \]

Let $f : X \to \overline{\mathbb{R}}$ be a proper lower semicontinuous (lsc) function. As usual, the effective domain of $f$ is denoted by
\[ D(f) := \{x \in X | f(x) < +\infty\}. \]

As in [21], we adopt the symbols $x \rightharpoonup S \bar{x}$ and $x^*_k \rightharpoonup u^* x^*$ to stand, respectively, for the convergence $x \rightharpoonup \bar{x}$ with $x \in S$ for a set $S$ and for the sequential weak* convergence in $X^*$. Let $F : X \rightrightarrows X^*$ be a set-valued mapping. The sequential Painlevé–Kuratowski upper/outer limit $\limsup_{x \to \bar{x}} F(x)$ of $F$ at $\bar{x}$ is defined by
\[ \limsup_{x \to \bar{x}} F(x) := \left\{ x^* \in X^* \left| \begin{array}{l} \text{there are }\{x_k\} \subset X \text{ and }\{x^*_k\} \subset X^* \text{ such that } \end{array} \right. \right. \]
\[ \left. \begin{array}{l} x^*_k \in F(x_k) \text{ and } x_k \rightharpoonup \bar{x} \text{ and } x^*_k \to u^* x^* \end{array} \right\}. \]

The notions of the limiting normal cone to nonconvex closed sets and the corresponding limiting subdifferential of lsc extended real-valued functions were first introduced by Mordukhovich in [27], where the original normal cone was given in finite-dimensional spaces via the Euclidean projections, while the subdifferential was defined geometrically via the normal cone to the epigraph of the function. In the present paper, we adopt the limiting forms as given in Definitions 2.1 and 2.2, which first appeared in Kruger and Mordukhovich [28,29] and now are broadly used as Mordukhovich subdifferentials and normal cones, see also [14,26,30,21,22,24].

**Definition 2.1.** Let $S$ be a nonempty subset of $X$ and $\varepsilon \geq 0$.

(a) The Fréchet type $\varepsilon$-normal cone $\hat{N}_\varepsilon(\bar{x}; S)$ to $S$ at $\bar{x} \in S$ is defined by
\[ \hat{N}_\varepsilon(\bar{x}; S) := \left\{ x^* \in X^* \left| \limsup_{x \to \bar{x}, x \in S} \frac{\langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \leq \varepsilon \right\}. \]

and by $\hat{N}_\varepsilon(\bar{x}; S) := \emptyset$ if $\bar{x} \notin S$.

(b) The limiting normal cone $N(\bar{x}; S)$ to $S$ at $\bar{x} \in S$ is defined by
\[ N(\bar{x}; S) := \limsup_{x \to \bar{x}, x \in S} \hat{N}_\varepsilon(\bar{x}; S). \]
Definition 2.2. Let $\bar{x} \in D(f)$ and $\varepsilon \geq 0$.
(a) The Fréchet type $\varepsilon$-subdifferential $\hat{\partial}_\varepsilon f(\bar{x})$ of $f$ at $\bar{x}$ is defined by
\[
\hat{\partial}_\varepsilon f(\bar{x}) := \left\{ x^* \in X^* \mid \liminf_{x \to \bar{x}} \frac{f(x) - f(\bar{x}) - \langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \geq -\varepsilon \right\}.
\]
(b) The limiting subdifferential $\hat{\partial} f(\bar{x})$ of $f$ at $\bar{x}$ is defined by
\[
\hat{\partial} f(\bar{x}) := \limsup_{x \to \bar{x}} \hat{\partial}_\varepsilon f(\bar{x}).
\]

Remark 2.1. (a) When $\varepsilon = 0$, the $\hat{\partial}_\varepsilon f(\bar{x})$ is reduced to the Fréchet subdifferential $\hat{\partial} f(\bar{x})$ and if additionally $f$ is convex, the Fréchet subdifferential $\hat{\partial}_\varepsilon f(\bar{x})$ coincides with the subdifferential $\hat{\partial} f(\bar{x})$ from convex analysis.
(b) In particular, if $f = \delta_S$, then the Fréchet type $\varepsilon$-subdifferential of $f$ at $\bar{x} \in S$ indicates the Fréchet type $\varepsilon$-normal cone to $S$ at $\bar{x}$, i.e.,
\[
\hat{N}_\varepsilon(\bar{x}; S) = \hat{\partial}_\varepsilon \delta_S(\bar{x}).
\]
Similarly, the limiting subdifferential of the indicator function of $S$ at $\bar{x} \in S$ indicates the limiting normal cone to $S$ at $\bar{x}$, i.e.,
\[
\hat{N}(\bar{x}; S) = \hat{\partial} \delta_S(\bar{x}).
\]

3. Limiting subdifferentials at points in set $S_0$

Throughout the whole paper, we assume that $S$ is a nonempty closed subset of $X$ and $J : S \to \mathbb{R}$ is a lsc function. Recall that the perturbed distance function is defined by (1.2):
\[
df'(x) = \inf_{s \in S} \{\|x - s\| + J(s)\} \quad \text{for each } x \in X.
\]
By [12], we know that $d_\varepsilon'$ is nonexpansive:
\[
|d_\varepsilon'(y) - d_\varepsilon'(x)| \leq \|y - x\| \quad \text{for each } x, y \in X. \quad (3.1)
\]
Following [9,12,25], let $S_0$ denote the set of all points $x \in S$ such that $x$ is a solution of the problem $\min_J(x, S)$, i.e.,
\[
S_0 := \{ x \in S | d_\varepsilon'(x) = J(x) \} = \{ x \in S | J(x) \leq d_\varepsilon'(x) \}. \quad (3.2)
\]
Obviously, $S_0$ is closed. For the remainder of the present paper, we assume that $S_0$ is non-empty. Note that, in the case when $J$ is nonexpansive, then $S_0 = S$.

This section is devoted to the study of the Fréchet type $\varepsilon$-subdifferential and to attain the limiting subdifferential at points in the set $S_0$. To this end, we apply Definition 2.2 to functions $d_\varepsilon'$ and $J + \delta_S$ to arrive at the following remark.

Remark 3.1. Let $\bar{x} \in X$, $x^* \in X^*$ and $\varepsilon \geq 0$. Then the following assertions hold:
(i) $x^* \in \hat{\partial}_\varepsilon d_\varepsilon'(\bar{x})$ if and only if, for any $\eta > 0$, there exists $\delta > 0$ such that
\[
\langle x^*, x - \bar{x} \rangle \leq d_\varepsilon'(x) - d_\varepsilon'(\bar{x}) + (\varepsilon + \eta)\|x - \bar{x}\| \quad \text{for each } x \in \bar{B}(\bar{x}, \delta). \quad (3.3)
\]
(ii) $x^* \in \hat{\partial} d_\varepsilon'(\bar{x})$ if and only if there exist sequences $\{\varepsilon_k\} \subset [0, 1]$, $\{x_k\} \subset X$ and $\{x^*_k\} \subset X^*$ such that $\varepsilon_k \downarrow 0$, $x_k \to \bar{x}$ and $x^*_k \rightharpoonup x^*$ with each $x^*_k \in \hat{\partial}_\varepsilon d_\varepsilon'(x_k)$. \quad (3.4)
(iii) Let $\bar{x} \in S$, $x^* \in \hat{\partial} (J + \delta_S)(\bar{x})$ if and only if, for any $\eta > 0$, there exists $\delta > 0$ such that
\[
\langle x^*, x - \bar{x} \rangle \leq J(x) - J(\bar{x}) + (\varepsilon + \eta)\|x - \bar{x}\| \quad \text{for each } x \in S \bigcap \bar{B}(\bar{x}, \delta). \quad (3.5)
\]
(iv) Let $\bar{x} \in S$, $x^* \in \hat{\partial} (J + \delta_S)(\bar{x})$ if and only if there exist sequences $\{\varepsilon_k\} \subset [0, 1]$, $\{x_k\} \subset X$ and $\{x^*_k\} \subset X^*$ such that $\varepsilon_k \downarrow 0$, $x_k \to \bar{x}$ and $x^*_k \rightharpoonup x^*$ with each $x^*_k \in \hat{\partial}_\varepsilon (J + \delta_S)(x_k)$. \quad (3.6)

