

WEAK SHARP MINIMA ON RIEMANNIAN MANIFOLDS

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Abstract. This is the first paper dealing with the study of weak sharp minima for constrained optimization problems on Riemannian manifolds, which are important in many applications. We consider the notions of local weak sharp minima, boundedly weak sharp minima, and global weak sharp minima for such problems and establish their complete characterizations in the case of convex problems on finite-dimensional Riemannian manifolds and Hadamard manifolds. A number of the results obtained in this paper are also new for the case of conventional problems in finite-dimensional Euclidean spaces. Our methods involve appropriate tools of variational analysis and generalized differentiation on Riemannian and Hadamard manifolds developed and efficiently implemented in this paper.

Key words. Variational analysis and optimization, Weak sharp minima, Riemannian manifolds, Hadamard manifolds, Convexity, Generalized differentiability

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1. Introduction. A vast majority of problems considered in optimization theory are formulated in finite-dimensional or infinite-dimensional Banach spaces, where the *linear structure* plays a crucial role to employ conventional tools of variational analysis and (classical or generalized) differentiation to deriving optimality conditions and then developing numerical algorithms. At the same time many optimization problems arising in various applications cannot be posed in linear spaces and require a *Riemannian manifold* (in particular, a *Hadamard manifold*) structure for their formalization and study. Among numerous problems of this type we mention geometric models for human spine [3], eigenvalue optimization problems [16, 49, 61], nonconvex and nonsmooth problems of constrained optimization in \mathbb{R}^n that can be reduced to convex and smooth unconstrained optimization problems on Riemannian manifolds as in [20, 28, 52, 58, 62], etc. We refer the reader to [2, 3, 7, 25, 34, 49, 53, 61, 62] and the bibliographies therein for various results, examples, discussions, and applications.

It is worth recalling that a strong interest in optimization problems formulated on Riemannian manifolds goes back to the very beginning of modern variational analysis; it was one of the *crucial motivations* for developing the fundamental *Ekeland variational principle* [26] in the framework of complete metric spaces, with *no linear structure*. The seminal Ekeland's paper [26] contains applications of his variational principle to the existence of minimal geodesics on Riemannian manifolds; see also [27] for further developments. More

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recently, a number of important results have been obtained on various aspects of optimization theory and applications for problems formulated on Riemannian and Hadamard manifolds as well as on other spaces with nonlinear structures; see, e.g., [1, 3, 10, 7, 21, 25, 30, 34, 47, 45, 49, 61, 62] and the references therein. Let us particularly mention Newton's method, the conjugate gradient method, the trust-region method, and their modifications extended from optimization problems on linear spaces to their Riemannian counterparts. On the other hand, the maximal monotonicity notion in Banach spaces extended to Riemannian manifolds makes it possible to develop a proximal-type method to find singular points for multivalued vector fields on Riemannian manifolds with nonpositive sectional curvatures, i.e., on Hadamard manifolds; see, e.g., [45] with other references. Furthermore, various derivative-like and subdifferential constructions for nondifferentiable functions on spaces with no linear structure are developed in [4, 6, 29, 41, 50, 51, 55] and applied therein to the study of constrained optimization problems, nonclassical problems of the calculus of variations and optimal control, and generalized solutions to the first-order partial differential equations on Riemannian manifolds and other important classes of spaces with no linearity.

This paper is devoted to the study of *weak sharp minimizers* for constrained optimization problems on *Riemannian* and *Hadamard manifolds*. To the best of our knowledge, it is the first work concerning the notions of this type for optimization problems on spaces with no linear structure. Recall that the notion of *sharp minima* is introduced by Polyak [57] in the case of finite-dimensional Euclidean spaces for the analysis of perturbation behavior of optimization problems and the convergence analysis of some numerical algorithms; a related notion of "strongly unique local minimum" can be found in the paper by Cromme [19]. Then Ferris [31] introduces in the same framework the notion of *weak sharp minima* to describe an extension of sharp minimizers in order to include the possibility of multiple solutions. The latter notion has been extensively studied by many authors in finite-dimensional and infinite-dimensional linear spaces. Primary motivations for these studies relate to sensitivity analysis [15, 16, 37, 43, 56, 66, 67, 68] and to convergence analysis of a broad range of optimization algorithms [11, 12, 17, 19, 32, 33, 35]. In particular, Burke and Ferris [11] derive necessary optimality conditions for weak sharp minimizers, obtaining also their full characterizations in the case of convex problems of unconstrained minimization, with applications to convex programming and convergence analysis in finite-dimensional Euclidean spaces. Then Burke and Deng [13] extend necessary optimality conditions and characterization results from [11] to problems of constrained optimization in Banach spaces, study asymptotic properties of weak sharp minima in terms of associated recession functions, and establish some new characterizations of local weak sharp minimizers and the so-called boundedly weak sharp minimizers. Furthermore, in [14] they explore relationships between the notions of weak sharp minima, linear regularity, and error bounds. Linear regularity has been extensively studied in [8, 9], where its importance for designing algorithms has been revealed. Note that linear regularity is closely related to metric regularity and error bounds for convex inequalities that have been comprehensively studied by many authors; see, e.g., [5, 22, 39, 40, 43, 46, 53, 54, 70, 71] and the references therein.

In the linear space setting the characterizations of weak sharp minimizers for convex optimization problems have been obtained in two interrelated terms: one via the directional derivative of convex functions and the other via the normal cone of convex analysis to the corresponding solution set \bar{S} ; see [11, 13]. The key ingredients to derive these characterizations are the following well-known representations in convex analysis on \mathbb{R}^n : of the *subdifferential* of the *distance function* $d_{\bar{S}}(\cdot)$ to \bar{S} given by

$$\partial d_{\bar{S}}(x) = \mathbb{B} \cap N_{\bar{S}}(x) \quad \text{for all } x \in \bar{S} \quad (1.1)$$

via the normal cone $N_{\bar{S}}(\cdot)$ to \bar{S} and the unit ball \mathbb{B} in the space in question, and of the *projection operator*

$P(\cdot|\bar{S})$ associated with the above solution set by

$$y \in P(x|\bar{S}) \iff \langle x - y, z - y \rangle \leq 0 \text{ for all } z \in \bar{S}. \quad (1.2)$$

One of the *primary goals* of this paper is to develop the aforementioned characterizations for appropriately defined notions of weak sharp minima for convex problems on Riemannian manifolds. However, significant technical difficulties arise in this way from the very beginning: the underlying representations are not known to hold on Riemannian manifolds. In particular, the distance function $d_{\bar{S}}(\cdot)$ may *not be convex* when the solution set \bar{S} is convex in the case Riemannian manifolds. Our approaches in this paper to characterizing weak sharp minimizers are largely different from those used under linear structures. We involve a Riemannian counterpart of (1.2) employing *variational fields*, which do not depend on local charts on Riemannian manifolds. Furthermore, an analog of equality (1.1) for convex sets is derived below for *weakly convex* sets in *Hadamard manifolds* exploiting their *nonpositive sectional curvatures*. Based on these and other developments, we establish *full characterizations* of global, local, and boundedly weak sharp minima for convex constrained optimization problems on Riemannian and Hadamard manifolds. Some of the characterizations obtained in this paper are appropriate extensions of known ones for spaces with linear structures, while a number of our results are new even for the case of finite-dimensional Euclidean spaces.

As follows from the above descriptions, the main impact of this paper is to the theory of constrained optimization on Riemannian and Hadamard manifolds, which is a highly demanded while largely underinvestigated area of modern optimization. In our future research we intend to address numerical applications of the results obtained in manifolds, taking into account well-recognized ones for optimization problems in linear spaces that in fact motivated theoretical developments on sharp and weak sharp minimizers.

The rest of the paper is organized as follows. In Section 2 we present some basic constructions and preliminaries in linear spaces, mostly for convex functions and sets, widely used in the sequel. Section 3 is devoted to the *Riemannian manifold theory* and contains, together with certain known constructions and facts important in what follows, some new results on Riemannian manifolds that play a crucial role for the subsequent characterizations of weak sharp minima. In particular, we establish new descriptions of *projections* for closed subsets in Riemannian manifolds, via verifiable conditions on minimizing geodesics, and obtain their complete characterizations and other useful consequences in the presence of convexity.

Sections 4 and 5 present the *main results* of the paper. In Section 4 we define the notions of local weak sharp minima, boundedly weak sharp minima, and global weak sharp minima on *general Riemannian manifolds* presenting also their equivalent descriptions in the case of *convex* problems of constrained optimization. Then we derive a number of their characterizations in terms of the appropriate directional derivative, subdifferential, and normal cone constructions of convex analysis on Riemannian manifolds. Section 5 is devoted to weak sharp minimizers and their modifications for convex constrained problems on *Hadamard manifolds*. We establish a Hadamard manifold counterpart of representation (1.1) and on its base derive new characterizations for the aforementioned notions of weak sharp minima.

In Section 6 we first illustrate how the developed theory applies to particular optimization problems on Riemannian manifolds by two *nontrivial examples*. Also in this section we present an application of our characterizations of weak sharp minima to *solution stability* with respect to perturbations. The final Section 7 contains *concluding discussions* of the main results obtained in the paper, their comparison with known results in the case of linear spaces, and also addresses some forthcoming developments for weak sharp minima in *nonconvex* problems on Riemannian and Hadamard manifolds extending recent results in this

direction for spaces with linear structures.

2. Some preliminaries in linear spaces. For the reader's convenience we review in this section some conventional notions, notation, and facts from *convex and variational analysis* in linear spaces used in what follows; see, e.g., [10, 53, 59] for more details. Let X be a normed space with the norm $\|\cdot\|$ and the canonical pairing $\langle \cdot, \cdot \rangle$ between X and its topological dual X^* . The symbol \mathbb{B} always stands for the closed unit ball in the space in question. Given a nonempty set $C \subset X$, we denote its *interior* and *closure* by $\text{int } C$ and $\text{cl } C$, respectively. The *conic hull* generated by C and the *polar* to X are defined, respectively, by

$$\text{cone } C := \bigcup_{\lambda \geq 0} \{\lambda C\} \quad \text{and} \quad C^\circ := \{x^* \in X^* \mid \langle x^*, x \rangle \leq 1 \text{ for all } x \in C\}.$$

The *indicator function* $\delta_C(\cdot)$ of the set $C \subset X$ is given by

$$\delta_C(x) := \begin{cases} 0 & x \in C, \\ \infty & \text{otherwise,} \end{cases}$$

and the *support function* $\sigma_C(\cdot)$ of the set C is defined by

$$\sigma_C(x^*) := \sup_{x \in C} \langle x^*, x \rangle \quad \text{for all } x^* \in X^*.$$

By $d_C(x) := \inf\{\|x - c\| \mid c \in C\}$ we denote the *distance function* of the set C . Note that the support function is always convex, while the indicator and distance function associated with the set are convex if and only if the set is convex. The results presented in the next proposition are proved in [13, Theorem A.1].

