Mann’s algorithm for nonexpansive mappings in \( \text{CAT}(\kappa) \) spaces

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ABSTRACT

In this paper, it is proved that the sequence defined by Mann’s algorithm \( \Delta \)-converges to a fixed point in complete \( \text{CAT}(\kappa) \) spaces.

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1. Introduction

Let \( X \) be a complete metric space and let \( T \) be a self-mapping defined on \( X \). It is well known that, if the mapping \( T \) is a contraction, then, by Banach’s contraction principle, for each \( x \in X \), the iterative sequence \( \{T^n x\} \) converges strongly to the unique fixed point of \( T \). But, if \( T \) is only assumed to be a nonexpansive mapping, we must add some additional conditions to ensure the existence of a fixed point of \( T \) and, even when a fixed point exists, the sequences of iterates in general do not converge to a fixed point. This fact explains why the study of the asymptotic behavior of nonexpansive mappings has become a very attractive research topic in nonlinear analysis.

One of the most successful methods to approach fixed points of a nonexpansive mapping is Mann’s algorithm, which, in the case when \( X \) is a linear space, generates a sequence \( \{x_n\} \) via the following iteration:

\[
x_{n+1} = t_n x_n + (1 - t_n) T(x_n), \quad n \geq 0.
\]

The convergence properties of this algorithm have been studied extensively; see, for example, [1–4] and the references therein.

Because of the convex structure of the algorithms, few results on approximation of fixed points have been obtained out of the setting of linear spaces except for ones in some special metric spaces. For example, Mann’s iteration has been studied in hyperbolic spaces (see [5,6]), the Hilbert ball (see [7]), and Hadamard manifolds (see [8]).
Our interest here is to study Mann’s iteration scheme in complete CAT(κ) spaces. Roughly speaking, CAT(κ) spaces are geodesic spaces of bounded curvature and generalizations of Riemannian manifolds of sectional curvature bounded above. The study of nonexpansive mappings in the realm of CAT(κ) spaces was initiated by Kirk [9,10]. In his seminal works, several results are proved, mainly concerning fixed point theorems in CAT(0) spaces. Recently, by using the Δ-convergence introduced by Lim [11], Dhompongsa and Panyanak established in [12] the Δ-convergence result of Mann’s iteration in CAT(0) spaces.

The aim of this paper is to establish in complete CAT(κ) spaces the Δ-convergence results of the sequence defined by Mann’s algorithm for any nonnegative number κ. Our results obtained in the present paper extend the corresponding ones in [8,12].

2. Preliminaries

In this section, we introduce some fundamental definitions, properties, and notations of model spaces $M^2$ and CAT(κ) spaces. We basically follow [13].

Let $(X, ρ)$ be a metric space and let $x, y \in X$. As usual, we use $B(x, δ)$ (respectively, $\overline{B}(x, δ)$) to denote the ball (respectively, closed ball) centered at $x$ with radius $δ$. A geodesic path joining $x$ to $y$ (or, more briefly, a geodesic from $x$ to $y$) is a map $c : [0, l] \subseteq \mathbb{R} \rightarrow X$ such that $c(0) = x, c(l) = y$, and $ρ(c(t), c(t′)) = |t − t′|$ for all $t, t′ \in [0, l]$. In particular, $c$ is an isometry between $[0, l]$ and $c([0, l])$, and $ρ(x, y) = l$. Usually, the image $c([0, l])$ of $c$ is called a geodesic segment (or metric segment) joining $x$ and $y$. A metric segment joining $x$ and $y$ is not necessarily unique in general. In particular, in the case when the geodesic segment joining $x$ and $y$ is unique, we use $[x, y]$ to denote the unique geodesic segment joining $x$ and $y$; this means that $z \in [x, y]$ if and only if there exists $t \in [0, 1]$ such that $ρ(z, x) = (1 − t)ρ(x, y)$ and $ρ(z, y) = tρ(x, y)$. In this case, we will write $z = tx + (1 − t)y$ for simplicity. For fixed $D ∈ (0, +∞]$, the space $(X, ρ)$ is called a $D$-geodesic space if any two points of $X$ with their distance smaller than $D$ are joined by a geodesic segment. An $∞$-geodesic space is simply called a geodesic space. Recall that a geodesic triangle $Δ := Δ(x, y, z)$ in the metric space $(X, ρ)$ consists of three points in $X$ (the vertices of $Δ$) and three geodesic segments between each pair of vertices (the edges of $Δ$). For the sake of saving printing space, we write $p \in Δ$ when a point $p \in X$ lies in the union of $[x, y], [x, z]$ and $[y, z]$.