Proposition 3.1. Let $\varepsilon \geq 0$ and let $\bar{x} \in X$. Then we have that
\[
\hat{\partial} d_\varepsilon'(\bar{x}) \subseteq \bar{B}_{1+\varepsilon}^*.
\]
Furthermore, if $\bar{x} \not\in S_0$, then we have that
\[
\hat{\partial} d_\varepsilon'(\bar{x}) \subseteq \bar{B}_{1+\varepsilon}^* \setminus \bar{U}_{1-\varepsilon}^*.
\]
Proof. Let \( x^* \in \hat{\partial}_J \, d^J(x) \) and let \( \eta > 0 \). Then by Remark 3.1(i), there exists \( \delta > 0 \) such that (3.3) holds. This together with (3.1) implies that
\[
(x^*, x - \tilde{x}) \leq \|x - \tilde{x}\| + (\varepsilon + \eta)\|x - \tilde{x}\| = (1 + \varepsilon + \eta)\|x - \tilde{x}\| \quad \text{for each} \ x \in B(\tilde{x}, \delta).
\]
Hence \( \|x^*\| \leq 1 + \varepsilon \) as \( \eta \) is arbitrary and inclusion (3.7) is proved.

Now we assume that \( \tilde{x} \not\in S_0 \). To show \( \|x^*\| \geq 1 - \varepsilon \), let \( t \in (0, \delta) \) and choose \( w_t \in S \) such that
\[
\|w_t - \tilde{x}\| + J(w_t) \leq d^J(x) + t^2. \tag{3.9}
\]
We claim that
\[
\lim_{t \to 0} \|w_t - \tilde{x}\| > 0. \tag{3.10}
\]
Otherwise, there exists sequence \( \{t_k\} \) with \( t_k \to 0 \) such that \( \|w_{t_k} - \tilde{x}\| \to 0 \). This together with (3.9) implies that \( J(\tilde{x}) \leq d^J(\tilde{x}) (\text{as } J \text{ is lsc}) \) and so \( \tilde{x} \in S_0 \). This contradicts the assumption that \( \tilde{x} \not\in S_0 \) and the claim stands. Let \( \tilde{t} := \frac{t}{\|w_t - \tilde{x}\|} \) and \( x_t := \tilde{x} + \tilde{t}(w_t - \tilde{x}) \). Then we have
\[
\|x_t - \tilde{x}\| = \tilde{t}\|w_t - \tilde{x}\| = t < \delta.
\]
Thus, by (3.3),
\[
(x^*, x_t - \tilde{x}) \leq d^J(x_t) - d^J(\tilde{x}) + (\varepsilon + \eta)\|x_t - \tilde{x}\|. \tag{3.11}
\]
Noting that \( w_t \in S \) and using (3.9), one gets that
\[
d^J(x_t) - d^J(\tilde{x}) \leq \|w_t - x_t\| + J(w_t) - \|w_t - \tilde{x}\| - J(w_t) + t^2.
\]
Since \( \tilde{t}(x^*, w_t - \tilde{x}) = (x^*, x_t - \tilde{x}) \), it follows from (3.11) that
\[
\tilde{t}(x^*, w_t - \tilde{x}) \leq \|w_t - x_t\| + J(w_t) - \|w_t - \tilde{x}\| - J(w_t) + t^2 + (\varepsilon + \eta)\|x_t - \tilde{x}\|
\]
\[
= (\varepsilon + \eta - 1)\tilde{t}\|w_t - \tilde{x}\| + t^2.
\]
Hence
\[
\left\langle -x^*, \frac{w_t - \tilde{x}}{\|w_t - \tilde{x}\|} \right\rangle \geq (1 - \varepsilon - \eta) - \frac{t^2}{\|w_t - \tilde{x}\|} = 1 - \varepsilon - \eta - t.
\]
This implies that \( \|x^*\| \geq 1 - \varepsilon \) as \( t \in (0, \delta) \) and \( \eta \) are arbitrary. The proof is complete. \( \square \)

Before starting the main theorems we introduce the notions of the Lipschitz conditions.

Definition 3.1. Let \( \tilde{x} \in S \). The function \( J \) is said to satisfy
(a) the center Lipschitz condition at \( \tilde{x} \) if the center Lipschitz constant at \( \tilde{x} \)
\[
L^C := \inf_{\rho > 0} \sup_{y \in B(\tilde{x}, \rho) \cap S} \frac{|J(y) - J(\tilde{x})|}{\|y - \tilde{x}\|} < +\infty.
\]
(b) the Lipschitz condition at \( \tilde{x} \) if the Lipschitz constant at \( \tilde{x} \)
\[
L := \inf_{\rho > 0} \sup_{x, y \in \partial B(\tilde{x}, \rho) \cap S} \frac{|J(y) - J(x)|}{\|y - x\|} < +\infty.
\]
(c) the center Lipschitz condition on \( S \) with respect to (w.r.t.) \( \tilde{x} \) if the center Lipschitz constant on \( S \) w.r.t. \( \tilde{x} \)
\[
L^C := \sup_{y \in S} \frac{|J(y) - J(\tilde{x})|}{\|y - \tilde{x}\|} < +\infty.
\]
(d) the Lipschitz condition on \( S \) if the Lipschitz constant on \( S \)
\[
L := \sup_{x, y \in S} \frac{|J(y) - J(x)|}{\|y - x\|} < +\infty.
\]

Recall from [9] that the problem \( \min_{y}(x, S) \) is well-posed (in the sense of Tykhonov) if \( \min_{y}(x, S) \) has a unique solution and every minimizing sequence of the problem \( \min_{y}(x, S) \) converges to this solution, where we say that a sequence \( \{z_n\} \subseteq S \) is a minimizing sequence of the problem \( \min_{y}(x, S) \) if
\[
\lim_{n \to +\infty} (\|x - z_n\| + J(z_n)) = \inf_{z \in S} (\|x - z\| + J(z)). \tag{3.12}
\]
We also need the following lemmas, which are known in [25, Lemmas 3.4 and 3.1].

**Lemma 3.1.** Let \( \bar{x} \in S_0 \). Assume that the center Lipschitz constant on \( W \) w.r.t. \( \bar{x} L^c \) < 1. Then the problem \( \min_j(\bar{x}, S) \) is well-posed.

**Lemma 3.2.** Let \( \eta > 0, \delta > 0 \) and let \( \bar{x} \in S_0 \). Assume that \( \min_j(\bar{x}, S) \) is well-posed. Then there exists \( r \in (0, 1) \) such that for any \( z \in B(\bar{x}, r) \) and \( y \in S \) if

\[
\|z - y\| + J(y) \leq d^*_f(z) + \eta\|z - \bar{x}\| \tag{3.13}
\]

holds, we have

\[
\|y - \bar{x}\| < \delta. \tag{3.14}
\]

Now we are ready to prove the first theorem of this section, which provides the upper and lower estimates for the \( \varepsilon \)-subdifferentials of the perturbed distance function.