PROPOSITION 2.1. (Properties of support and distance functions of convex sets). *Let E, F be two convex subsets of X^* , and let K be a nonempty closed convex cone in X . The following assertions hold:*

- (i) $\sigma_E(x) \leq \sigma_F(x)$ for each $x \in K$ if and only if $E \subset \text{cl}(F + K^\circ)$.
- (ii) For all $x \in X$ we have the relationship $d_K(x) = \sigma_{\mathbb{B} \cap K^\circ}(x)$.

Consider now an extended-real-valued function $g: X \rightarrow \overline{\mathbb{R}} := (-\infty, \infty]$ with the *effective domain* $\text{dom } g := \{x \in X \mid g(x) < \infty\}$ and the *epigraph* of g defined by

$$\text{epi } g := \{(x, r) \in X \times \mathbb{R} \mid g(x) \leq r\}.$$

Unless otherwise stated, we assume in what follows that g is *proper*, i.e., $\text{dom } g \neq \emptyset$.

Recall further that a function $g: X \rightarrow \overline{\mathbb{R}}$ is *lower semicontinuous* (l.s.c.) on X if its epigraph $\text{epi } g$ is closed in $X \times \mathbb{R}$. The *lower semicontinuous hull*, or the *closure* of g , is the function $\text{cl } g: X \rightarrow \overline{\mathbb{R}}$ with

$$\text{epi}(\text{cl } g) = \text{cl}(\text{epi } g),$$

which is the greatest l.s.c. function not exceeding g .

Let $g: X \rightarrow \overline{\mathbb{R}}$ be convex and proper. The *directional derivative* of g at the point $x \in \text{dom } g$ in the direction $v \in X$ is defined by

$$g'(x; v) := \lim_{t \rightarrow 0^+} \frac{g(x + tv) - g(x)}{t} \tag{2.1}$$

while the *subdifferential* of g at $x \in \text{dom } g$ is constructed by

$$\partial g(x) := \{x^* \in X^* \mid \langle x^*, y - x \rangle \leq g(y) - g(x) \text{ for all } y \in \text{dom } g\}.$$

Note that the limit in (2.1) exists and equals

$$\lim_{t \rightarrow 0^+} \frac{g(x + tv) - g(x)}{t} = \inf_{t > 0} \frac{g(x + tv) - g(x)}{t}$$

if there is $t > 0$ such that $x + tv \in \text{dom } g$; otherwise we have $g'(x; v) = \infty$ in (2.1).

The following proposition is also well known in convex analysis; see, e.g., [69, Corollary 2.4.15].

PROPOSITION 2.2. (Relationship between the subdifferentials and the directional derivatives of convex functions). *Let $g: X \rightarrow \overline{\mathbb{R}}$ be convex and proper, and let $x \in \text{dom } g$ be such that $\partial g(x) \neq \emptyset$. Then we have*

$$\sigma_{\partial g(x)}(\cdot) = \text{cl}g'(x; \cdot). \quad (2.2)$$

3. Auxiliary results in Riemannian manifolds. This section contains necessary material in Riemannian manifolds needed for obtaining the main results on weak sharp minima in the subsequent sections. We start with basic definitions and reviewing the required known facts referring the reader to [24, 36] for more details and then derive new results of their own interest that play a crucial role in what follows. For simplicity our considerations are confined to *finite-dimensional* Riemannian manifolds, while it is worth mentioning that the major results obtained below admit natural extensions to infinite-dimensional settings by using advanced variational principles and techniques of modern *variational analysis* in infinite-dimensional spaces; see, e.g., [26, 27, 53] and the references therein.

Let M be a complete connected m -dimensional Riemannian manifold. By ∇ we denote the *Levi-Civita connection* on M . The collection of all *tangent vectors* of M at p forms an m -dimensional vector space and is denoted by $T_p M$. The union $\bigcup_{p \in M} (\{p\} \times T_p M)$ forms a new manifold, which is called the *tangent bundle* of M and is denoted by TM . Recall that a *Riemannian metric* on a smooth manifold M is a 2-tensor field that is symmetric and positively definite. Every Riemannian metric thus determines an inner product and a norm on each tangent space $T_p M$, which are typically written as $\langle \cdot, \cdot \rangle_p$ and $\| \cdot \|_p$, where the subscript p may be omitted if no confusion arises. In this way we can treat the tangent space $T_p M$ for each $p \in M$ as a usual finite-dimensional space denoting by \mathbb{B}_p the closed unit ball of $T_p M$, i.e., $\mathbb{B}_p := \{v \in T_p M \mid \|v\| \leq 1\}$.

Given two points $p, q \in M$, let $\gamma: [0, 1] \rightarrow M$ be a piecewise smooth curve connecting p and q . Then we define the *arc-length* $l(\gamma)$ of γ and the *Riemannian distance* from p to q by, respectively,

$$l(\gamma) := \int_0^1 \|\gamma'(t)\| dt \quad \text{and} \quad d(p, q) := \inf_{\gamma} l(\gamma),$$

where the infimum is taken over all piecewise smooth curves $\gamma: [0, 1] \rightarrow M$ connecting p and q . Thus (M, d) is a *complete metric space* by the Hopf-Rinow theorem; see, e.g., [24, p.146, Theorem 2.8]. Taking into account that M is complete, the *exponential map* at p denoted by $\exp_p: T_p M \rightarrow M$ is well-defined on $T_p M$. Recall further that a geodesic $\gamma(\cdot)$ on M connecting p and q is called a *minimizing geodesic* if its arc-length equals the Riemannian distance between p and q . It is easy to see that a curve $\gamma: [0, 1] \rightarrow M$ is a minimizing geodesic connecting p and q *if and only if* there is a vector $v \in T_p M$ such that

$$\|v\| = d(p, q) \quad \text{and} \quad \gamma(t) = \exp_p(tv) \quad \text{for each } t \in [0, 1].$$

The symbols $\mathbf{B}(p, r)$ and $\overline{\mathbf{B}(p, r)}$ denote, respectively, the *open metric ball* and the *closed metric ball* centered at the point $p \in M$ with radius $r > 0$, i.e.,

$$\mathbf{B}(p, r) := \{q \in M \mid d(p, q) < r\} \quad \text{and} \quad \overline{\mathbf{B}(p, r)} := \{q \in M \mid d(p, q) \leq r\}.$$

Given a nonempty subset D of M , define the *distance function* $d_D(\cdot): M \rightarrow [0, \infty)$ associated with D by

$$d_D(x) := \inf \{d(x, y) \mid y \in D\} \quad \text{for each } x \in M,$$

and consider the *projection* $P(x|D)$ of $x \in M$ on the set D formed by all points of D closest to x as measured by the corresponding distance, i.e.,

$$P(x|D) := \{y \in D \mid d(x, y) = d_D(x)\}.$$

Observe that $P(x|D) \neq \emptyset$ whenever D is closed due to the assumed finite dimensionality of M . It is not hard to show, similarly to the proof for the case of the standard distance function on linear spaces, that the Riemannian counterpart $d_D(x)$ is *globally Lipschitzian* on M .

PROPOSITION 3.1. (Lipschitz continuity of the distance function on Riemannian manifolds.) *Whenever $\emptyset \neq D \subset M$ in the Riemannian manifold M , the associated distance function $d_D(\cdot)$ satisfies the global Lipschitz condition on M with Lipschitz constant $\ell = 1$, i.e.,*

$$|d_D(x) - d_D(y)| \leq d(x, y) \quad \text{for all } x, y \in M. \quad (3.1)$$

Proof. Let $x, y \in M$. Then for each $z \in D$ we have

$$d_D(x) \leq d(x, z) \leq d(y, z) + d(x, y),$$

which implies that $d_D(x) \leq d_D(y) + d(x, y)$ and similarly we have $d_D(y) \leq d_D(x) + d(x, y)$. Combining the latter two inequalities we arrive at (3.1) and complete the proof of the proposition. \square

Following [63], we recall now the notions of *weakly convex*, *strongly convex*, and *locally convex* subsets of Riemannian manifolds that play a significant role in the paper. Note to this end that the *uniqueness* of geodesics is always understood up to an equivalent parameter transformation.

DEFINITION 3.2. (Convexity notions for subsets of Riemannian manifolds.) *Let D be a nonempty subset of the Riemannian manifold M . We say that:*

(a) D is **WEAKLY CONVEX** if for any $x, y \in D$ there is a minimizing geodesic of M connecting x, y , and this geodesic entirely belongs to D .

(b) D is **STRONGLY CONVEX** if for any $x, y \in D$ there is a unique minimizing geodesic on M connecting x, y , and this geodesic entirely belongs to D .

(c) D is **LOCALLY CONVEX** if for any $x \in \text{cl } D$ there is a positive $\varepsilon > 0$ such that the set $D \cap \mathbf{B}(x, \varepsilon)$ is strongly convex.

We clearly have the following implications held for subsets of Riemannian manifolds: strong convexity \implies weak convexity \implies local convexity. All these implications are generally strict. Note, in particular, that the union of two closed disjoint strongly convex sets is locally convex while not weakly convex. Observe also

that the manifold M itself is weakly convex, since it is assumed to be complete and connected, but may not be strongly convex.