Let $m \in \mathbb{N}$ and let $(\cdot | \cdot)$ denote the Euclidean scalar product in $\mathbb{R}^m$, that is, $(x | y) = \sum_{i=1}^{m} x_i y_i$, where $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_m)$. Let $S^m$ denote the $m$-dimensional sphere defined by

$$S^m := \{x \in \mathbb{R}^{m+1} : (x | x) = 1\}.$$  \hspace{1cm} (2.1)

Define $d : S^m \times S^m \rightarrow [0, +∞)$ by

$$d(x, y) = \arccos(x | y), \quad \forall (x, y) \in S^m \times S^m.$$  \hspace{1cm} (2.2)

From now on, we always assume that $κ \geq 0$, and write

$$D_κ := \begin{cases} \mathbb{R} \setminus \{κ\} & \text{if } κ > 0, \\ \{κ\} & \text{if } κ = 0. \end{cases}$$

Let $M^m_κ$ denote the metric space obtained from $(S^m, d)$ by multiplying the distance function by the constant $\frac{1}{κ}$ if $κ > 0$ and the $m$-dimensional Euclidean space $\mathbb{R}^m$ if $κ = 0$. We use the same symbol $d$ to denote the distance in $M^m_κ$ if no confusion arises. Clearly, $M^m_κ$ is a geodesic metric space.

Consider triangles $Δ := Δ(x, y, z) \subseteq X$ and $\overline{Δ} := Δ(\tilde{x}, \tilde{y}, \tilde{z}) \subseteq \overline{M^2}$. The triangle $\overline{Δ}$ is called a comparison triangle for $Δ$ if

$$\rho(x, y) = d(\tilde{x}, \tilde{y}), \quad \rho(x, z) = d(\tilde{x}, \tilde{z}) \quad \text{and} \quad \rho(z, y) = d(\tilde{z}, \tilde{y}).$$

By [13, Lemma 2.14, page 24], a comparison triangle for $Δ$ always exists provided that the perimeter $ρ(x, y) + ρ(y, z) + ρ(z, x) < 2D_κ$. A point $\tilde{p} \in [\tilde{x}, \tilde{y}] \subseteq \overline{Δ}$ is called a comparison point for $p \in [x, y] \subseteq Δ$ if $d(\tilde{p}, \tilde{x}) = d(p, x)$.

Recall that a geodesic triangle $Δ$ in $X$ with perimeter less than $2D_κ$ is said to satisfy the CAT(κ) inequality if, given $\overline{Δ}$ a comparison triangle in $\overline{M^2}$ for $Δ$, one has that

$$\rho(p, q) \leq d(\tilde{p}, \tilde{q}), \quad \forall p, q \in Δ,$$

where $\tilde{p}$ and $\tilde{q}$ are respectively the comparison points of $p$ and $q$. Thus we are ready to introduce the notion of a CAT(κ) space in the following definition taken from [13].

Definition 2.1. The metric space $(X, ρ)$ is called a CAT(κ) space if it is $D_κ$-geodesic and any geodesic triangle in $X$ of perimeter less than $2D_κ$ satisfies the CAT(κ) inequality.
By definition, it is clear that $(X, \rho)$ is a CAT(0) space if and only if it is geodesic and any geodesic triangle in $X$ satisfies the CAT(0) inequality.

Recall that a set $Y \subseteq X$ is said to be convex if any two points $x, y \in Y$ can be joined by a geodesic in $X$ and all geodesics joining them are contained in $Y$. Then the following proposition is known in [13, Proposition 1.4, page 160] and will be used in the next section.

**Proposition 2.1.** Let $X$ be a CAT($\kappa$) space. Then any ball in $X$ of radius smaller than $\frac{\kappa}{2}$ is convex. In particular, any ball in a CAT(0) space is convex.

**Remark 2.1.** For $\kappa < 0$, a CAT($\kappa$) space is defined in terms of comparison triangles in the hyperbolic plane; see [13] for details. Here, for the sake of simplicity, we omit its definition, since it is known (see [13, page 165]) that any CAT($\kappa_1$) space is also a CAT($\kappa_2$) space for any pair ($\kappa_1, \kappa_2$) with $\kappa_2 \geq \kappa_1$. This means that any CAT($\kappa$) space with $\kappa < 0$ is a CAT(0) space.