**Theorem 3.1.** Let \( \bar{x} \in S_0 \) and let \( \varepsilon \geq 0 \). Then the following assertions hold.

(i) We have

\[
\partial_\varepsilon d^*_f(\bar{x}) \subset \partial_\varepsilon (J + \delta_\varepsilon)(\bar{x}) \cap B^*_1(\bar{x}). \tag{3.15}
\]

(ii) Assume that \( \min_j(\bar{x}, S) \) is well-posed and the center Lipschitz constant at \( \bar{x} L^c \) < 1. Then we have

\[
\partial_\varepsilon (J + \delta_\varepsilon)(\bar{x}) \cap B^*_1(\bar{x}) \subset \partial_\alpha d^*_f(\bar{x}), \tag{3.16}
\]

where \( \alpha = 1 + \frac{4}{1-L^c} \).

**Proof.** (i) By (3.7), we only need to show

\[
\partial_\varepsilon d^*_f(\bar{x}) \subset \partial_\varepsilon (J + \delta_\varepsilon)(\bar{x}). \tag{3.17}
\]

To do this, let \( x^* \in \partial_\varepsilon d^*_f(\bar{x}) \) and let \( \eta > 0 \). Then by Remark 3.1(ii), there exists \( \delta > 0 \) such that \( (3.3) \) holds. Note that

\[
d^*_f(\bar{x}) = J(\bar{x}) \quad \text{and} \quad d^*_f(x) \leq J(x) \quad \text{for any} \ x \in S. \tag{3.18}
\]

It follows from \( (3.3) \) that

\[
(x^*, x - \bar{x}) \leq J(x) - J(\bar{x}) + (\varepsilon + \eta)\|x - \bar{x}\| \quad \text{for each} \ x \in S \cap B(\bar{x}, \delta). \tag{3.19}
\]

Hence \( x^* \in \partial_\varepsilon (J + \delta_\varepsilon)(\bar{x}) \) by Remark 3.1(iii) as \( \eta > 0 \) is arbitrary and the proof of \( (3.17) \) is complete.

(ii) Let \( x^* \in \partial_\varepsilon (J + \delta_\varepsilon)(\bar{x}) \) with \( \|x^*\| \leq 1 + \varepsilon \). To show \( x^* \in \partial_\alpha d^*_f(\bar{x}) \), let \( \eta > 0 \). Then by the assumption, there exists \( \delta > 0 \) such that \( (3.5) \) holds and

\[
|J(x) - J(\bar{x})| \leq (L^c_\varepsilon + \eta)\|x - \bar{x}\| \quad \text{for each} \ x \in S \cap B(\bar{x}, \delta). \tag{3.20}
\]

Let \( r \in (0, 1) \) be determined by Lemma 3.2. Take \( z \in B(\bar{x}, r) \setminus \{\bar{x}\} \) and choose \( y_2 \in S \) such that

\[
\|y_2 - z\| + J(y_2) \leq d^*_f(z) + \eta\|z - \bar{x}\|. \tag{3.21}
\]

Then Lemma 3.2 is applied to get \( \|y_2 - \bar{x}\| < \delta. \) Hence

\[
y_2 \in S \cap B(\bar{x}, \delta). \tag{3.22}
\]

By the first assertion in \( (3.18) \) and \( (3.1) \),

\[
d^*_f(z) \leq d^*_f(\bar{x}) + \|z - \bar{x}\| = J(\bar{x}) + \|z - \bar{x}\|.
\]

This together with \( (3.21) \) implies that

\[
\|y_2 - \bar{x}\| \leq \|y_2 - z\| + \|z - \bar{x}\|
\leq d^*_f(z) - J(y_2) + \eta\|z - \bar{x}\| + \|z - \bar{x}\|
\leq (2 + \eta)\|z - \bar{x}\| + J(\bar{x}) - J(y_2).
\]

By \( (3.22) \), we obtain from \( (3.20) \) that

\[
\|y_2 - \bar{x}\| \leq (2 + \eta)\|z - \bar{x}\| + J(\bar{x}) - J(y_2) \leq (2 + \eta)\|z - \bar{x}\| + (L^c_\varepsilon + \eta)\|y_2 - \bar{x}\|.
\]
Therefore, we arrive at
\[ \|y_2 - \tilde{x}\| < \frac{2 + \eta}{1 - L^0_\xi - \eta} \|z - \tilde{x}\|. \] (3.23)

On the other hand, by (3.21),
\[ (1 + \varepsilon)\|z - y_2\| = \|z - y_2\| + \varepsilon\|z - y_2\| \]
\[ \leq d_\varepsilon^J(z) - J(y_2) + \eta\|z - \tilde{x}\| + \varepsilon\|y_2 - \tilde{x}\| \]
\[ \leq d_\varepsilon^J(z) - J(y_2) + (\varepsilon + \eta)\|z - \tilde{x}\| + \varepsilon\|y_2 - \tilde{x}\|. \]

Hence
\[ (x^*, z - y_2) \leq (1 + \varepsilon)\|z - y_2\| \leq d_\varepsilon^J(z) - J(y_2) + (\varepsilon + \eta)\|z - \tilde{x}\| + \varepsilon\|y_2 - \tilde{x}\|. \] (3.24)

thanks to the fact that \(\|x^*\| \leq 1 + \varepsilon\). Moreover, by (3.22), we can apply (3.5) to get
\[ (x^*, y_2 - \tilde{x}) \leq J(y_2) - J(\tilde{x}) + (\varepsilon + \eta)\|y_2 - \tilde{x}\|. \] (3.25)

Thus combining this and (3.24) gives that
\[ (x^*, z - \tilde{x}) = (x^*, z - y_2) + (x^*, y_2 - \tilde{x}) \]
\[ \leq d_\varepsilon^J(z) - J(\tilde{x}) + (\varepsilon + \eta)\|z - \tilde{x}\| + (2\varepsilon + \eta)\|y_2 - \tilde{x}\|. \] (3.26)

By the first assertion in (3.18) and using (3.23), one has that
\[ (x^*, z - \tilde{x}) \leq d_\varepsilon^J(z) - d_\varepsilon^J(\tilde{x}) + (\varepsilon + \eta)\|z - \tilde{x}\| + (2\varepsilon + \eta)\|z - \tilde{x}\| \]
\[ = d_\varepsilon^J(z) - d_\varepsilon^J(\tilde{x}) + (\varepsilon + \eta)\|z - \tilde{x}\|, \] (3.27)

where
\[ \tilde{\eta} = \left(1 + \frac{(6 - 2\varepsilon)\varepsilon}{(1 - (L^0_\xi + \eta))(1 - L^0_\xi)} + \frac{2 + \eta}{1 - (L^0_\xi + \eta)}\right)\eta > 0. \]

This shows that \(x^* \in \hat{\alpha}_\varepsilon d_\varepsilon^J(\tilde{x})\) by Remark 3.1(i) because \(z \in B(\tilde{x}, r)\) is arbitrary and \(\tilde{\eta} \to 0\) as \(\eta \to 0\). The proof is complete. \(\square\)

The following corollary is a direct consequence of Theorem 3.1 and Lemma 3.1.