Let $x \in M$ and $r > 0$. Recall from [24, p. 72] that $\mathbf{B}(x, r)$ is a *totally normal ball* around x if there is $\eta > 0$ such that for each $y \in \mathbf{B}(x, r)$ we have $\exp_y(\mathbf{B}(0, \eta)) \supset \mathbf{B}(x, r)$ and $\exp_y(\cdot)$ is a diffeomorphism on $\mathbf{B}(0, \eta) \subset T_y M$. The supremum of the radii r of totally normal balls around x is denoted by \mathbf{r}_x , i.e.,

$$\mathbf{r}_x := \sup \{r > 0 \mid \mathbf{B}(x, r) \text{ is a totally normal ball around } x\}. \quad (3.2)$$

Here we call \mathbf{r}_x the *totally normal radius* of x . By [24, Theorem 3.7] the totally normal radius \mathbf{r}_x is well defined and positive. The next proposition follows directly from [64, Lemma 3.6] and is useful in the sequel.

PROPOSITION 3.3. (Gradients of distance functions on Riemannian manifolds.) *Let $z_0 \in M$, and let \mathbf{r}_{z_0} be the totally normal radius of z_0 . Then the gradient of the distance function is computed by*

$$\text{grad } d(z_0, \cdot)(z) = -\frac{\exp_z^{-1} z_0}{d(z_0, z)} \quad \text{for each } z \in \mathbf{B}(z_0, \mathbf{r}_{z_0}) \setminus \{z_0\}. \quad (3.3)$$

Throughout the paper we denote by Γ_{xy} the set of all geodesics $\gamma: [0, 1] \rightarrow M$ with $\gamma(0) = x$ and $\gamma(1) = y$; the symbol Γ_{xy}^D stands for the set of geodesics satisfying the conditions $\gamma \in \Gamma_{xy}$ and $\gamma \subset D$. Note that Γ_{xy} is nonempty for all $x, y \in M$, since M is assumed to be complete and connected.

The next result, that plays an important technical role in our subsequent considerations, has been recently proved in [48, Theorems 3.1 and 3.2]. Following the referee's suggestion, we present a new and much shorter proof of the main part of this result.

PROPOSITION 3.4. (Characterizations of projections on convex subsets of Riemannian manifolds.) *Let D be a closed subset of M , and let $y \in D$. The following assertions hold:*

(i) *Pick a point $x \in M$ and a minimizing geodesic $\gamma_{xy} \in \Gamma_{xy}$. Then the inclusion $y \in P(x|D)$ yields*

$$\langle \gamma'_{xy}(1), \gamma'_{yz}(0) \rangle \geq 0 \quad \text{for all } z \in D \quad \text{and } \gamma_{yz} \in \Gamma_{yz}^D. \quad (3.4)$$

(ii) *If furthermore D is weakly convex, then there is $\varepsilon > 0$ such that for each $x \in \mathbf{B}(y, \varepsilon)$ we have the inclusion $y \in P(x|D)$ whenever (3.4) holds for some minimizing geodesic $\gamma_{xy} \in \Gamma_{xy}$.*

Proof. To justify (i), take $y \in P(x|D)$ and let $r := \mathbf{r}_y$ be the totally normal radius of y . Then $r > 0$ as noted above. Consider further a minimizing geodesic $\gamma \in \Gamma_{xy}$ connecting x, y and let $\bar{x} := \gamma(\bar{t})$ for some $\bar{t} \in (0, 1)$ be such that $\bar{x} \in \mathbf{B}(y, r)$. Then we have that $y \in P(\bar{x}|D)$ and $(1 - \bar{t})\gamma'_{xy}(1) = -\exp_y^{-1} \bar{x}$ by definitions. Define the function $h: \mathbf{B}(y, r) \rightarrow \mathbb{R}$ by

$$h(z) := \frac{1}{2} d^2(\bar{x}, z) \quad \text{for each } z \in \mathbf{B}(y, r)$$

and conclude by Proposition 3.3 when $z \neq \bar{x}$ and by the definition when $z = \bar{x}$ that

$$\text{grad } h(z) = -\exp_z^{-1} \bar{x} \quad \text{for each } z \in \mathbf{B}(y, r). \quad (3.5)$$

Taking $z \in D$ and $\gamma_{yz} \in \Gamma_{yz}^D$ gives us by elementary calculations that

$$\left. \frac{dh(\gamma_{yz}(t))}{dt} \right|_{t=0} = \langle -\exp_y^{-1} \bar{x}, \gamma'_{yz}(0) \rangle = \langle (1 - \bar{t})\gamma'_{xy}(1), \gamma'_{yz}(0) \rangle. \quad (3.6)$$

The inclusion $y \in P(x|D)$ means that the function $h(\gamma_{yz}(\cdot))$ attains its local minimum at zero. Combining the latter with (3.6), we get that $\langle \gamma'_{xy}(1), \gamma'_{yz}(0) \rangle \geq 0$, which yields (3.4) and thus completes the proof of assertion (i) of the proposition. Assertion (ii) follows from [48, Theorem 3.2]. \square

Let D be a weakly convex subset in M , and let $x \in D$. Recall that a vector $v \in T_x M$ is *tangent* to D if there is a smooth curve $\gamma: [0, \varepsilon) \rightarrow D$ such that $\gamma(0) = x$ and $\gamma'(0) = v$. Then the collection $T_x D$ of all tangent vectors to D at x is a *convex cone* in the space $T_x M$; see [62, p. 71]). This implies that $(N_D(x))^\circ = \text{cl}(T_x D)$ for the *dual/polar cone* $N_D(x) := (T_x D)^\circ$ to $T_x D$ given by

$$N_D(x) := \{w \in T_x M \mid \langle w, v \rangle \leq 0 \text{ for all } v \in T_x D\}. \quad (3.7)$$

Let further $\mathcal{F}_x D$ stand for the set of *geodesic feasible directions* of D at x given by

$$\mathcal{F}_x D := \{v \in T_x M \mid \text{there exists } \bar{t} > 0 \text{ such that } \exp_x tv \in D \text{ for all } t \in [0, \bar{t})\}.$$

The following proposition clarifies the usefulness of the latter construction, which is applied below.

PROPOSITION 3.5. (Equivalence expression of normal cones.) *Suppose that D is a weakly convex subset of M , and let $x \in D$. Then*

$$\mathcal{F}_x D \subset T_x D \subset \text{cl}(\mathcal{F}_x D) \text{ and thus } N_D(x) = (\mathcal{F}_x D)^\circ. \quad (3.8)$$

Proof. The second equality in (3.8) is a direct consequence of the first assertion—the only one we need to prove. It immediately follows from the definitions that $T_x D \supset \mathcal{F}_x D$. To justify the inclusion $T_x D \subset \text{cl}(\mathcal{F}_x D)$, let $v \in T_x D$ and find a smooth curve $\gamma: [0, \varepsilon) \rightarrow D$ satisfying $\gamma(0) = x$ and $\gamma'(0) = v$.

For any number $\eta \in (0, \varepsilon)$, denote $v_\eta := \exp_{\gamma(0)}^{-1} \gamma(\eta)$ and then show that

$$v = \lim_{\eta \rightarrow 0} \frac{v_\eta}{\eta}, \quad (3.9)$$

which clearly implies $T_x D \subset \text{cl}(\mathcal{F}_x D)$ due to $\frac{v_\eta}{\eta} \in \mathcal{F}_x D$ for all $\eta > 0$ sufficiently small.

To prove (3.9), take $f \in C^1(M)$ and get by its smoothness that

$$f(\gamma(\eta)) = f(\exp_{\gamma(0)} v_\eta) = f(\gamma(0)) + v_\eta(f) + o(\|v_\eta\|), \quad (3.10)$$

which implies in turn that

$$v(f) = \lim_{\eta \rightarrow 0} \frac{f(\gamma(\eta)) - f(\gamma(0))}{\eta} = \lim_{\eta \rightarrow 0} \frac{v_\eta(f)}{\eta} + \lim_{\eta \rightarrow 0} \frac{o(\|v_\eta\|)}{\eta}. \quad (3.11)$$

Observing further the relationships

$$\|v_\eta\| = d(\gamma(0), \gamma(\eta)) \leq \int_0^\eta \|\gamma'(t)\| dt \leq \max_{s \in (0, \eta)} \|\gamma'(s)\| \eta,$$

we get that $\{\frac{\|v_\eta\|}{\eta}\}$ is bounded as $\eta \rightarrow 0$. The latter implies, together with $\lim_{\eta \rightarrow 0} v_\eta = 0$ and (3.11), that (3.9) holds because $f \in C^1(M)$ was chosen arbitrarily. Thus the proof is complete. \square

REMARK 3.6. (Discussions on normal and tangent cones to closed subsets of Riemannian manifolds). The following observations are useful to better understand relationships between normal and tangent cone notions for subsets of Riemannian manifolds.

(a) The given definition of the normal cone $N_D(x)$ at $x \in D$ for a weakly convex set D is indeed in the sense of convex analysis. For a general (not necessarily weakly convex) closed subset D and a point $x \in D$ the Fréchet normal cone $\widehat{N}(x; D)$ to D at x was defined in [41, Definition 3.3]. It follows from this definition and [41, Remark 3.2] that we have $w \in \widehat{N}(x; D)$ if and only if there exist a neighborhood U_x of x and a function $g \in C^1(U_x)$ such that $w = dg(x) := \text{grad } g(x)$ and that function $\delta_D - g$ attains its local minimum at x . In the case when D is weakly convex, it is not hard to shown similarly to the Euclidean space setting that the Fréchet normal cone $\widehat{N}(x; D)$ and the normal cone $N_D(x)$ agree, i.e.,

$$\widehat{N}(x, D) = N_D(x). \quad (3.12)$$

Indeed, pick $w \in \widehat{N}(x; D)$ and let U_x and $g \in C^1(U_x)$ be the corresponding neighborhood and function such that $w = dg(x)$ and $\delta_D - g$ attains its local minimum at x . Then

$$\langle w, v \rangle = \langle dg(x), v \rangle = \lim_{t \rightarrow 0^+} \frac{g(\exp_x tv) - g(x)}{t} \leq 0 \quad \text{for each } v \in \mathcal{F}_x D \quad (3.13)$$

since $\exp_x tv \in D$ for all $t > 0$ sufficiently small by the assumed convexity of D . This implies that $w \in (\mathcal{F}_x D)^\circ$ and so $\widehat{N}(x; D) \subset (\mathcal{F}_x D)^\circ = N_D(x)$. Conversely, let $w \in N_D(x)$ and let U_x be a neighborhood of x such that the function $g(\cdot) := \langle w, \exp_x^{-1}(\cdot) \rangle$ is well defined on U_x and that $g \in C^1(U_x)$. Clearly $dg(x) = w$ by the definition. Moreover, since $w \in N_D(x)$ and D is weakly convex, it is easy to check that the function $\delta_D - g$ attains its local minimum at x . Hence $w \in \widehat{N}(x; D)$, which justifies the converse inclusion.