The remainder of this section is concerned with the notion of $\Delta$-convergence and some relative properties that are useful for our study in the next section. Let $\{x_n\}$ be a bounded sequence in $X$. We define

$$r(x, \{x_n\}) := \limsup_{n \to \infty} \rho(x, x_n), \quad \forall x \in X.$$  

The asymptotic radius $r(\{x_n\})$ and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ are defined respectively by

$$r(\{x_n\}) := \inf \{r(x, \{x_n\}) : x \in X\}$$

and

$$A(\{x_n\}) := \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$  

Therefore, the following equivalence holds for any point $u \in X$:

$$u \in A(\{x_n\}) \iff \limsup_{n \to \infty} \rho(u, x_n) \leq \liminf_{n \to \infty} \rho(x, x_n), \quad \forall x \in X.$$  

(2.3)

The notion of $\Delta$-convergence in the following definition is taken from [11].

**Definition 2.2.** Let $\{x_n\} \subseteq X$ be a bounded sequence and let $x \in X$. The sequence $\{x_n\}$ is said to $\Delta$-converge to $x$ if, for any subsequence $\{x_{n_k}\}$ of $\{x_n\}$, the point $x$ is the unique asymptotic center of $\{x_{n_k}\}$. In this case, we write $\Delta - \lim x_n = x$ and call $x$ the $\Delta$-limit of $\{x_n\}$.

The notion of $\Delta$-convergence was introduced by Lim [11] in general metric spaces with some restrictions. In [14], it was shown that CAT(0) spaces are a natural realm for this notion, and this notion was applied to study the fixed point theory in CAT(0). Recently, Espínola and Fernández-León [15] extended this study to CAT($\kappa$) spaces and proved that this notion of convergence coincides, in this setting, with one of two notions of weak convergence introduced by Sosov [16] in geodesic metric spaces, and with weak convergence in Hilbert spaces.

Let $C$ be a subset of $X$. We use $\text{conv}(C)$ and $\text{diam}(C)$ to denote the closed convex hull and the diameter of $C$, which are respectively defined by

$$\text{conv}(C) := \bigcap \{E : E \supseteq C \text{ and } E \text{ is closed convex}\}$$

and

$$\text{diam}(C) := \sup \{\rho(x, y) : x, y \in C\}.$$  

The following proposition was proved in [15]; see [15, Proposition 4.1] for assertion (i) and [15, Corollary 4.4] for assertion (ii).

**Proposition 2.2.** Let $X$ be a complete CAT($\kappa$) space and let $\{x_n\}$ be a sequence in $X$ with $r(\{x_n\}) < \frac{\kappa}{2}$. Then the following assertions hold.

(i) $A(\{x_n\})$ consists of exactly one point.

(ii) $\{x_n\}$ has a $\Delta$-convergent subsequence.

Assertion (2.4) in the following proposition was proved in [15, Proposition 4.5] under the additional assumption that $\text{diam}(X) < D_\kappa$.

**Proposition 2.3.** Let $X$ be a complete CAT($\kappa$) space and let $p \in X$. Suppose that $\{x_n\} \subseteq X$ satisfies $r(p, \{x_n\}) < \frac{\kappa}{2}$ and that $\{x_n\}$ $\Delta$-converges to $x$. Then

$$x \in \bigcap_{n=1}^{\infty} \text{conv}(\{x_n, x_{n+1}, \ldots\})$$  

(2.4)

and

$$\rho(x, p) \leq \liminf_{n \to \infty} \rho(x_n, p).$$  

(2.5)
Proof. Without loss of generality, we assume that $\kappa = 1$. Then $D_{\kappa} = \pi$. Let $r(p, \{x_n\}) < \tilde{r} < \frac{\pi}{2}$ and let $p_0 := \frac{1}{2} p \oplus \frac{1}{2} x$. Consider the subset $X_0 := \overline{B}(p_0, \tilde{r})$. Then $\text{diam}(X_0) < \pi$, and $X_0$ is a CAT(1) space by [13, Example 1.15(1), page 167]. By the choice of $\tilde{r}$, there exists $N \in \mathbb{N}$ such that
\[
\max \{ \rho(x_n, p), \rho(x_n, x) \} < \tilde{r}, \quad \forall n > N;
\]
that is, $x, p \in B(x_n, \tilde{r})$ for each $n > N$. This, together with Proposition 2.1, implies that $p_0 \in B(x_n, \tilde{r})$, and so $\{x_n : n > N\} \subset X_0$. Moreover, since
\[
\rho(x, p) \leq r(x, \{x_n\}) + r(p, \{x_n\}) \leq 2r(x, \{x_n\}) < 2\tilde{r},
\]
it follows that $\rho(x, p_0) < \tilde{r}$ and $x \in X_0$. Hence $\{x_n\}$ $\Delta$-converges to $x$ in $X_0$ as $\{x_n\} \subset X_0$ does in $X$. Thus [15, Proposition 4.5] is applicable to $X_0$ and $\{x_n : n > N\}$ in place of $X$ and $\{x_n\}$, allowing us to conclude that
\[
x \in \bigcap_{n>N} \text{conv}(\{x_n, x_{n+1}, \ldots \}) = \bigcap_{n=1}^{\infty} \text{conv}(\{x_n, x_{n+1}, \ldots \}),
\]
and assertion (2.4) is proved.