**Corollary 3.1.** Let \(\varepsilon > 0\) and let \(\tilde{x} \in S_0\). Assume that the center Lipschitz constant on \(S\) w.r.t. \(\tilde{x} L^0\) \(< 1\). Then we have
\[ \hat{\alpha}_\varepsilon d_\varepsilon^J(\tilde{x}) \subset \hat{\alpha}_\varepsilon (J + \delta_\varepsilon)(\tilde{x}) \bigcap \mathbb{B}^{\alpha}_1(\tilde{x}) \subset \hat{\alpha}_\varepsilon d_\varepsilon^J(\tilde{x}) \]
where \(\alpha = 1 + \frac{4}{1 - \varepsilon R^0}\).

In particular, letting \(\varepsilon = 0\), we get the following corollary, which was proved in [25].

**Corollary 3.2.** Let \(\tilde{x} \in S_0\). Then the following assertions hold.

(i) We have
\[ \hat{\alpha} d_\varepsilon^J(\tilde{x}) \subset \hat{\alpha}(J + \delta_\varepsilon)(\tilde{x}) \cap \mathbb{B}^*. \]

(ii) Assume that \(\min_\gamma(\tilde{x}, S)\) is well-posed and the center Lipschitz constant at \(\tilde{x} L^0\) \(< 1\). Then we have
\[ \hat{\alpha} d_\varepsilon^J(\tilde{x}) = \hat{\alpha}(J + \delta_\varepsilon)(\tilde{x}) \cap \mathbb{B}^*. \]

For the next theorems, we recall the well-known Ekeland’s variational principle from [31,32]; see also [21, Theorem 2.26].

**Proposition 3.2.** Let \((E, d)\) be a complete metric space and let \(\varphi : E \to \mathbb{R}\) be a proper lsc function bounded from below. Let \(\tilde{\eta} > 0\) and \(\tilde{w} \in E\) be such that
\[ \varphi(\tilde{w}) = \inf_{w \in E} \varphi(w) + \tilde{\eta}. \] (3.28)

Then, for any \(\lambda > 0\), there is \(\tilde{w} \in E\) satisfying
\[ \varphi(\tilde{w}) \leq \varphi(\tilde{w}), \quad d(\tilde{w}, \tilde{w}) \leq \lambda. \] (3.29)
and
\[ \varphi(\tilde{w}) \leq \varphi(w) + \frac{\tilde{\eta}}{\lambda} d(w, \tilde{w}) \quad \text{for each } w \in E. \] (3.30)

Lemma 3.3. Let \( \varepsilon \geq 0, \eta > 0 \) and \( \bar{x} \not\in S_0 \). Let \( x^* \in \hat{\delta}(\bar{x}) \). Then there exist \( \bar{w} \in S \) and \( \tilde{w} \in S \) such that
\[ \| \bar{w} - \bar{x} \| \leq \eta, \quad x^* \in \hat{\delta}_{\varepsilon+\eta}(f + \delta_2)(\tilde{w}) \] (3.31)
and
\[ \| \tilde{w} - \bar{x} \| + J(\tilde{w}) \leq d_2^*(\bar{x}) + \eta. \] (3.32)

Proof. By Remark 3.1(i), there exists \( \delta > 0 \) such that
\[ \langle x^*, x - \bar{x} \rangle \leq d_2^*(x) - d_2^*(\bar{x}) + \left( \varepsilon + \frac{\eta}{2} \right) \| x - \bar{x} \| \quad \text{for each } x \in B(\bar{x}, \delta). \] (3.33)

Set \( \tilde{\eta} := \min \left\{ \frac{\eta}{2}, \frac{\delta}{2}, 1 \right\} \) and let \( \tilde{w} \in S \) be such that
\[ \| \tilde{w} - \bar{w} \| + J(\tilde{w}) \leq d_2^*(\bar{x}) + \tilde{\eta}^2. \] (3.34)

Now we consider the metric space \( E := S \cap B(\tilde{w}, \delta) \) and the function \( \varphi : E \to \mathbb{R} \) defined by
\[ \varphi(w) := -\langle x^*, w - \tilde{w} \rangle + J(\tilde{w}) - J(w) + \tilde{\eta}^2 + \left( \varepsilon + \frac{\eta}{2} \right) \| w - \tilde{w} \| \quad \text{for each } w \in E. \] (3.35)

Clearly, \( E \) is a complete metric space and \( \varphi \) is a lsc function. Then to apply the Ekeland’s variational principle, we have to check condition (3.28). Note that \( \varphi(\tilde{w}) = \tilde{\eta}^2 \). It suffices to check that \( \varphi(w) \geq 0 \) for any \( w \in E \). To this end, let \( w \in E \). Then \( w - \tilde{w} + \bar{x} \in B(\bar{x}, \delta) \). Thus, by (3.33) and (3.34), we have
\[ \langle x^*, w - \tilde{w} \rangle \leq d_2^*(w - \tilde{w} + \bar{x}) - d_2^*(\bar{x}) + \left( \varepsilon + \frac{\eta}{2} \right) \| w - \tilde{w} \| \]
\[ \leq d_2^*(w - \tilde{w} + \bar{x}) - \| \tilde{x} - \tilde{w} \| - J(\tilde{w}) + \tilde{\eta}^2 + \left( \varepsilon + \frac{\eta}{2} \right) \| w - \tilde{w} \| \]
\[ \leq J(w) - J(\tilde{w}) + \tilde{\eta}^2 + \left( \varepsilon + \frac{\eta}{2} \right) \| w - \tilde{w} \|, \]
where the last inequality holds because
\[ d_2^*(w - \tilde{w} + \bar{x}) \leq \| w - \tilde{w} + \bar{x} - w \| + J(w) = \| \tilde{x} - \tilde{w} \| + J(w). \]

Hence \( \varphi(w) \geq 0 \) and condition (3.28) is checked. Then Proposition 3.2 is applicable (with \( \tilde{\eta}^2, \tilde{\eta} \) placing of \( \tilde{\eta}, \lambda \), respectively) and there exists \( \bar{w} \in E \) such that
\[ \| \bar{w} - \tilde{w} \| \leq \tilde{\eta} \] (3.36)
and
\[ \varphi(\bar{w}) \leq \varphi(w) + \tilde{\eta} \| w - \tilde{w} \| \quad \text{for each } w \in S \cap B(\tilde{w}, \delta). \] (3.37)

By the construction of \( \varphi \), (3.37) is equivalent to the following condition which holds for any \( w \in S \cap B(\tilde{w}, \delta) \):
\[ \langle x^*, w - \bar{w} \rangle \leq J(w) - J(\tilde{w}) + (\varepsilon + \eta) \| w - \tilde{w} \|. \] (3.38)

Since
\[ \| w - \bar{w} \| \leq \| w - \tilde{w} \| + \| \tilde{w} - \bar{w} \| \leq 2\tilde{\eta} \leq \delta \quad \text{for any } w \in S \cap B(\tilde{w}, \tilde{\eta}), \]
it follows that \( S \cap B(\tilde{w}, \tilde{\eta}) \subset S \cap B(\tilde{w}, \delta) \). This together with (3.37) implies that (3.38) holds for any \( w \in S \cap B(\tilde{w}, \tilde{\eta}) \) and so \( x^* \in \hat{\delta}_{\varepsilon+\eta}(f + \delta_2)(\bar{w}) \) by Remark 3.1(iii). Combining this and (3.36) shows (3.31). Moreover, by (3.34) and (3.36), we have
\[ \| \bar{w} - \tilde{x} \| + J(\tilde{w}) \leq \| \tilde{x} - \tilde{w} \| + \| \tilde{w} - \bar{w} \| + J(\tilde{w}) \]
\[ \leq d_2^*(\tilde{x}) + \tilde{\eta}^2 + \tilde{\eta} \]
\[ \leq d_2^*(\bar{x}) + \tilde{\eta}. \]
Hence (3.32) holds and the proof is complete. \( \Box \)
Lemma 3.4. Let $\bar{x} \in S_0$ be such that $\min_J(\bar{x}, S)$ is well-posed. Let $\{x_k\} \subset S$ be a sequence converging to $\bar{x}$. Assume that the Lipschitz constant $L_\delta < 1$. Then there exists $k_0 > 0$ such that $x_k \in S_0$ for each $k > k_0$.