(b) For a general closed subset D of a Riemannian manifold and a point $x \in D$ the notion of the Bouligand tangent/contingent cone $T_x^B(D)$ was introduced in [41, Definition 3.8]. It seems to us that this definition as given is incomplete and requires some additional restrictions on a sequence $\{c_{v_i}\}$ for a vector $v \in T_x^B(D)$ therein, which were in fact used by the authors in the [41, Proposition 3.9]. They are: $\lim_{i \rightarrow \infty} c_{v_i}(t_i) = x$ and $\lim_{i \rightarrow \infty} c'_{v_i}(t_i) = v$. Clearly in this case the cone $T_x^B(D)$ is closed. Furthermore, if the set D is weakly convex we have that $T_x^B(D) = \text{cl}(T_x D)$ since

$$T_x^B(D) \subset (\widehat{N}(x, D))^\circ = (N_D(x))^\circ = \text{cl}(T_x D) \quad (3.14)$$

due to [41, Proposition 3.9] and (3.12) while the converse inclusion holds trivially by the definition.

Taking into account the normal cone expression in Proposition 3.5, we can reformulate condition (3.4) of Proposition 3.4 in the following equivalent *dual form*: if $\gamma_{yx} \in \Gamma_{yx}$ is minimizing, then

$$(3.4) \text{ holds} \iff [\langle \gamma'_{yx}(0), v \rangle \leq 0 \text{ for all } v \in \mathcal{F}_y D] \iff \gamma'_{yx}(0) \in N_D(y). \quad (3.15)$$

Consider next an extended-real-valued function $f: M \rightarrow \overline{\mathbb{R}}$ on a Riemannian manifold. The *effective domain* $\text{dom } f$ and the *properness* of f are defined similarly to the standard case of linear spaces. We use the symbol Γ_{xy}^f to denote the set of all $\gamma \in \Gamma_{xy}$ with $\gamma \subset \text{dom } f$.

DEFINITION 3.7. (Convex functions on Riemannian manifolds.) *Let $f: M \rightarrow \overline{\mathbb{R}}$ be a proper function on Riemannian manifold M with weakly convex effective domain. We say that f is CONVEX on M if for any $x, y \in \text{dom } f$ and $\gamma \in \Gamma_{xy}^f$ the composition $(f \circ \gamma): [0, 1] \rightarrow \mathbb{R}$ is a convex function on $[0, 1]$, i.e.,*

$$f(\gamma(t)) \leq (1-t)f(x) + tf(y) \quad \text{for all } t \in [0, 1].$$

In what follows we study proper convex functions $f: M \rightarrow \overline{\mathbb{R}}$ on Riemannian manifolds. The *directional derivative* of f at the point $x \in \text{dom } f$ in the direction $v \in T_x M$ is defined by

$$f'(x; v) := \lim_{t \rightarrow 0^+} \frac{f(\exp_x tv) - f(x)}{t},$$

while the *subdifferential* of f at $x \in \text{dom } f$ is constructed by

$$\partial f(x) := \{w \in T_x M \mid f(y) \geq f(x) + \langle w, \gamma'(0) \rangle \text{ for all } y \in \text{dom } f \text{ and } \gamma \in \Gamma_{xy}^f\}.$$

Observe that $\text{dom}(f'(x; \cdot)) = \mathcal{F}_x D$ with $D := \text{dom } f$. It is worth mentioning that the subdifferential set $\partial f(x)$ is nonempty, convex, and compact in the space $T_x M$ for any point $x \in \text{int}(\text{dom } f)$. In particular, if S is a closed weakly convex subset of M , then we have from Proposition 3.5 that

$$\partial \delta_S(x) = N_S(x) \quad \text{for each } x \in S. \quad (3.16)$$

The next proposition presents some useful properties of the directional derivative and subdifferential of convex functions on Riemannian manifolds that are similar to known ones on linear spaces. Assertion (i) of this theorem was proved in [62, p. 71] for convex functions with totally convex effective domains. However, the proof therein holds true in the case of proper convex functions with weakly convex effective domains; so we do not give it here.

PROPOSITION 3.8. (Properties of the directional derivative and subdifferential of convex functions on Riemannian manifolds.) *Let $f: M \rightarrow \overline{\mathbb{R}}$ be a proper convex function on a Riemannian manifold, and let $x \in D := \text{dom } f$. The following assertions hold.*

(i) *The directional derivative $f'(x; \cdot): T_x M \rightarrow \overline{\mathbb{R}}$ is convex and positively homogeneous with respect to directions, i.e., it satisfies the conditions*

$$f'(x; v_1 + v_2) \leq f'(x; v_1) + f'(x; v_2) \quad \text{for all } v_1, v_2 \in T_x M,$$

$$f'(x; sv) = s f'(x; v) \quad \text{for all } v \in T_x M \text{ and } s > 0. \quad (3.17)$$

Moreover, it possesses the properties

$$f'(x; 0) = 0 \quad \text{and} \quad -f'(x; -v) \leq f'(x; v) \quad \text{whenever } v \in T_x M.$$

(ii) *We have the subdifferential representation*

$$\partial f(x) = \{w \in T_x M \mid \langle w, v \rangle \leq f'(x; v) \text{ for all } v \in T_x M\}. \quad (3.18)$$

(iii) *The support function of the subdifferential $\sigma_{\partial f(x)}(\cdot)$ is the lower semicontinuous hull*

$$\sigma_{\partial f(x)}(\cdot) = \text{cl } f'(x; \cdot) \quad (3.19)$$

of the directional derivative $f'(x; \cdot)$ of f at x .

Proof. We need to justify assertions (ii) and (iii). Starting with (ii), take any w belonging to the set on the right-hand side of (3.18). Then we have

$$\langle w, v \rangle \leq f'(x; v) \quad \text{for all } v \in T_x M.$$

Pick now an arbitrary element $y \in \text{dom } f$ and consider a geodesic $\gamma \in \Gamma_{xy}^f$. Then $\gamma'(0) \in T_x M$ and thus

$$\langle w, \gamma'(0) \rangle \leq f'(x; \gamma'(0)). \quad (3.20)$$

Since f is convex, we have the relationships

$$f'(x; \gamma'(0)) = \inf_{t>0} \frac{f(\exp_x t\gamma'(0)) - f(x)}{t} \leq f(\exp_x \gamma'(0)) - f(x) = f(y) - f(x).$$

This yields together with (3.20) that

$$\langle w, \gamma'(0) \rangle \leq f'(x; \gamma'(0)) \leq f(y) - f(x).$$

The latter implies by the subdifferential definition that $w \in \partial f(x)$, and thus the subdifferential $\partial f(x)$ contains the set on the right-hand side of (3.18).

To justify the opposite inclusion “ \subset ” in (3.18), take an arbitrary subgradient $w \in \partial f(x)$ and then pick some $v \in T_x M$. Note that $f'(x; v) = \infty$ if $v \notin \mathcal{F}_x D$, which allows us to assume without loss of generality that $v \in \mathcal{F}_x D$. Then there exists $t_0 > 0$ such that

$$c_t(s) := \exp_x(stv) \in D \text{ for all } s \in [0, 1] \text{ and } t \in (0, t_0).$$

The latter implies that $c_t(1) = \exp_x(tv)$ and $c_t \in \Gamma_{x \exp_x(tv)}^f$. Since $c_t'(0) = tv$, we have furthermore by the subdifferential definition that

$$\langle w, tv \rangle = \langle w, c_t'(0) \rangle \leq f(c_t(1)) - f(x) = f(\exp_x tv) - f(x).$$

This implies, by using the directional derivative construction, that

$$\langle w, v \rangle \leq \lim_{t \rightarrow 0^+} \frac{f(\exp_x tv) - f(x)}{t} = f'(x; v),$$

which shows that the subgradient w belongs to the set on the right-hand side of (3.18) due to the arbitrary choice of $v \in T_x M$. This completes the proof of assertion (ii) of the proposition.

It remains to justify assertion (iii). To proceed, define a proper convex function $g: T_x M \rightarrow \overline{\mathbb{R}}$ by $g(v) := f'(x; v)$ as $v \in T_x M$. By (3.18) we have $\partial f(x) = \partial g(0)$. Note that $g'(0; \cdot) = f'(x; \cdot)$. Then employing Proposition 2.2, we arrive at (3.19). \square

We end this section with a short remark about convex functions on Riemannian manifolds.

REMARK 3.9. (Existence of convex continuous functions on Riemannian manifolds). It is known in the theory of Riemannian manifolds that there is no nontrivial continuous convex functions on complete manifolds with finite volume; see, e.g., [65]. The latter class includes the Stiefel manifolds (in particular, spheres and orthogonal groups) and the Grassmann manifolds, which have become popular in the computational mathematics community; see, e.g., [2, 25]. However, there are always nontrivial proper convex functions with *weakly convex* effective domains on any Riemannian manifold studied and employed in this paper; see [38] for specific examples and more discussions.

4. Weak sharp minima on Riemannian manifolds. Given a function $f: M \rightarrow \overline{\mathbb{R}}$ and a subset S of M , consider the *constrained optimization problem*

$$\begin{aligned} & \text{minimize } f(x) \\ & \text{subject to } x \in S \end{aligned} \quad (4.1)$$

with the *cost function* f and the *constraint set* S . Our *standing assumptions* in Sections 4–6 are that the function f is *proper and convex* on M and that both sets S and $S \cap \text{dom} f$ are *weakly convex* in M . Let \bar{S} be the set of *optimal solutions* to problem (4.1), i.e.,

$$\bar{S} := \operatorname{argmin}_S f = \{x \in S \mid f(x) = \min_{y \in S} f(y)\}. \quad (4.2)$$

It is easy to check that \bar{S} is *weakly convex* under the assumptions made. Throughout the paper we suppose that the solution set \bar{S} in (4.2) is *nonempty and closed* in M . The following definitions extend and modify the corresponding notions of weak sharp minima from linear spaces (cf. [13] with somewhat different terminology and also the discussion in Section 1) to the Riemannian manifold setting under consideration.