To show assertion (2.5), set $r := \liminf_n \rho(x_n, p)$, and let $\epsilon > 0$ be such that $r + \epsilon < \frac{\pi}{2}$. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that
\[
\rho(x_{n_k}, p) < r + \epsilon < \frac{\pi}{2}, \quad \forall k \in \mathbb{N}.
\]
This means that $\{x_{n_k}\} \subset \overline{B}(p, r + \epsilon)$. By Proposition 2.1, $B(p, r + \epsilon)$ is convex. It follows that $\text{conv}(\{x_{n_k}\}) \subset \overline{B}(p, r + \epsilon)$. Applying (2.4) to $\{x_{n_k}\}$ in place of $\{u_n\}$, we have that
\[
x \in \bigcap_{k=1}^{\infty} \text{conv}(\{x_{n_k}, x_{n_k+1}, \ldots \}) \subset \text{conv}(\{x_{n_k}\}) \subset \overline{B}(p, r + \epsilon).
\]
Hence $\rho(x, p) \leq r + \epsilon$. Since $\epsilon > 0$ is arbitrary, it follows that $\rho(x, p) \leq \liminf_n \rho(x_n, p)$. The proof is complete. \quad $\Box$

3. Mann’s algorithm in CAT($k$) spaces

In the case when $X$ is a normed linear space and $T : X \rightarrow X$ is a nonexpansive mapping, for $\{x_n\} \subset [0, 1]$, Mann introduced in [1] the iteration process $\{x_n\}$ defined by the algorithm (1.1), which is known as Mann’s iteration for finding fixed points of nonexpansive mappings. Ishikawa proved in [2] that, if $0 < b \leq t_n < 1$ for each $n$ and $\sum_{n=1}^{\infty} t_n = \infty$, then $\|x_n - T(x_n)\| \rightarrow 0$ as $n \rightarrow \infty$, which implies the convergence of $\{x_n\}$ to a fixed point of $T$ if the range of $T$ lies in a compact subset of $X$, and Reich obtained in [3] the weak convergence of the sequence in a uniformly convex Banach space, assuming that $\{x_n\} \subset (0, 1)$ and $\sum_{n=1}^{\infty} t_n(1-t_n) = \infty$. For further results, see [4] and the references therein.

In the framework of metric spaces ($X, \rho$), Goebel and Kirk in [5,9] and Reich and Shafrir in [17] provided a iterative method for finding fixed points of nonexpansive mappings in spaces of hyperbolic type, whose algorithm is defined by
\[
x_{n+1} \in [x_n, T(x_n)], \quad \rho(x_n, T(x_n)) = t_n \rho(x_n, x_{n+1}), \quad \forall n \geq 0,
\]
where each $t_n \in (0, 1)$; that is, using the notation fixed in the preliminary section,
\[
x_{n+1} = t_n x_n \oplus (1-t_n)T(x_n), \quad \forall n \geq 0.
\]
(3.1)

The $\Delta$-convergence of the sequence defined by algorithm (3.1) has been studied in [12] for CAT(0) spaces. Motivated by these results, we studied here the $\Delta$-convergence of this sequence in CAT($k$) spaces. In order to get the main result, we need the following definition and lemmas.

Definition 3.1. Let $X$ be a complete metric space and let $C \subset X$ be a nonempty set. A sequence $\{x_n\} \subset X$ is called Fejér monotone with respect to $C$ if
\[
\rho(x_{n+1}, y) \leq \rho(x_n, y), \quad \forall y \in C \text{ and } \forall n \geq 0.
\]

For convenience, we say that a point $x \in X$ is a $\Delta$-cluster point of the sequence $\{x_n\}$ if there exists a subsequence of $\{x_n\}$ that $\Delta$-converges to $x$.

Lemma 3.1. Let $X$ be a complete metric space and let $C \subset X$ be a nonempty set. Suppose that the sequence $\{x_n\} \subset X$ is Fejér monotone with respect to $C$. Let $\{x_n\}$ be a subsequence of $\{x_n\}$. Then we have that $A(\{x_n\}) \cap C \subset A(\{x_n\})$. In particular, if $x \in C$ is a $\Delta$-cluster point of $\{x_n\}$ then $x \in A(\{x_n\})$. 