**Proof.** By (3.2), it is sufficient to show that there exists $k_0 > 0$ such that

$$J(x_k) \leq d_\epsilon^J(x_k) \quad \text{for each } k > k_0. \quad (3.39)$$

Since $L_\delta < 1$ and since $\{x_k\}$ converges to $\bar{x}$, there exist $\delta > 0$ and $k_1 > 0$ such that

$$J(x_k) \leq \|y - x_k\| + J(y) \quad \text{for any } y \in S \cap B(\bar{x}, \delta) \text{ and } k > k_1. \quad (3.40)$$

Below we show that there exists $k_2 > 0$ such that

$$\|\bar{x} - x_k\| + J(\bar{x}) < \|y - x_k\| + J(y) \quad \text{for any } y \in S \setminus B(\bar{x}, \delta) \text{ and } k > k_2. \quad (3.41)$$

Granting this, together with (3.40), one has that (3.39) holds with $k_0 := \max\{k_1, k_2\}$ because, for any $k > k_0$,

$$J(x_k) \leq \|y - x_k\| + J(y) \quad \text{for any } y \in S,$$

which holds by (3.40) if $y \in S \cap B(\bar{x}, \delta)$, and holds by (3.40) and (3.41) if $y \in S \setminus B(\bar{x}, \delta)$ as

$$J(x_k) \leq \|\bar{x} - x_k\| + J(\bar{x}) < \|y - x_k\| + J(y) \quad \text{for any } y \in S \setminus B(\bar{x}, \delta). \quad (3.42)$$

To check (3.41), we suppose on the contrary that there exists a sequence $\{y_k\} \subset S \setminus B(\bar{x}, \delta)$ satisfying

$$\|y_k - x_k\| + J(y_k) \leq \|\bar{x} - x_k\| + J(\bar{x}). \quad (3.43)$$

Then

$$d_\epsilon^J(\bar{x}) \leq \|y_k - \bar{x}\| + J(y_k) \leq \|y_k - x_k\| + \|x_k - \bar{x}\| + J(y_k) \leq 2\|\bar{x} - x_k\| + J(\bar{x}).$$

Taking limits and noting that $J(\bar{x}) = d_\epsilon^J(\bar{x})$ as $\bar{x} \in S_0$ implies that $\{y_k\} \subset S$ is a minimizing sequence of the problem $\min_J(\bar{x}, S)$. By the well-posedness assumption, we obtain $y_k \to \bar{x}$, which contradicts the choice of the sequence $\{y_k\}$. Therefore, (3.41) holds and the proof is complete. □

Now we are ready to prove the following theorem, which provides estimates for the limiting subdifferentials of the perturbed distance function at points in $S_0$.

**Theorem 3.2.** Let $\bar{x} \in S_0$ be such that $\min_J(\bar{x}, S)$ is well-posed. Then the following assertions hold.

(i) We have

$$\partial d_\epsilon^J(\bar{x}) \subset \partial (J + \delta_\epsilon)(\bar{x}) \bigcap B^* \quad (3.44)$$

(ii) If the Lipschitz constant $L_\delta = 0$, then we have

$$\bigcup_{\lambda > 0} \lambda \partial d_\epsilon^J(\bar{x}) = \bigcup_{\lambda > 0} \lambda \left[ \partial (J + \delta_\epsilon)(\bar{x}) \bigcap B^* \right] = \bigcup_{\lambda > 0} \lambda \partial (J + \delta_\epsilon)(\bar{x}). \quad (3.45)$$

**Proof.** (i) Let $x^* \in \partial d_\epsilon^J(\bar{x})$. Then by Remark 3.1(ii), there exist sequences $\{\epsilon_k\} \subset [0, 1], \{x_k\} \subset X, \{x_k^*\} \subset X^*$ such that (3.4) holds. By (3.7), we have $\|x_k^*\| \leq 1 + \epsilon_k$ for each $k \in \mathbb{N}$ and so $x^* \in B^*$ as $\|x^*\| \leq \liminf_{k \to \infty} \|x_k^*\| \leq 1$. Thus we only need to show that

$$x^* \in \partial (J + \delta_\epsilon)(\bar{x}). \quad (3.46)$$

If there exists a subsequence $\{x_{k_l}\}$ satisfying $\{x_{k_l}\} \subset S_0$, then we have by (3.15) and (3.4) that

$$x^*_{k_l} \in \tilde{\partial}_{\epsilon_{k_l}}(J + \delta_\epsilon)(x_{k_l}) \quad \text{for each } k \in \mathbb{N}. \quad (3.47)$$

This means (3.46) holds by Remark 3.1(iv). It remains to consider the case where there exists $k_0 > 0$ such that $x_k \notin S_0$ for all $k > k_0$. Let $k > k_0$ and note that $x_k^* \in \tilde{\partial}_{\epsilon_{k}} d_\epsilon^J(x_k)$. Then Lemma 3.3 is applicable to conclude that there exist $w_k \in S$ and $\tilde{w}_k \in S$ such that

$$\|w_k - \tilde{w}_k\| \leq \frac{1}{k}, \quad \|w_k - x_k\| + J(\tilde{w}_k) \leq d_\epsilon^J(x_k) + \frac{1}{k} \quad (3.48)$$

and

$$x_k^* \in \tilde{\partial}_{\epsilon_{k}}(J + \delta_\epsilon)(w_k). \quad (3.48)$$
Then
\[
\begin{align*}
    d_{\epsilon}^j(x) & \leq \|\tilde{w}_k - x\| + J(\tilde{w}_k) \\
    & \leq \|\tilde{w}_k - w_k\| + \|w_k - x_k\| + \|x_k - x\| + J(\tilde{w}_k) \\
    & \leq d_{\epsilon}^j(x_k) + \frac{2}{k} + \|x_k - x\|.
\end{align*}
\]

Passing to limits and using the continuity of \(d_{\epsilon}^j(\cdot)\) implies that \(\{\tilde{w}_k\}\) is a minimizing sequence of the problem \(\min_{y}(\bar{x}, S)\). Since \(\bar{x} \in S_0\) and \(\min_{y}(\bar{x}, S)\) is well-posed, it follows that \(\tilde{w}_k \to \bar{x}\). This together with (3.47) implies that \(w_k \to \bar{x}\). Thus (3.46) follows from (3.48) and Remark 3.1(iv), and we attain the estimate (3.44).