DEFINITION 4.1. (Versions of weak sharp minima on Riemannian manifolds.) *Let \bar{S} be the solution set for problem (4.1). Then we say that:*

(a) $\bar{x} \in \bar{S}$ is a **LOCAL WEAK SHARP MINIMIZER** for problem (4.1) with modulus $\alpha > 0$ if there is $\varepsilon > 0$ such that for all $x \in S \cap \mathbf{B}(\bar{x}, \varepsilon)$ we have the estimate

$$f(x) \geq f(\bar{x}) + \alpha d_{\bar{S}}(x). \quad (4.3)$$

(b) \bar{S} is the set of **LOCAL WEAK SHARP MINIMA** for problem (4.1) if each $\bar{x} \in \bar{S}$ is a local weak sharp minimizer for problem (4.1) with some modulus $\alpha > 0$.

(c) \bar{S} is the set of **BOUNDEDLY WEAK SHARP MINIMA** for problem (4.1) if for every bounded set $W \subset M$ with $W \cap \bar{S} \neq \emptyset$ there is $\alpha = \alpha_W > 0$ such that (4.3) holds with this modulus α for all $\bar{x} \in \bar{S}$ and $x \in S \cap W$.

(d) \bar{S} is the set of **GLOBAL WEAK SHARP MINIMA** for problem (4.1) with the uniform modulus $\alpha > 0$ if estimate (4.3) holds for all $\bar{x} \in \bar{S}$ and $x \in S$.

To conduct the study of all the versions of weak sharp minima from Definition 4.1, consider an extended-real-valued function $f_0: M \rightarrow \bar{\mathbb{R}}$ given by

$$f_0(x) := f(x) + \delta_S(x) = \begin{cases} f(x) & \text{if } x \in S, \\ \infty & \text{otherwise} \end{cases} \quad (4.4)$$

and observe that the initial constrained optimization problem (4.1) can be rewritten in *unconstrained* form:

$$\text{minimize } f_0(x) \text{ subject to } x \in M.$$

We have furthermore that for all $x \in S$ condition (4.3) is equivalent to

$$f_0(x) \geq f_0(\bar{x}) + \alpha d_{\bar{S}}(x), \quad (4.5)$$

where the set $\text{dom } f_0 = S \cap \text{dom } f$ is weakly convex and the function f_0 is proper and convex. The following *qualification condition* plays an important role for deriving the subsequent results in this paper.

DEFINITION 4.2. (Mild qualification condition.) *Given a proper convex function $f: M \rightarrow \bar{\mathbb{R}}$ and a subset S of M satisfying the standing assumptions made, we say that the pair $\{f, S\}$ satisfies the **MILD QUALIFICATION CONDITION (MQC, for short)** at $x \in (\text{dom } f) \cap S$ if*

$$\partial(f + \delta_S)(x) = \text{cl}(\partial f(x) + N_S(x)). \quad (4.6)$$

Note that the MQC (4.6) is a Riemannian manifold counterpart of the one used in [13] in the linear space setting. It is indeed a “mild” qualification condition ensuring a version of the *subdifferential sum rule* for the summation function f_0 in (4.4). Besides [13], we refer the reader to the recent papers [23, 44] and the bibliographies therein for “closedness qualification conditions” of this type, their relationships with more conventional qualification conditions in convex analysis, and applications to various classes of optimization problems on linear spaces. The following proposition, which is a Riemannian counterpart of the well-known result in Euclidean spaces, provides an easily verifiable property ensuring the validity of the MQC from Definition 4.2.

PROPOSITION 4.3. (Sufficient condition of the mild qualification condition.) *Let $f_1, f_2 : M \rightarrow \overline{\mathbb{R}}$ be proper and convex such that $\text{dom } f_1 \cap \text{dom } f_2$ is weakly convex, and let $x \in \text{int}(\text{dom } f_1) \cap \text{dom } f_2$. Then we have the subdifferential sum rule*

$$\partial(f_1 + f_2)(x) = \partial f_1(x) + \partial f_2(x). \quad (4.7)$$

In particular, for a convex function f with $\text{dom } f$ having nonempty interior and a nonempty weakly convex set S such that $S \cap \text{dom } f$ is weakly convex, it follows that

$$\partial(f + \delta_S)(x) = \partial f(x) + N_S(x) \quad \text{whenever } x \in \text{int}(\text{dom } f) \cap S. \quad (4.8)$$

Proof. It follows from definition (3.18) of the subdifferential that

$$\partial f_i(x) = \partial f'_i(x; \cdot)(0) \quad \text{and} \quad \partial(f_1 + f_2)(x) = \partial[f'_1(x; \cdot) + f'_2(x; \cdot)](0).$$

for each $i = 1, 2$. Noting that $0 \in \text{int}(\text{dom } f'_1(x; \cdot))$ under the assumption made, we have from the classical Moreau-Rockafellar theorem in linear space (see, e.g., [69, Theorem 2.8.7, p. 127]) that

$$\partial[f'(x; \cdot) + f'_2(x; \cdot)](0) = \partial f'_1(x; \cdot)(0) + \partial f'_2(x; \cdot)(0) = \partial f_1(x) + \partial f_2(x).$$

This readily implies the subdifferential sum rule (4.7). Taking finally $f_1 = f$, $f_2 = \delta_S$ and noting that $\partial \delta_S(x) = N_S(x)$, we arrive at (4.8) under the assumptions made. \square

To proceed further, observe first the following obvious while useful relationships held for any closed weakly convex subset \overline{S} of M , $z \in \overline{S}$ and any $\alpha > 0$:

$$\alpha \|v\| = \sigma_{\alpha \mathbb{B}_z}(v) = \sigma_{\alpha \mathbb{B}_z \cap N_{\overline{S}}(z)}(v) \quad \text{for all } v \in N_{\overline{S}}(z). \quad (4.9)$$

The next lemma is important to establish the main results of this section. Recall that $\text{cl} f'_0(z; \cdot)$ and $\text{cl} f'(z; \cdot)$ denote the lower semicontinuous hulls of $f'_0(z; \cdot)$ and $f'(z; \cdot)$, respectively.

LEMMA 4.4. (Some relationships under the mild qualification condition.) *Let \overline{S} be a nonempty closed weakly convex subset of $S \cap \text{dom } f$. Assume that the MQC (4.6) holds for the pair $\{f, S\}$ at a given point $z \in \overline{S}$. Fix $\alpha > 0$ and consider the following conditions:*

$$\text{cl} f'_0(z; v) \geq \alpha \|v\| \quad \text{for all } v \in N_{\overline{S}}(z); \quad (4.10)$$

$$\alpha \mathbb{B}_z \subset \text{cl}(\partial f(z) + N_S(z) + T_z \overline{S}); \quad (4.11)$$

$$\alpha \mathbb{B}_z \cap N_{\overline{S}}(z) \subset \text{cl}(\partial f(z) + N_S(z) + T_z \overline{S}); \quad (4.12)$$

$$\text{cl}f'(z; v) \geq \alpha\|v\| \quad \text{for all } v \in T_z S \cap N_{\overline{S}}(z); \quad (4.13)$$

$$\widehat{\alpha}\mathbb{B}_z \subset \partial f(z) + (T_z S \cap N_{\overline{S}}(z))^\circ \quad \text{for all } \widehat{\alpha} \in (0, \alpha); \quad (4.14)$$

$$\widehat{\alpha}\mathbb{B}_z \cap N_{\overline{S}}(z) \subset \partial f(z) + (T_z S \cap N_{\overline{S}}(z))^\circ \quad \text{for all } \widehat{\alpha} \in (0, \alpha). \quad (4.15)$$

Then we have the relationships between these conditions:

$$(4.10) \iff (4.11) \iff (4.12) \implies (4.13) \iff (4.14) \iff (4.15).$$

Proof. First we justify the equivalencies: (4.10) \iff (4.11) \iff (4.12). Indeed, it follows from Proposition 3.8(iii) and the relationships in (4.9) that condition (4.10) is equivalent to

$$\sigma_{\partial f_0(z)}(v) \geq \alpha\|v\| = \sigma_{\alpha\mathbb{B}_z \cap N_{\overline{S}}(z)}(v) \quad \text{for all } v \in N_{\overline{S}}(z). \quad (4.16)$$

Invoking Proposition 2.1 and the MQC assumption (4.6), we get the equivalence between condition (4.16) and

$$\alpha\mathbb{B}_z \cap N_{\overline{S}}(z) \subset \text{cl}(\partial f_0(z) + (N_{\overline{S}}(z))^\circ) = \text{cl}(\partial f(z) + N_S(z) + T_z \overline{S}).$$

Hence (4.10) is equivalent to (4.12). Similarly we derive the equivalence between (4.10) and (4.11).

To check next implication (4.12) \implies (4.13), observe from Proposition 3.8(iii) and the relationships in (4.9) that (4.13) is equivalent to

$$\sigma_{\alpha\mathbb{B}_z \cap N_{\overline{S}}(z)}(v) = \alpha\|v\| \leq \sigma_{\partial f(z)}(v) \quad \text{for all } v \in T_z S \cap N_{\overline{S}}(z). \quad (4.17)$$

Note that by (4.12) we surely have

$$\sigma_{\alpha\mathbb{B}_z \cap N_{\overline{S}}(z)}(v) \leq \sigma_{\text{cl}(\partial f(z) + N_S(z) + T_z \overline{S})}(v) = \sigma_{(\partial f(z) + N_S(z) + T_z \overline{S})}(v) \quad \text{for all } v \in T_z M. \quad (4.18)$$

Picking now $v \in T_z S \cap N_{\overline{S}}(z)$ and $w \in \partial f(z) + N_S(z) + T_z \overline{S}$, observe that there are $w_1 \in \partial f(z)$, $w_2 \in N_S(z)$, $w_3 \in T_z \overline{S} = (N_{\overline{S}}(z))^\circ$ such that $w = w_1 + w_2 + w_3$ and

$$\langle w, v \rangle = \langle w_1 + w_2 + w_3, v \rangle \leq \langle w_1, v \rangle. \quad (4.19)$$

Since w is arbitrary, (4.17) follows from (4.18) and (4.19), and hence we get (4.13).