Proof. Let $\bar{u} \in A(\{u_n\}) \cap C$. By the assumed Fejér monotonicity, one has that $\lim_{n \to \infty} \rho(x_n, \bar{u}) = \lim_{n \to \infty} \rho(x_n, \bar{u})$. It follows from definition that, for any $u \in X$,
\[
\lim_{n \to \infty} \rho(x_n, \bar{u}) = \lim_{n \to \infty} \rho(u_n, \bar{u}) \leq \limsup_{n \to \infty} \rho(u_n, u) \leq \limsup_{n \to \infty} \rho(x_n, u).
\]
Hence $\bar{u} \in A(\{u_n\})$ by definition (see (2.3)), and the proof is complete. \hfill $\square$

Lemma 3.2. Let $X$ be a complete CAT$(\kappa)$ space and let $C \subseteq X$ be a nonempty set. Suppose that the sequence $\{u_n\} \subseteq X$ is Fejér monotone with respect to $C$ and satisfies that $r(\{u_n\}) < \frac{\kappa}{2}$. Suppose also that any $\Delta$-cluster point $x$ of $\{u_n\}$ belongs to $C$. Then $\{u_n\}$ $\Delta$-converges to a point in $C$.

Proof. By Proposition 2.2(ii), the $\Delta$-cluster point set of $\{u_n\}$ is nonempty. Let $x$ be a $\Delta$-cluster point of $\{u_n\}$. Then $x \in C$ by assumption, and so $A(\{u_n\}) = \{x\}$ thanks to Lemma 3.1 and Proposition 2.2(i). Below, we show that $\{u_n\}$ $\Delta$-converges to $x$. For this purpose, let $\{u_k\}$ be an arbitrary subsequence of $\{u_n\}$. By Proposition 2.2(i), we have that $A(\{u_k\}) = \{u\}$ for some $u \in X$. Applying Proposition 2.2(ii) to the sequence $\{u_n\}$, we conclude that $\{u_n\}$ has a $\Delta$-cluster point $u$. Clearly, $u$ is also a $\Delta$-cluster point of $\{u_n\}$, and so $u \in C$ by assumptions. It follows from Lemma 3.1 that $u \in A(\{u_n\}) \cap A(\{u_n\})$, and so $\bar{u} = u = x$. Thus, by definition, $\{u_n\}$ $\Delta$-converges to $x \in C$, and the proof is complete. \hfill $\square$

Now we give our main result in this section.

Theorem 3.1. Let $X$ be a complete CAT$(\kappa)$ space and let $T : X \to X$ be a nonexpansive mapping such that $F := \text{Fix}(T) \neq \emptyset$. Suppose that $\{x_n\} \subseteq (0, 1)$ satisfies that
\[
\sum_{n=0}^{\infty} t_n(1-t_n) = \infty. \tag{3.2}
\]
Then, for each $x_0 \in X$ with $\rho(x_0, F) < \frac{\kappa}{4}$, the sequence $\{x_n\}$ defined by (3.1) $\Delta$-converges to a point of $F$.

Proof. Without loss of generality, we assume that $\kappa = 1$. Let $\{x_n\}$ be the sequence generated by the algorithm (3.1) and let $x_0 \in X$ be such that $\rho(x_0, F) < \frac{\pi}{2}$. Write $F_0 := F \cap B(x_0, \frac{\pi}{3})$. Let $n \geq 0$ and let $q \in F_0$ be fixed. Since $\rho(T(x_0), q) \leq \rho(x_0, q)$ and since the open ball in $X$ with center $q$ and radius less than $\frac{\pi}{2}$ is convex (see Proposition 2.1), it follows that
\[
\rho(x_1, q) = \rho(t_0 x_0 \oplus (1-t_0) T(x_0), q) \leq \rho(x_0, q).
\]
By mathematical induction, we can easily show that
\[
\rho(x_{n+1}, q) \leq \rho(x_n, q) \leq \rho(x_0, q), \quad \forall n \geq 0. \tag{3.3}
\]
Hence $\{x_n\}$ is Fejér monotone with respect to $F_0$. In particular, choose $p \in F$ such that $\rho(x_0, p) < \frac{\pi}{2}$. Then $p \in F_0$ and
\[
\rho(x_{n+1}, p) \leq \rho(x_n, p) \leq \rho(x_0, p) < \frac{\pi}{2}, \quad \forall n \in \mathbb{N}. \tag{3.4}
\]
This implies that $r(\{x_n\}) < \frac{\pi}{2}$. Thus, by Lemma 3.2, it suffices to show that any $\Delta$-cluster point of $\{x_n\}$ belongs to $F_0$. For this end, let $\bar{x} \in X$ be a $\Delta$-cluster point of $\{x_n\}$. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which $\Delta$-converges to $\bar{x}$. Below, we will show that $\bar{x} \in F_0$. We first note by (3.4) that $r(\{x_{n_k}\}) \leq \rho(x_0, p) < \frac{\pi}{2}$. It follows from (2.5) in Proposition 2.3 that
\[
\rho(\bar{x}, x_0) \leq \rho(\bar{x}, p) + \rho(x_0, p) \leq \liminf_k \rho(x_{n_k}, p) + \rho(x_0, p) < \frac{\pi}{2};
\]
hence $\bar{x} \in B(x_0, \frac{\pi}{2})$. To show that $\bar{x} \in F$, it suffices to prove that
\[
\lim_{k \to \infty} \rho(x_{n_k}, T(x_n)) = 0. \tag{3.5}
\]
Indeed, if (3.5) holds, then
\[
\limsup_k \rho(T(\bar{x}), x_{n_k}) \leq \limsup_k \rho(T(\bar{x}), T(x_{n_k})) + \limsup_k \rho(T(x_{n_k}), x_{n_k}) 
\leq \limsup_k \rho(\bar{x}, x_{n_k});
\]
hence $T(\bar{x}) \in A(\{x_{n_k}\})$ by definition, and so $T(\bar{x}) = \bar{x}$; that is, $\bar{x} \in F$.