(ii) By (3.44), the following inclusions are clear:
\[
\bigcup_{\lambda > 0} \lambda \partial d_{\epsilon}^j(\bar{x}) \subset \bigcup_{\lambda > 0} \lambda \partial (f + \delta_S)(\bar{x}).
\]

Thus, to complete the proof, it is sufficient to show that
\[
\bigcup_{\lambda > 0} \lambda \partial (f + \delta_S)(\bar{x}) \subset \bigcup_{\lambda > 0} \lambda \partial d_{\epsilon}^j(\bar{x}),
\]
assuming \(L_\chi = 0\). To do this, assume that \(L_\chi = 0\) and let \(x^* \in \partial (J + \delta_S)(\bar{x})\). Then by Remark 3.1(iv), there exist sequences \(\{\epsilon_k\} \subset [0, 1], \{x_k\} \subset X, \{x_k^*\} \subset X^*\) such that (3.6) holds. Then \(x_k^* \in \partial_{N_\epsilon} (J + \delta_S)(x_k)\) for each \(k \in \mathbb{N}\). It follows that \(\{x_k\} \subset S\) as \(\partial_{N_\epsilon} (J + \delta_S)(x_k) = \infty\) if \(x_k \notin S\). Since \(x_k \to \bar{x}\), \(L_\chi \to 0\) and \(\min_{y}(\bar{x}, S)\) is well-posed, it follows from Lemma 3.4 that \(\{x_k\}_{k \geq k_0} \subset S_0\) for some \(k_0 > 0\). Without loss of generality, we assume that \(\{x_k\} \subset S_0\).

If there exists a subsequence of \(\{k\}\), denoted by itself, such that \(\|x_k\| \leq 1 + \epsilon_k\) for each \(k \in \mathbb{N}\), then
\[
x_k^* \in \partial_{N_\epsilon} (J + \delta_S)(x_k) \cap B_{1 + \epsilon_k} \subset \partial_{N_\epsilon} d_{\epsilon}^j(x_k) \quad \text{for each} \ k \in \mathbb{N},
\]
by (3.16), where \(\alpha = 1 + \frac{4}{\epsilon_k}\); hence \(x^* \in \partial d_{\epsilon}^j(\bar{x})\) thanks to Remark 3.1(ii).

Thus, without loss of generality, we assume that \(\|x_k\| > 1 + \epsilon_k\) for each \(k \in \mathbb{N}\). Since \(L_\chi = 0\) and \(x_k \to \bar{x}\), there exists \(k_0 > 0\) and \(\delta_k > 0\) such that
\[
|f(y) - J(x_k)| \leq \epsilon_k \|y - x_k\| \quad \text{for each} \ k > k_0 \text{ and } y \in B(\bar{x}, \delta_k).
\]
Let \(k > k_0\). Then
\[
\{0, y - x_k\} = 0 \leq f(y) - J(x_k) + \epsilon_k \|y - x_k\| \quad \text{for each} \ y \in S \cap B(\bar{x}, \delta_k).
\]
Letting \(k \to \infty\), this together with Remark 3.1(iv) implies that
\[
0 \in \partial_{N_\epsilon} (J + \delta_S)(x_k).
\]
Write \(\lambda_k := \frac{1 + \epsilon_k}{\epsilon_k}\); then \(\lambda_k \epsilon_k \in B_{1 + \epsilon_k}^*\) and \(\lambda_k < 1\). Noting that \(\partial_{N_\epsilon} (J + \delta_S)(x_k)\) is convex, we have
\[
\lambda_k x_k^* = (1 - \lambda_k)0 + \lambda_k x_k^* \in \partial_{N_\epsilon} (J + \delta_S)(x_k).
\]
Hence
\[
\lambda_k x_k^* \in \partial_{N_\epsilon} (J + \delta_S)(x_k) \cap B_{1 + \epsilon_k}^* \subset \partial_{N_\epsilon} d_{\epsilon}^j(x_k)
\]
thanks to (3.16). Without loss of generality, assume that \(\lambda_k \to \lambda\) for some \(\lambda > 0\). Then \(\lambda x^* \in \partial d_{\epsilon}^j(\bar{x})\) by Remark 3.1(ii) and (3.49) is established. The proof is complete. \(\square\)

Conclusion (ii) in Theorem 3.2 is improved in the following theorem for the special case where \(X\) is a finite-dimensional space.

**Theorem 3.3.** Let \(\bar{x} \in S_0\) be such that \(\min_{y}(\bar{x}, S)\) is well-posed. Suppose that \(X\) is a finite-dimensional space and the Lipschitz constant \(L_\chi < 1\). Then we have
\[
\partial d_{\epsilon}^j(\bar{x}) = \partial (J + \delta_S)(\bar{x}) \cap B^*.
\]

**Proof.** By (3.44), we only need to show that
\[
\partial (J + \delta_S)(\bar{x}) \cap B^* \subset \partial d_{\epsilon}^j(\bar{x}).
\]
Let \( x^* \in \partial (f + \delta S)(x) \) with \( ||x^*|| \leq 1 \). Then by Remark 3.1(iv), there exist sequences \( \{ \varepsilon_k \} \subset [0,1], \{ x_k \} \subset X, \{ x_k^* \} \subset X^* \) such that (3.6) holds. As in the proof of Theorem 3.2, we may assume that \( \{ x_k \} \subset S_0 \). Moreover, since \( X \) is of finite dimension, we have by (3.6) that \( ||x_k^*|| \to ||x^*|| \). Thus if \( ||x^*|| < 1 \), then we have \( ||x_k^*|| \leq 1 + \varepsilon_k \) for sufficiently large \( k \in \mathbb{N} \). Therefore, by (3.16),

\[
x_k^* \in \hat{\partial} \delta_j (f + \delta S)(x_k) \cap \mathbb{B}_{1+\varepsilon_k}^* \subset \hat{\partial} \delta_j d_j^S(x_k)
\]

for sufficiently large \( k \in \mathbb{N} \), where \( \alpha = 1 + \frac{1}{4} \varepsilon_k \).

If \( ||x^*|| = 1 \), let \( \delta_k := \max \{ 1 - ||x_k^*||, \varepsilon_k \} \). Then \( \delta_k \downarrow 0 \) and \( x_k^* \in \mathbb{B}_{1+\delta_k}^* \) for each \( k \in \mathbb{N} \). Therefore, again by (3.16), we have that

\[
x_k^* \in \hat{\partial} \delta_j (f + \delta S)(x_k) \cap \mathbb{B}_{1+\delta_k}^* \subset \hat{\partial} \delta_j d_j^S(x_k)
\]

for each \( k \in \mathbb{N} \). Thus in either cases, one sees from Remark 3.1(ii) that \( x^* \in \partial \alpha d_j^S(x) \) and the proof is now complete. \( \square \)

Combining Theorem 3.3 and Lemma 3.1, we have the following corollary, which extends the corresponding one in [26, Problem 7.17(c), P173] for the special case where \( J \equiv 0 \).