It remains to prove equivalences (4.13) \iff (4.14) \iff (4.15), which hold in fact without the MQC assumption. Indeed, by Proposition 3.8(iii) and the relationships in (4.9), condition (4.13) is equivalent to

$$\sigma_{\alpha\mathbb{B}_z}(v) = \alpha\|v\| \leq \sigma_{\partial f(z)}(v) \quad \text{for all } v \in T_z S \cap N_{\overline{S}}(z).$$

Furthermore, by Proposition 2.1 the latter inequality is equivalent to the inclusion

$$\alpha\mathbb{B}_z \subset \text{cl}[\partial f(z) + (T_z S \cap N_{\overline{S}}(z))^\circ],$$

which in turn is equivalent to the one

$$\text{int}(\alpha\mathbb{B}_z) \subset \text{int}[\text{cl}(\partial f(z) + (T_z S \cap N_{\overline{S}}(z))^\circ)] = \text{int}[\partial f(z) + (T_z S \cap N_{\overline{S}}(z))^\circ]$$

by the convexity of the sets involved. This ensures the equivalence between (4.13) and (4.14). To justify equivalence (4.14) \iff (4.15), we proceed similarly and thus complete the proof of the lemma. \square

The next result establishes close relationships between the generalized differential conditions of Lemma 4.4 and the underlying weak sharp inequality (4.3) in the setting under consideration. These relationships are of their own independent interest while playing a crucial role in the characterizations of weak sharp minimizers on Riemannian manifolds derived in this section.

THEOREM 4.5. (Weak sharp inequality via generalized differentiation.) *Let \bar{S} be the solution set for problem (4.1). Let $0 < r \leq \infty$, $\bar{x} \in \bar{S}$ and suppose that the MQC (4.6) holds for the pair $\{f, S\}$ at any point $z \in \bar{S} \cap \mathbf{B}(\bar{x}, r)$. Fix arbitrary $\alpha > 0$, consider the following assertions:*

- (i) for each $x \in S \cap \mathbf{B}(\bar{x}, r)$ condition (4.3) holds;
- (ii) for each $z \in \bar{S} \cap \mathbf{B}(\bar{x}, r)$ condition (4.10) holds;
- (iii) for each $z \in \bar{S} \cap \mathbf{B}(\bar{x}, r)$ condition (4.13) holds;
- (iv) for each $x \in S \cap \mathbf{B}(\bar{x}, \frac{r}{2})$ condition (4.3) holds.

Then we have implications (i) \implies (ii) \implies (iii) \implies (iv).

Proof. To justify implication (i) \implies (ii), take arbitrary $z \in \bar{S} \cap \mathbf{B}(\bar{x}, r)$ and observe that

$$f_0(z) = f_0(\bar{x}) = f(\bar{x}) \quad (4.20)$$

due to construction (4.4). Define the generalized directional derivative

$$d_{\bar{S}}^{\circ}(z; v) := \limsup_{t \rightarrow 0^+} \frac{d_{\bar{S}}(\exp_z tv) - d_{\bar{S}}(z)}{t} \quad \text{for all } v \in T_z M. \quad (4.21)$$

By the Lipschitz continuity of $d_{\bar{S}}(\cdot)$ it is easy to verify that $d_{\bar{S}}^{\circ}(z; \cdot)$ is continuous on $T_z M$. Picking $v \in T_z M$ and taking into account that z belongs to the open ball $\mathbf{B}(\bar{x}, r)$, we find $\xi > 0$ such that

$$\{\exp_z tv \mid t \in [0, \xi]\} \subset \mathbf{B}(\bar{x}, r).$$

By assertion (i) assumed to hold we have condition (4.5). It follows further from (4.20) that

$$f_0(\exp_z tv) - f_0(z) = f_0(\exp_z tv) - f_0(\bar{x}) \geq \alpha d_{\bar{S}}(\exp_z tv) \quad \text{for all } t \in [0, \xi], \quad (4.22)$$

and thus (4.21) implies the relationships

$$f'_0(z; v) = \lim_{t \rightarrow 0^+} \frac{f_0(\exp_z tv) - f_0(z)}{t} \geq \alpha d_{\bar{S}}^{\circ}(z; v) \quad \text{for all } v \in T_z M. \quad (4.23)$$

This gives therefore the estimate

$$\text{cl}f'_0(z, v) \geq \alpha d_{\bar{S}}^{\circ}(z; v) \quad \text{for all } v \in T_z M \quad (4.24)$$

due to the continuity of the function $d_{\bar{S}}^{\circ}(z; \cdot)$. We are going to show below that

$$d_{\bar{S}}^{\circ}(z; v) = \|v\| \quad \text{whenever } v \in N_{\bar{S}}(z), \quad (4.25)$$

which yields with (4.24) assertion (ii), and thus complete the proof of implication (i) \implies (ii).

To justify (4.25), take $v \in N_{\bar{S}}(z)$ and define a geodesic $\gamma: [0, 1] \rightarrow M$ by

$$\gamma(t) := \exp_z tv \text{ for all } t \in [0, 1].$$

Since $z \in \bar{S}$ and since \bar{S} is a closed weakly convex subset of M , we get from Proposition 3.4(ii) that there exists $\varepsilon_0 > 0$ such that for each $p \in \mathbf{B}(z, \varepsilon_0)$ the following implication holds:

$$[\langle \gamma'_{pz}(1), \gamma'_{zy}(0) \rangle \geq 0 \text{ for all } y \in \bar{S}, \gamma_{zy} \in \Gamma_{zy}^{\bar{S}}] \implies z \in P(p|\bar{S}), \quad (4.26)$$

where $\gamma_{pz} \in \Gamma_{pz}$ is a minimizing geodesic and where the constructions involved in (4.26) are defined in Section 3. Choose further $\eta \in (0, 1)$ such that

$$\gamma([0, \eta]) \subset \mathbf{B}(z, \varepsilon_0) \text{ and } \gamma|_{[0, \eta]} \text{ is minimizing} \quad (4.27)$$

and verify the fulfillment of the equality

$$d_{\bar{S}}(\exp_z tv) = t\|v\| \text{ for all } t \in [0, \eta]. \quad (4.28)$$

To this end we take $t \in [0, \eta]$, denote $p := \exp_z tv$, and define a geodesic $\gamma_{pz}: [0, 1] \rightarrow M$ by

$$\gamma_{pz}(s) := \exp_z(1-s)tv \text{ for all } s \in [0, 1].$$

Employing then the conditions in (4.27) held due to the choice of η , we get

$$p \in \mathbf{B}(z, \varepsilon_0) \text{ and } \gamma_{pz} \in \Gamma_{pz} \text{ is minimizing.} \quad (4.29)$$

Since $v \in N_{\bar{S}}(z)$, it follows from the above that

$$\langle \gamma'_{pz}(1), \gamma'_{zy}(0) \rangle = \langle -v, \gamma'_{zy}(0) \rangle \geq 0 \text{ for all } y \in \bar{S} \text{ and } \gamma_{zy} \in \Gamma_{zy}^{\bar{S}}$$

due to $\gamma'_{zy}(0) \in T_z\bar{S}$. Thus we get $z \in P(p|\bar{S})$ by conditions (4.26) and (4.29), and hence justify the equalities

$$d_{\bar{S}}(\exp_z tv) = d_{\bar{S}}(p) = d(z, p) = d(z, \exp_z tv) = t\|v\|,$$

which yield in turn the fulfillment of (4.28) by the above arbitrary choice of $t \in [0, \eta]$. Then the relationship in (4.25) follows from (4.28) and the definition of $d_{\bar{S}}^{\circ}(z; \cdot)$.

The next implication (ii) \implies (iii) follows immediately from implication (4.10) \implies (4.13) in Lemma 4.4.

It remains to justify implication (iii) \implies (iv). Pick $x \in S \cap \mathbf{B}(\bar{x}, \frac{r}{2})$ and $\bar{y} \in P(x|\bar{S})$. Without loss of generality, assume that $x \in \text{dom } f$ and consider a minimizing geodesic $\gamma_{\bar{y}x} \in \Gamma_{\bar{y}x}$ with $\gamma_{\bar{y}x} \subset S \cap \text{dom } f$. Then we have the equalities

$$\|\gamma'_{\bar{y}x}(0)\| = l(\gamma_{\bar{y}x}) = d(x, \bar{y}) = d_{\bar{S}}(x). \quad (4.30)$$

It follows from Proposition 3.4(i) and the dual form in (3.15) (\bar{y} and \bar{S} in place of y and D) that $\gamma'_{\bar{y}x}(0) \in N_{\bar{S}}(\bar{y})$. Noting that $\gamma'_{\bar{y}x}(0) \in T_{\bar{y}}S$ by $\gamma_{\bar{y}x} \subset S$, we get

$$\gamma'_{\bar{y}x}(0) \in T_{\bar{y}}S \cap N_{\bar{S}}(\bar{y}). \quad (4.31)$$

It follows further from assertion (iii) and the obvious inequalities

$$d(\bar{y}, \bar{x}) \leq d(\bar{y}, x) + d(x, \bar{x}) \leq 2d(x, \bar{x}) < 2 \cdot \frac{r}{2} = r$$

that (4.13) holds with $z = \bar{y}$. Hence we have

$$f'(\bar{y}; v) \geq \text{cl}f'(\bar{y}; v) \geq \alpha\|v\| \quad \text{for all } v \in T_{\bar{y}}S \cap N_{\bar{S}}(\bar{y}).$$

Combining the latter with (4.30) and (4.31) gives us

$$f'(\bar{y}; \gamma'_{\bar{y}x}(0)) \geq \alpha\|\gamma'_{\bar{y}x}(0)\| = \alpha d_{\bar{S}}(x). \quad (4.32)$$