To show (3.5), we only need to show that $\{\rho(x_{n_k}, T(x_{n_k}))\}$ is monotone and that
\[
\lim_{n \to \infty} \rho(x_{n_k}, T(x_n)) = 0. \tag{3.6}
\]
Using the nonexpansivity of $T$ and the definition of $x_{n+1}$, we conclude that
\[
\rho(x_{n+1}, T(x_{n+1})) \leq \rho(x_{n+1}, T(x_n)) + \rho(T(x_n), T(x_{n+1})) \\
\leq \rho(x_{n+1}, T(x_n)) + \rho(x_n, x_{n+1}) \\
= \rho(x_n, T(x_n)) + (1 - t_n)\rho(x_n, T(x_n)) \\
= \rho(x_n, T(x_n)).
\]
This means that $\{\rho(x_n, T(x_n))\}$ is monotone. Moreover, we have by (3.4) that
\[
\rho(x_0, T(x_0)) \leq \rho(x_0, p) + \rho(p, T(x_0)) \leq 2\rho(x_0, p) < \frac{\pi}{2},
\]
and so
\[
\rho(x_n, T(x_n)) \leq \rho(x_0, T(x_0)) < \frac{\pi}{2}, \quad \forall n \in \mathbb{N}. \tag{3.7}
\]
It remains to show assertion (3.6). For this end, we fix $n \in \mathbb{N}$ and write $q_n := T(x_n)$. Then $\rho(p, q_n) \leq \rho(p, x_n) < \frac{\pi}{4}$ by (3.4) and the nonexpansivity of $T$. Therefore, we get that
\[
\rho(x_n, p) + \rho(x_n, q_n) + \rho(q_n, p) \leq 4\rho(x_n, p) \leq 4\rho(x_0, p) < \pi.
\]
Thus we can take $\Delta(\bar{x}_n, \bar{p}, \bar{q}_n)$ to be a comparison triangle of $\Delta(x_n, p, q_n)$ in $S^2$. Hence,
\[
\rho(x_n, p) = d(\bar{x}_n, \bar{p}), \quad \rho(q_n, p) = d(\bar{q}_n, \bar{p}) \quad \text{and} \quad \rho(x_n, q_n) = d(\bar{x}_n, \bar{q}_n). \tag{3.8}
\]
Let $\tilde{y}_{n+1} := t_n\bar{x}_n \oplus (1 - t_n)\bar{q}_n \in [\bar{x}_n, \bar{q}_n]$ be the comparison point of $x_{n+1} = t_nx_n \oplus (1 - t_n)q_n$. Then
\[
d(\tilde{y}_{n+1}, \bar{x}_n) = (1 - t_n)d(\bar{x}_n, \bar{q}_n) \quad \text{and} \quad d(\tilde{y}_{n+1}, \bar{q}_n) = t_n d(\bar{x}_n, \bar{q}_n). \tag{3.9}
\]
Moreover, by the CAT(1) inequality, we have that
\[
\rho(x_{n+1}, p) \leq d(\tilde{y}_{n+1}, \bar{p}). \tag{3.10}
\]
Note that points $\bar{x}_n$, $\bar{q}_n$ and $\tilde{y}_{n+1}$ are located in the two-dimensional subspace of the Euclidean space $\mathbb{R}^3$ spanned by $\{\bar{x}_n, \bar{q}_n\}$, and that the point $\tilde{y}_{n+1}$ is in the arc connecting $\bar{x}_n$ and $\bar{q}_n$. Then, by elementary geometry, $\tilde{y}_{n+1}$ can be expressed as