**Corollary 3.3.** Suppose that \( X \) is a finite-dimensional space and that \( J \) satisfies the Lipschitz condition on \( S \) with the Lipschitz constant \( L < 1 \). Then (3.50) holds for each \( x \in S \). \( \square \)

### 4. Limiting subdifferentials at points out of set \( S_0 \)

The present section is devoted to the study of estimates for subdifferentials of the perturbed distance functions at points out of the set \( S_0 \). We begin with the following proposition. For \( x \in X \), let \( P_j^S(x) \) denote the set of all solutions of the problem (1.1), that is,

\[
P_j^S(x) := \{ w \in S ||x - w|| + J(w) = d_j^S(x) \}.
\]

**Proposition 4.1.** Let \( x \in X \) and let \( \bar{w} \in P_j^S(\bar{x}) \). Then we have \( \bar{w} \in P_j^S(t\bar{w} + (1 - t)\bar{x}) \) for each \( t \in (0,1] \). Consequently,

\[
d_j^S(t\bar{w} + (1 - t)\bar{x}) = (1 - t)d_j^S(\bar{x}) + tf(\bar{w}) \quad \text{for each } t \in (0,1].
\]

**Proof.** Let \( t \in (0,1] \) and set \( x_t := t\bar{w} + (1 - t)\bar{x} \).

Then for any \( w \in S \), we have that

\[
||\bar{w} - x_t|| + J(\bar{w}) = d_j^S(\bar{x}) - t||\bar{w} - \bar{x}|| \\
\leq ||\bar{w} - \bar{x}|| + J(w) - t||\bar{w} - \bar{x}|| \\
\leq ||w - x_t|| + J(w).
\]

This means that \( \bar{w} \in P_j^S(x_t) \). Consequently,

\[
d_j^S(x_t) = ||\bar{w} - x_t|| + J(\bar{w}) = (1 - t)d_j^S(\bar{x}) + tf(\bar{w}).
\]

Hence (4.1) is proved. \( \square \)

**Theorem 4.1.** Let \( x \notin S_0 \) be such that \( P_j^S(x) \neq \emptyset \) and let \( \varepsilon \geq 0 \). Then we have

\[
\hat{\partial}_\varepsilon d_j^S(\bar{x}) \subset \bigcap_{w \in P_j^S(\bar{x})} \bigcap_{t \in (0,1]} \hat{\partial}_\varepsilon d_j^S(tw + (1 - t)\bar{x}) \cap (\mathbb{B}_{1+\varepsilon}^* \setminus \mathbb{U}_{1-\varepsilon}^*).
\]

In particular,

\[
\hat{\partial}_\varepsilon d_j^S(\bar{x}) \subset \bigcap_{w \in P_j^S(\bar{x})} \hat{\partial}_\varepsilon d_j^S(w) \cap (\mathbb{B}_{1+\varepsilon}^* \setminus \mathbb{U}_{1-\varepsilon}^*).
\]

**Proof.** Let \( w \in P_j^S(\bar{x}) \) and let \( t \in (0,1] \). By (3.8), we only need to prove

\[
\hat{\partial}_\varepsilon d_j^S(\bar{x}) \subset \hat{\partial}_\varepsilon d_j^S(tw + (1 - t)\bar{x}).
\]

To do this, let \( x^* \in \hat{\partial}d_j^S(\bar{x}) \) and let \( \eta > 0 \). By Remark 3.1(i), there exists \( \delta > 0 \) such that (3.3) holds. Write \( x_t := tw + (1 - t)\bar{x} \).

Then, by (4.1) and noting that \( w \in P_j^S(\bar{x}) \),

\[
d_j^S(x_t) = d_j^S(\bar{x}) + t(f(w) - d_j^S(\bar{x})) = d_j^S(\bar{x}) - t||w - \bar{x}||.
\]
Note that, for any \(x \in B(x_t, \delta)\), we have \(x - x_t = x - t(w - \bar{x}) - \bar{x}\) and so \(x - t(w - \bar{x}) \in B(\bar{x}, \delta)\). This together with (3.3) implies that
\[
\langle x^*, x - x_t \rangle \leq d^*_f(x - t(w - \bar{x})) - d^*_f(\bar{x}) + (\varepsilon + \eta)\|x - x_t\|.
\] (4.8)

Using \(d^*_f(x - t(w - \bar{x})) \leq d^*_f(x) + t\|w - \bar{x}\|\) by (3.1) and by (4.7), we have from (4.8) that
\[
\langle x^*, x - x_t \rangle \leq d^*_f(x) - d^*_f(x_t) + (\varepsilon + \eta)\|x - x_t\|.
\] (4.9)

This means that \(x^* \in \hat{\partial}_d d^*_f(x_t)\) by Remark 3.1(i). Thus estimate (4.6) is established and the proof is complete. □

To consider the case where the projection set \(P^d_f(\bar{x})\) may be empty, we introduce the set-valued mapping \(P^d_{S, \varepsilon}(-) : \mathbb{R}_+ \times X \to S\) by
\[
P^d_{S, \varepsilon}(x) := \{w \in S \| w - x \| \leq d^*_f(x) - J(w) + \varepsilon\} \quad \text{for each } \varepsilon \in \mathbb{R}_+ \text{ and } x \in X.
\]

Then we have the following proposition, which shows that the mapping \(P^d_{S, \varepsilon}(-)\) is closed at each point \((0, \bar{x}, \bar{w}) \in \mathbb{R}_+ \times X \times S\), that is, for any sequences \(\varepsilon_k \subset [0, 1], \{x_k\} \subset X, \{\bar{w}_k\} \subset S\) with each \(\bar{w}_k \in P^d_{S, \varepsilon_k}(x_k)\) satisfying
\[
\varepsilon_k \downarrow 0, \quad x_k \to \bar{x} \text{ and } \bar{w}_k \to \bar{w},
\] (4.10)
we have \(\bar{w} \in P^d_{S, 0}(\bar{x}) = P^d_f(\bar{x})\).

**Proposition 4.2.** Suppose that \(J\) is continuous on \(S\). Then \(P^d_{S, \varepsilon}(-)\) is closed at each point \((0, \bar{x}, \bar{w}) \in \mathbb{R}_+ \times X \times S\).

**Proof.** Let \((0, \bar{x}, \bar{w}) \in \mathbb{R}_+ \times X \times S\) and let \(\varepsilon_k \subset [0, 1], \{x_k\} \subset X, \{\bar{w}_k\} \subset S\) be sequences with each \(\bar{w}_k \in P^d_{S, \varepsilon_k}(x_k)\) satisfying (4.10). Then, for each \(k \in \mathbb{N}, \|\bar{w}_k - x_k\| \leq d^*_f(x_k) + \varepsilon_k\) and it follows that
\[
\|\bar{w} - \bar{x}\| + J(\bar{w}) \leq \|\bar{w} - \bar{w}_k\| + \|x_k - \bar{x}\| + d^*_f(x_k) + \varepsilon_k + J(\bar{w}) - J(\bar{w}_k).
\] (4.11)

Passing to limits, we have
\[
\|\bar{w} - \bar{x}\| + J(\bar{w}) \leq d^*_f(\bar{x}).
\] (4.12)

This shows that \(\bar{w} \in P^d_f(\bar{x})\). □

**Theorem 4.2.** Let \(\bar{x} \not\in S_0\) and let \(\varepsilon \geq 0\). Assume that \(J\) is uniformly continuous on \(S\). Then we have
\[
\hat{\partial}_d d^*_f(\bar{x}) \subset \bigcap_{\eta > 0} \bigcup_{w \in P^d_{S, \varepsilon}(\bar{x})} \left(\hat{\partial}_{d+\eta}(J + \delta_x)(w) \cap (B^*_f + \eta) \setminus U_{\varepsilon}^*(\bar{x})\right).
\] (4.13)

**Proof.** Let \(x^* \in \hat{\partial}_d d^*_f(\bar{x})\) and \(\eta > 0\). By (3.8), it suffices to show that there exists \(\bar{w} \in P^d_{S, \varepsilon}(\bar{x})\) such that \(x^* \in \hat{\partial}_{d+\eta}(J + \delta_x)(\bar{w})\).