Remembering finally that $\gamma_{\bar{y}x} \subset \text{dom } f$ and that f is a convex function on a Riemannian manifold, we get from (4.32) the relationships

$$f(x) - f(\bar{y}) = f(\gamma_{\bar{y}x}(1)) - f(\gamma_{\bar{y}x}(0)) \geq f'(\bar{y}; \gamma'_{\bar{y}x}(0)) \geq \alpha d_{\bar{S}}(x),$$

(noting that we did not use the assumption that \bar{S} is a solution set for problem (4.1)). Hence

$$f(x) - f(\bar{x}) = f(x) - f(\bar{y}) \geq \alpha d_{\bar{S}}(x)$$

as \bar{S} is a solution set for problem (4.1), which justifies (4.3) and thus completes the proof of the theorem. \square

Let \bar{S} be a nonempty closed weakly convex subset of $S \cap \text{dom } f$ such that the pair $\{f, S\}$ satisfies the MQC (4.6) on \bar{S} . Then we have relationships (4.10) \iff (4.11) \implies (4.13) for each $z \in \bar{S}$ by Lemma 4.4, and hence it follows from the proof of implication (iii) \implies (iv) in Theorem 4.5 that

$$[(4.11) \text{ holds for each } z \in \bar{S}] \implies [f(x) \geq f(\bar{x}) + \alpha d_{\bar{S}}(x) \quad \text{for any } x \in S \text{ and } \bar{x} \in P(x|\bar{S})]. \quad (4.33)$$

In the rest of this section we employ the assertions established above as well as auxiliary results from Section 3 and some related constructions to derive *comprehensive characterizations* of all the types of weak sharp minima from Definition 4.1 via the defined notions of generalized differentiation on Riemannian manifolds.

Let us start with characterizing the set of *global weak sharp minima*. Observe that in the standard case of finite-dimensional Euclidean spaces the equivalences between assertions (i), (iv), (vi), and (vii) of the next theorem were obtained in [13, Theorem 2.3] while the equivalences between assertions (i), (iv) and (vi) were obtained in [11, Theorem 2.6]. The equivalences between assertions (i) (ii), (iii), and (v) in Theorem 4.6 seem to be new even for the classical setting of \mathbb{R}^n .

THEOREM 4.6. (Characterizations of the set of global weak sharp minima on Riemannian manifolds.) *Let \bar{S} be the solution set for problem (4.1). Suppose in addition that the pair $\{f, S\}$ satisfies the MQC (4.6) at every point $x \in \bar{S}$. Then, given a number $\alpha > 0$, the following assertions are equivalent:*

- (i) \bar{S} is the set of global weak sharp minima for problem (4.1) with the uniform modulus $\alpha > 0$.
- (ii) For each $\bar{x} \in \bar{S}$ we have the inclusion

$$\alpha \mathbb{B}_{\bar{x}} \subset \text{cl}(\partial f(\bar{x}) + N_S(\bar{x}) + T_{\bar{x}}\bar{S}).$$

- (iii) For each $\bar{x} \in \bar{S}$ we have the inclusion

$$\alpha \mathbb{B}_{\bar{x}} \cap N_{\bar{S}}(\bar{x}) \subset \text{cl}(\partial f(\bar{x}) + N_S(\bar{x}) + T_{\bar{x}}\bar{S}).$$

- (iv) For each $\bar{x} \in \bar{S}$ we have the inclusion

$$\hat{\alpha} \mathbb{B}_{\bar{x}} \subset \partial f(\bar{x}) + (T_{\bar{x}}S \cap N_{\bar{S}}(\bar{x}))^\circ \quad \text{whenever } \hat{\alpha} \in (0, \alpha).$$

(v) For each $\bar{x} \in \bar{S}$ we have the inclusion

$$\hat{\alpha}\mathbb{B}_{\bar{x}} \cap N_{\bar{S}}(\bar{x}) \subset \partial f(\bar{x}) + (T_{\bar{x}}S \cap N_{\bar{S}}(\bar{x}))^\circ \quad \text{whenever } \hat{\alpha} \in (0, \alpha).$$

(vi) For each $\bar{x} \in \bar{S}$ we have the estimate

$$f'(\bar{x}; v) \geq \alpha \|v\| \quad \text{whenever } v \in T_{\bar{x}}S \cap N_{\bar{S}}(\bar{x}).$$

(vii) For each $x \in S$, each $\bar{x} \in P(x|\bar{S})$, and each minimizing geodesic $\gamma \in \Gamma_{\bar{x}x}^S$ we have

$$f'(\bar{x}; \gamma'(0)) \geq \alpha d_{\bar{S}}(x).$$

Proof. Let us first verify implication (i) \implies (ii) of the theorem. By Definition 4.1(d) the weak sharp inequality (4.3) is satisfied with the given $\alpha > 0$ for each $x \in S$ and $\bar{x} \in \bar{S}$. It follows from implication (i) \implies (ii) of Theorem 4.5 with $r = \infty$ that the directional derivative estimate (4.10) holds on \bar{S} . Now the validity of assertion (ii) of the theorem follows from the equivalency relationship (4.10) \iff (4.11) in Lemma 4.4. Furthermore, we have:

Equivalence (ii) \iff (iii) follows from the corresponding equivalence (4.11) \iff (4.12) of Lemma 4.4.

Implication (iii) \implies (iv) follows from the corresponding implication (4.12) \implies (4.14) of Lemma 4.4.

Equivalence (iv) \iff (v) follows from the corresponding equivalence (4.14) \iff (4.15) of Lemma 4.4.

Implication (v) \implies (vi) follows from the corresponding implication (4.15) \implies (4.13) of Lemma 4.4 and the obvious implication (4.13) \implies (vi).

To justify implication (vi) \implies (vii), pick any points $x \in S$ and $\bar{x} \in P(x|\bar{S})$ and consider a minimizing geodesic $\gamma \in \Gamma_{\bar{x}x}^S$. Then we have the equalities

$$\|\gamma'(0)\| = d(x, \bar{x}) = d_{\bar{S}}(x). \quad (4.34)$$

Employing now assertion (i) of Proposition 3.4 allows us to conclude that condition (3.4) holds with \bar{x} and \bar{S} in place of y and D , which together with the equivalences in (3.15) implies in turn the inclusion $\gamma'(0) \in N_{\bar{S}}(\bar{x})$. Combining the latter with the inclusion $\gamma \subset S$ gives us that $\gamma'(0) \in T_{\bar{x}}S \cap N_{\bar{S}}(\bar{x})$. It follows therefore from the assumed assertion (vi) of the theorem that

$$f'(\bar{x}; \gamma'(0)) \geq \alpha \|\gamma'(0)\| = \alpha d_{\bar{S}}(x),$$

where the last equality holds due to (4.34). Thus we arrive at assertion (vii).

It remains to verify implication (vii) \implies (i) of the theorem. Take $x \in S$ and $\bar{x} \in P(x|\bar{S})$. Without loss of generality, assume that $x \in \text{dom } f$. Pick a minimizing geodesic $\gamma \in \Gamma_{\bar{x}x}^S$ with $\gamma \subset S \cap \text{dom } f$. Then by (vii) we get that $f'(\bar{x}; \gamma'(0)) \geq \alpha d_{\bar{S}}(x)$. Since the function f is convex, the latter gives

$$f(x) - f(\bar{x}) = f(\gamma(1)) - f(\bar{x}) \geq f'(\bar{x}; \gamma'(0)) \geq \alpha d_{\bar{S}}(x),$$

which ensures (i) and completes the proof of the theorem. \square

The next theorem provides similar characterizations of *boundedly weak sharp minima* for problem (4.1). All the results of this theorem are new even in the classical case of finite-dimensional Euclidean spaces.

THEOREM 4.7. (Characterizations of the set of boundedly weak sharp minima on Riemannian manifolds.) *Let \bar{S} be the solution set for problem (4.1), and let all the assumptions of Theorem 4.6 be satisfied. Then the following assertions are equivalent:*

(i) \bar{S} is the set of boundedly weak sharp minima for problem (4.1).

(ii) For every $\bar{x} \in \bar{S}$ and every $r > 0$ there is $\alpha(r) > 0$ such that

$$\alpha(r)\mathbb{B}_z \subset \text{cl}(\partial f(\bar{x}) + N_S(z) + T_z\bar{S}) \quad \text{whenever } z \in \bar{S} \cap \mathbf{B}(\bar{x}, r). \quad (4.35)$$

(iii) For every $\bar{x} \in \bar{S}$ and every $r > 0$ there is $\alpha(r) > 0$ such that

$$\alpha(r)\mathbb{B}_z \cap N_{\bar{S}}(z) \subset \text{cl}(\partial f(z) + N_S(z) + T_z\bar{S}) \quad \text{whenever } z \in \bar{S} \cap \mathbf{B}(\bar{x}, r). \quad (4.36)$$

(iv) For every $\bar{x} \in \bar{S}$ and $r > 0$ there is $\alpha(r) > 0$ such that

$$\hat{\alpha}\mathbb{B}_z \subset \partial f(z) + (T_z S \cap N_{\bar{S}}(z))^\circ \quad \text{whenever } z \in \bar{S} \cap \mathbf{B}(\bar{x}, r) \text{ and } 0 \leq \hat{\alpha} < \alpha(r). \quad (4.37)$$

(v) For every $\bar{x} \in \bar{S}$ and $r > 0$ there is $\alpha(r) > 0$ such that

$$\hat{\alpha}\mathbb{B}_z \cap N_{\bar{S}}(z) \subset \partial f(z) + (T_z S \cap N_{\bar{S}}(z))^\circ \quad \text{whenever } z \in \bar{S} \cap \mathbf{B}(\bar{x}, r) \text{ and } 0 \leq \hat{\alpha} < \alpha(r). \quad (4.38)$$

(vi) For every $\bar{x} \in \bar{S}$ and every $r > 0$ there is $\alpha(r) > 0$ such that

$$f'(z; v) \geq \alpha(r)\|v\| \quad \text{whenever } z \in \bar{S} \cap \mathbf{B}(\bar{x}, r) \text{ and } v \in T_z S \cap N_{\bar{S}}(z). \quad (4.39)$$

(vii) For every $\bar{x} \in \bar{S}$ and every $r > 0$ there is $\alpha(r) > 0$ such that

$$f'(z; \gamma'(0)) \geq \alpha(r)d_{\bar{S}}(x) \quad \text{whenever } x \in S \cap \mathbf{B}(\bar{x}, r) \text{ and } z \in P(x|\bar{S}) \quad (4.40)$$

independently of the choice of a minimizing geodesic $\gamma \in \Gamma_{zx}^S$.