\[
\tilde{y}_{n+1} = \alpha_n\bar{x}_n + \beta_n\bar{q}_n, \tag{3.11}
\]
where $\alpha_n$ and $\beta_n$ are positive numbers satisfying $\alpha_n + \beta_n \geq 1$. By the definition of the metric in (2.1),
\[
\cos d(\tilde{y}_{n+1}, \bar{x}_n) = \langle \tilde{y}_{n+1}, \bar{x}_n \rangle = \langle \alpha_n\bar{x}_n + \beta_n\bar{q}_n, \bar{x}_n \rangle = \alpha_n + \beta_n \cos d(\bar{x}_n, \bar{q}_n). \tag{3.12}
\]
and
\[
\cos d(\tilde{y}_{n+1}, \bar{q}_n) = \langle \tilde{y}_{n+1}, \bar{q}_n \rangle = \langle \alpha_n\bar{x}_n + \beta_n\bar{q}_n, \bar{q}_n \rangle = \beta_n + \alpha_n \cos d(\bar{x}_n, \bar{q}_n). \tag{3.13}
\]
Combining the two equalities above and (3.9) gives that
\[
\begin{cases}
\alpha_n + \beta_n \cos d(\bar{x}_n, \bar{q}_n) = \cos((1 - t_n)d(\bar{x}_n, \bar{q}_n)) \\
\beta_n + \alpha_n \cos d(\bar{x}_n, \bar{q}_n) = \cos(t_n d(\bar{x}_n, \bar{q}_n)).
\end{cases}
\]
Hence,
\[
\alpha_n + \beta_n = \frac{\cos(t_n d(\bar{x}_n, \bar{q}_n)) + \cos((1 - t_n)d(\bar{x}_n, \bar{q}_n))}{1 + \cos d(\bar{x}_n, \bar{q}_n)}. \tag{3.14}
\]
By (2.1) and (3.11), we conclude that
\[
\cos d(\tilde{y}_{n+1}, \bar{p}) = \langle \tilde{y}_{n+1}, \bar{p} \rangle = \alpha_n\langle \bar{x}_n, \bar{p} \rangle + \beta_n\langle \bar{q}_n, \bar{p} \rangle = \alpha_n \cos d(\bar{x}_n, \bar{p}) + \beta_n \cos d(\bar{q}_n, \bar{p}). \tag{3.15}
\]
By the nonexpansivity of $T$ and (3.8), one sees that
\[
0 \leq d(\bar{q}_n, \bar{p}) = \rho(q_n, p) \leq \rho(x_n, p) \leq d(\bar{x}_n, \bar{p}) < \frac{\pi}{4}.
\]
It follows that
\[
0 \leq d(\bar{q}_n, \bar{p}) \geq \cos d(\bar{x}_n, \bar{p}) \geq \cos d(\bar{x}_n, \bar{p}) > \frac{\sqrt{2}}{2}. \tag{3.16}
\]
The first inequality in (3.16) together with (3.15) entails that
\[
\cos d(\tilde{y}_{n+1}, \bar{p}) \geq (\alpha_n + \beta_n) \cos d(\bar{x}_n, \bar{p}).
Substituting expression (3.14) of $\alpha_n + \beta_n$ into the above inequality, we get that
\[
\cos d(\tilde{y}_{n+1}, \tilde{p}) \geq \frac{\cos(t_0 d(\tilde{x}_n, \tilde{q}_n)) + \cos((1 - t_0) d(\tilde{x}_n, \tilde{q}_n))}{1 + \cos d(\tilde{x}_n, \tilde{q}_n)} \cos d(\tilde{x}_n, \tilde{p}).
\] (3.17)