Since \(J\) is uniformly continuous on \(S\), there exists \(0 \leq \tilde{\eta} \leq \frac{\varepsilon}{2}\) such that
\[
|J(\bar{w}) - J(\bar{w})| < \frac{\eta}{2} \quad \text{for any } \bar{w}, \bar{w} \in S \text{ with } \|\bar{w} - \bar{w}\| < \tilde{\eta}.
\] (4.14)

By Lemma 3.3, there exist \(\bar{w}\) and \(\bar{w}\) such that
\[
\|\bar{w} - \bar{w}\| \leq \tilde{\eta}, \quad \|\bar{w} - \bar{x}\| \leq d^*_f(\bar{x}) - J(\bar{w}) + \tilde{\eta}
\] (4.15)
and
\[
x^* \in \hat{\partial}_{d+\eta}(J + \delta_x)(\bar{w}) \subset \hat{\partial}_{d+\eta}(J + \delta_x)(\bar{w})
\] (noting that \(\tilde{\eta} \leq \frac{\varepsilon}{2}\)). Thus, by (4.15) and (4.14), we get that
\[
\|\bar{w} - \bar{x}\| \leq d^*_f(\bar{x}) - J(\bar{w}) + J(\bar{w}) - J(\bar{w}) + \tilde{\eta} \leq d^*_f(\bar{x}) - J(\bar{w}) + \eta.
\]

Hence, \(\bar{w} \in P^d_{S, \varepsilon}(\bar{x})\) and \(x^* \in \hat{\partial}_{d+\eta}(J + \delta_x)(\bar{w})\) by (4.16). The proof is complete. □

For the next theorem, we need to extend the notion of the well-posedness due to Mordukhovich (cf. [21, P.109, Definition 1.104]) from the best approximation problem to the perturbed optimization problem.
Definition 4.1. Let $\bar{x} \notin S_0$. The problem $\min J(\bar{x}, S)$ is said to be well-posed in the sense of Mordukhovich if either one of the following properties holds:

(a) Every minimizing sequence $\{x_k\} \subseteq S$ of the problem $\min J(x, S)$ contains a convergent subsequence.

(b) For every sequence $\{x_k\}$ converging to $\bar{x}$ and $\delta_k d^*_J(x_k) \neq \emptyset$ as $\varepsilon_k \downarrow 0$, there is a sequence $\{w_k\}$ with each $w_k \in P^*_S(x_k)$ that contains a convergent subsequence.

Remark 3.1

Proof. Let $x^* \in \partial d^*_J(\bar{x})$. Then by Remark 3.1(ii), there exist sequences $\{\varepsilon_k\} \subseteq [0, 1], \{x_k\} \subseteq X$ and $\{x^*_k\} \subseteq X^*$ such that (3.4) holds. Without loss of generation, we may assume that each $x_k \notin S_0$. By (3.8), we have $x^*_k \in B^*_{1+\varepsilon_k} \setminus U^*_{1-\varepsilon_k}$ for each $k \in \mathbb{N}$ and so $x^* \in B^*$. Thus it remains to show that

$$x^* \in \partial (f + \delta_J)(\bar{w}) \quad \text{for some } \bar{w} \in P^*_S(\bar{x}). \quad (4.18)$$

To this end, we first assume that the well-posedness property (a) holds. Let $k \in \mathbb{N}$ and then, by Lemma 3.3, there exist $w_k$ and $\bar{w}_k \in S$ such that

$$\|w_k - \bar{w}_k\| \leq \varepsilon_k, \quad \|w_k - x_k\| \leq d^*_J(x_k) - f(\bar{w}) + \varepsilon_k \quad (4.19)$$

and

$$x^*_k \in \hat{\partial}_{2\varepsilon_k} (f + \delta_J)(w_k). \quad (4.20)$$

Since $\|\bar{w}_k - \bar{x}\| \leq \|\bar{w}_k - w_k\| + \|w_k - x_k\| + \|x_k - \bar{x}\|$ and $\bar{w}_k \in S$, (4.19) implies that

$$\|\bar{w}_k - \bar{x}\| + f(\bar{w}) \leq d^*_J(x_k) + \|x_k - \bar{x}\| + 2\varepsilon_k. \quad (4.21)$$

This means that $\{\bar{w}_k\}$ is a minimizing sequence of $\min J(\bar{x}, S)$ because $x_k \to \bar{x}$ and $d^*_J$ is continuous. By using the well-posedness property (a), there is $\bar{w} \in S$ such that $\bar{w}_k \to \bar{w}$ (using a subsequence if necessary). This together with (4.19) implies that $w_k \to \bar{w}$. Furthermore, we have that $\bar{w}_k \in P^*_S(\bar{x})$ because, by (4.19), we have

$$\|\bar{w}_k - x_k\| \leq \|\bar{w}_k - w_k\| + \|w_k - x_k\| \leq d^*_J(x_k) - f(\bar{w}) + 2\varepsilon_k. \quad (4.22)$$

Thus Proposition 4.2 is applicable and $\bar{w} \in P^*_S(\bar{x})$. Since $\varepsilon_k \downarrow 0$ by (3.4), it follows from Remark 3.1(iv) and (4.20) that $x^* \in \partial (f + \delta_J)(\bar{w})$ and so (4.18) is shown.

Now we assume that the well-posedness property (b) holds. Since $x_k \to \bar{x}, \varepsilon_k \downarrow 0$ and $x^*_k \in \hat{\partial}_{\varepsilon_k} d^*_J(x_k) \neq \emptyset$, it follows that there is a sequence $\{w_k\}$ with each $w_k \in P^*_S(x_k)$ converging to some $\bar{w}$. Then $\bar{w} \in P^*_S(\bar{x})$ by Proposition 4.2. Below we show that $x^*_k \in \hat{\partial}_{\varepsilon_k} (f + \delta_J)(w_k)$ for each $k \in \mathbb{N}$. Granting this, (4.18) is seen to hold. Let $k \in \mathbb{N}$ and let $\eta > 0$. Since $x^*_k \in \hat{\partial}_{\varepsilon_k} d^*_J(w_k)$ by (4.5), Remark 3.1(i) yields that there exists $\delta > 0$ such that

$$\langle x^*_k, w - w_k \rangle \leq d^*_J(w) - d^*_J(w_k) + (\varepsilon_k + \eta)\|w - w_k\| \quad \text{for each } w \in B(w_k, \delta). \quad (4.23)$$

Note that $d^*_J(x_k) - d^*_J(w_k) \leq \|x_k - w_k\|$ by (3.1) and note also that $d^*_J(x_k) = \|w_k - x_k\| + f(w_k)$ as $w_k \in P^*_S(x_k)$. We have

$$d^*_J(w) - d^*_J(w_k) \leq d^*_J(w) - d^*_J(x_k) + \|w_k - x_k\| = d^*_J(w) - f(w). \quad (4.24)$$

Hence, (4.23) implies that

$$\langle x^*_k, w - w_k \rangle \leq d^*_J(w) - f(w) + (\varepsilon_k + \eta)\|w - w_k\| \leq f(w) - f(w_k) + (\varepsilon_k + \eta)\|w - w_k\| \quad (4.25)$$

for each $w \in S \cap B(w_k, \delta)$, where the last inequality holds because $d^*_J(w) \leq f(w)$ for all $w \in S$. This yields that $x^*_k \in \hat{\partial}_{\varepsilon_k} (f + \delta_J)(w_k)$ thanks to Remark 3.1(iii). The proof is now complete. \(\square\)
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