For simplicity, we write
\[
a_n := d(\tilde{x}_n, \tilde{q}_n)
\]
and
\[
s_n := \frac{\cos(t_0 a_n) + \cos((1 - t_0) a_n) - 1 - \cos a_n}{1 + \cos a_n}.
\]

Then $a_n \in [0, \frac{x}{2})$ by (3.7) and, by elementary trigonometry,
\[
s_n = \frac{2 \sin \frac{t_0 a_n}{2} \sin \left(\frac{1 - t_0 a_n}{2}\right)}{\cos \frac{a_n}{2}} \geq 0.
\] (3.18)

Furthermore, by (3.8) and the definition of $q_n$, condition (3.6) is equivalent to
\[
\liminf_{n \to +\infty} a_n = 0.
\] (3.19)

Thus, to complete the proof, we only need to show that condition (3.19) holds. For this purpose, we note that (3.8), (3.10), (3.16) and (3.17) imply that
\[
\cos \rho(x_{n+1}, p) - \cos \rho(x_n, p) \geq \cos d(\tilde{y}_{n+1}, \tilde{p}) - \cos d(\tilde{x}_n, \tilde{p}) \geq s_n \cos d(\tilde{x}_n, \tilde{p}) > \frac{\sqrt{2}}{2} s_n.
\] (3.20)

It follows that
\[
\sum_{n=0}^{\infty} s_n \leq \frac{\sqrt{2}}{2} \sum_{n=0}^{\infty} (\cos \rho(x_{n+1}, p) - \cos \rho(x_n, p)) < \infty.
\] (3.21)

Now, we suppose on the contrary that (3.19) does not hold. Then, there exist $a > 0$ and integer $N \in \mathbb{N}$ such that $a_n > a$ holds for each $n > N$. Since each $a_n < \frac{x}{2}$ as noted earlier, it follows from (3.18) that
\[
s_n \geq \frac{2 \sin \frac{t_0 a_n}{2} \sin \left(\frac{1 - t_0 a_n}{2}\right)}{t_0(1 - t_0) \cos \frac{a_n}{2}}, \quad \forall t \geq N.
\] (3.22)

Consider the function $h$ on $(0, 1)$ defined by
\[
h(t) := \frac{2 \sin t \cdot \sin \left(\frac{1-t}{2}\right)}{t(1-t)}, \quad \forall t \in (0, 1).
\]

Then $h$ is positive, continuous on $(0, 1)$, and
\[
\lim_{t \to 0^+} h(t) = \lim_{t \to 1^-} h(t) = a \sin \frac{a}{2}.
\]

Hence, $h_{\min} := \inf_{t \in (0, 1)} h(t) > 0$. This, together with (3.22), implies that
\[
\liminf_{n \to +\infty} \frac{s_n}{t_0(1 - t_0) \cos \frac{a_n}{2}} \geq \frac{h_{\min}}{2} > 0.
\]

Since $\sum_{n=0}^{\infty} t_0(1 - t_0) = \infty$ by assumption (3.2), it follows that $\sum_{n=0}^{\infty} s_n = \infty$, which contradicts (3.21). Therefore, (3.19) holds, and the proof is complete. $\Box$

Recall that $X$ is boundedly compact if each closed bounded subset of $X$ is compact.

**Corollary 3.1.** Let $X$ be a boundedly compact, complete CAT(κ) space and let $T : X \to X$ be a nonexpansive mapping such that $F := \text{Fix}(T^\infty) = \emptyset$. Suppose that $\{\tilde{x}_n\} \subset (0, 1)$ satisfies (3.2). Then, for each $x_0 \in X$ with $\rho(x_0, F) < \frac{\kappa}{4}$, the sequence $\{\tilde{x}_n\}$ defined by (3.1) converges strongly to a point of $F$.

**Proof.** Let $\{\tilde{x}_n\}$ be the sequence generated by (3.1) and let $x_0 \in X$ be such that $\rho(x_0, F) < \frac{\kappa}{4}$. Then, by **Theorem 3.1**, $\{\tilde{x}_n\}$ is bounded and $\Delta$-converges to $x \in F$. Suppose on the contrary that $\{\tilde{x}_n\}$ does not converge strongly to $x$. Then, by the
compactness assumption, one can find a converging subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a point $\bar{x} \in X$ with $\bar{x}' = x$ such that $x_{n_k} \to \bar{x}$ strongly. Therefore,
\[
\lim_{k \to \infty} \rho(x_{n_k}, \bar{x}) = 0 \leq \lim_{k \to \infty} \rho(x_{n_k}, x).
\]
Since $x$ is the unique asymptotic center of $\{x_{n_k}\}$, it follows that $\bar{x} \in A(\{x_{n_k}\})$ and $\bar{x} = x$, which is a contradiction. Thus we complete the proof. □

The following corollary is a direct consequence of Theorem 3.1 and Corollary 3.1, which extends the corresponding ones in [8,12].

**Corollary 3.2.** Let $X$ be a complete CAT(0) space and let $T : X \to X$ be a nonexpansive mapping such that $\text{Fix}(T) = \emptyset$ Suppose that $\{x_n\} \subset (0, 1)$ is such that (3.2) holds. Then, for each $x_0 \in X$, the sequence $\{x_n\}$ defined by (3.1) $\Delta$-converges to a fixed point of $T$. Furthermore, if $X$ is additionally boundedly compact, then the sequence $\{x_n\}$ converges strongly to a fixed point of $T$.

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**References**