Proof. We first justify implication (i) \implies (ii) of the theorem. Observe that Definition 4.1(c) of the set of boundedly weak sharp minima can be equivalently formulated as follows: for every $\bar{x} \in \bar{S}$ and every $r > 0$ there is a modulus $\alpha(r) > 0$ such that

$$f(x) \geq f(\bar{x}) + \alpha(r)d_{\bar{S}}(x) \quad \text{whenever } x \in S \cap \mathbf{B}(\bar{x}, r). \quad (4.41)$$

Using now implication (i) \implies (ii) of Theorem 4.5, we conclude that assertion (i) of the theorem yields the validity of condition (4.10) for all $z \in \bar{S} \cap \mathbf{B}(\bar{x}, r)$ with the same number $\alpha = \alpha(r)$ as in (4.41). Thus it follows from implication (4.10) \implies (4.11) in Lemma 4.4 that assertion (ii) of this theorem holds.

Observe further that relationships (ii) \iff (iii) \implies (iv) \iff (v) \implies (vi) of the theorem follow directly from the corresponding results of Lemma 4.4 similarly to the proof of Theorem 4.6.

To verify next implication (vi) \implies (vii) of the theorem, pick any $\bar{x} \in \bar{S}$ and $r > 0$ and find by (vi) a number $\alpha = \alpha(r) > 0$ such that

$$[z \in \bar{S} \cap \mathbf{B}(\bar{x}, 2r) \text{ and } v \in T_z S \cap N_{\bar{S}}(z)] \implies [f'(z; v) \geq \alpha(r)\|v\|]. \quad (4.42)$$

Taking any $x \in S \cap \mathbf{B}(\bar{x}, r)$ and $z \in P(x|\bar{S})$, we have the equalities

$$\|\gamma'(0)\| = d(x, z) = d_{\bar{S}}(x), \quad (4.43)$$

where $\gamma \in \Gamma_{zx}^S$ is a minimizing geodesic. Applying then Proposition 3.4(i) gives us condition (3.4) with z and \bar{S} in place of y and D . It follows then from the equivalence in (3.15) and the inclusion $\gamma \subset S$ that

$$\gamma'(0) \in T_z S \cap N_{\bar{S}}(z). \quad (4.44)$$

Furthermore, we get the inclusion $z \in \bar{S} \cap \mathbf{B}(\bar{x}, 2r)$ from the obvious inequalities

$$d(z, \bar{x}) \leq d(z, x) + d(x, \bar{x}) \leq 2d(x, \bar{x}) \leq 2r.$$

Combining this with relationships (4.42) and (4.44) allows us to conclude that

$$f'(z; \gamma'(0)) \geq \alpha(r)\|\gamma'(0)\| = \alpha(r)d_{\bar{S}}(x),$$

where the last equality holds due to (4.43). Thus we arrive at assertion (vii) of the theorem.

Let us finally justify the remaining implication (vii) \implies (i). Take $\bar{x} \in \bar{S}$ and $r > 0$ and by (vii) find a number $\alpha(r) > 0$ such that

$$f'(z; \gamma'(0)) \geq \alpha(r)d_{\bar{S}}(x)$$

for each $x \in S \cap \mathbf{B}(\bar{x}, r)$, $z \in P(x|\bar{S})$, and the minimizing geodesic $\gamma \in \Gamma_{zx}$ with $\gamma \subset S \cap \text{dom} f$. It follows further from the convexity of the cost function f in problem (4.1) that

$$f(x) - f(\bar{x}) = f(\gamma(1)) - f(z) \geq f'(z; \gamma'(0)) \geq \alpha(r)d_{\bar{S}}(x) \quad \text{for each } x \in S \cap \mathbf{B}(\bar{x}, r) \cap \text{dom} f,$$

which gives the underlying estimate (4.41) in (i) (noting that estimate (4.41) is trivial if $x \notin \text{dom} f$) and thus completes the proof of the theorem. \square

Our next step is to derive efficient characterizations of *local weak sharp minimizers* for problem (4.1) in the sense of Definition 4.1(a). To proceed, we need the following proposition taken from [63, Theorem 2], which ensures the local Lipschitz continuity of the projection mapping.

PROPOSITION 4.8. (Local Lipschitz continuity of projections on locally convex subsets of Riemannian manifolds.) *Let D be a locally convex and closed nonempty subset of a Riemannian manifold M . Then there exists an open set $U \supset D$ such that the projection mapping $P(\cdot|D)$ is locally Lipschitz continuous on U , that is, for each $x \in U$, there are numbers $r_x > 0$ and $\ell_x \geq 1$ such that*

$$d(P(y|D), P(z|D)) \leq \ell_x d(y, z)$$

for each pair $(y, z) \in U \times U$ with $d(y, x) < r_x$ and $d(z, x) < r_x$.

Combining this proposition with the methods and results developed above allows us to establish the following characterizations of local weak sharp minimizers. All the results of the next theorem are new even in the classical setting of finite-dimensional Euclidean spaces.

THEOREM 4.9. (Characterizations of local weak sharp minimizers for convex problems on Riemannian manifolds.) *Let \bar{S} be the solution set for problem (4.1) and let $\bar{x} \in \bar{S}$. Suppose in addition*

that the MQC (4.6) holds on a relative neighborhood of \bar{x} in \bar{S} . Then, given $\alpha > 0$, the following assertions are equivalent:

(i) \bar{x} is a local weak sharp minimizer for problem (4.1) with modulus α .

(ii) There is $\varepsilon > 0$ such that we have the inclusion

$$\alpha\mathbb{B}_z \subset \text{cl}(\partial f(z) + N_S(z) + T_z\bar{S}) \quad \text{for all } z \in \bar{S} \cap \mathbf{B}(\bar{x}, \varepsilon).$$

(iii) There is $\varepsilon > 0$ such that we have the inclusion

$$\alpha\mathbb{B}_z \cap N_{\bar{S}}(z) \subset \text{cl}(\partial f(z) + N_S(z) + T_z\bar{S}) \quad \text{for all } z \in \bar{S} \cap \mathbf{B}(\bar{x}, \varepsilon).$$

(iv) There is $\varepsilon > 0$ such that we have the inclusion

$$\hat{\alpha}\mathbb{B}_z \subset \partial f(z) + (T_z S \cap N_{\bar{S}}(z))^\circ \quad \text{for all } z \in \bar{S} \cap \mathbf{B}(\bar{x}, \varepsilon) \quad \text{and } 0 \leq \hat{\alpha} < \alpha.$$

(v) There is $\varepsilon > 0$ such that we have the inclusion

$$\hat{\alpha}\mathbb{B}_z \cap N_{\bar{S}}(z) \subset \partial f(z) + (T_z S \cap N_{\bar{S}}(z))^\circ \quad \text{for all } z \in \bar{S} \cap \mathbf{B}(\bar{x}, \varepsilon) \quad \text{and } 0 \leq \hat{\alpha} < \alpha.$$

(vi) There is $\varepsilon > 0$ such that we have the estimate

$$f'(z; v) \geq \alpha\|v\| \quad \text{for all } z \in \bar{S} \cap \mathbf{B}(\bar{x}, \varepsilon) \quad \text{and } v \in T_z S \cap N_{\bar{S}}(z).$$

(vii) There is $\varepsilon > 0$ such that we have the estimate

$$f'(z; \gamma'(0)) \geq \alpha d_{\bar{S}}(x) \quad \text{for all } x \in S \cap \mathbf{B}(\bar{x}, \varepsilon) \quad \text{and } z \in P(x|\bar{S}),$$

where $\gamma \in \Gamma_{zx}$ is a minimizing geodesic.

Proof. Observe that the imposed MQC (4.6) on $\{f, S\}$ around \bar{x} means there is a number $\varepsilon_1 > 0$ such that

$$\partial(f + \delta_S)(z) = \text{cl}(\partial f(z) + N_S(z)) \quad \text{for all } z \in \bar{S} \cap \mathbf{B}(\bar{x}, \varepsilon_1). \quad (4.45)$$

To verify implication (i) \implies (ii) of the theorem, find by assertion (i) such a number $\varepsilon_2 > 0$ that the weak sharp inequality (4.3) holds on $S \cap \mathbf{B}(\bar{x}, \varepsilon_2)$. Letting $\varepsilon := \min\{\varepsilon_1, \varepsilon_2\}$, we have by implication (i) \implies (ii) of Theorem 4.5 that condition (4.10) is satisfied for all $z \in \bar{S} \cap \mathbf{B}(\bar{x}, \varepsilon)$. Using now (4.45) and equivalence (4.10) \iff (4.11) from Lemma 4.4, we arrive at assertion (ii).

As above, relationships (ii) \iff (iii) \implies (iv) \iff (v) \implies (vi) in the theorem follow directly from the corresponding relationships of Lemma 4.4.

The proof of implication (vi) \implies (vii) of this theorem requires some change in comparison with the similar implications of Theorem 4.6 and Theorem 4.7. To proceed, assume that assertion (vi) of the theorem holds, i.e., there is $\varepsilon_1 > 0$ such that

$$f'(z; v) \geq \alpha\|v\| \quad \text{for all } z \in \bar{S} \cap \mathbf{B}(\bar{x}, \varepsilon_1) \quad \text{and } v \in T_z S \cap N_{\bar{S}}(z). \quad (4.46)$$

Since \bar{S} is *locally convex*, Proposition 4.8 allows us to conclude that there are $r > 0$ and $\ell \geq 1$ such that

$$d(P(x|\bar{S}), P(y|\bar{S})) \leq \ell d(x, y) \text{ whenever } d(x, \bar{x}) \leq r \text{ and } d(y, \bar{x}) \leq r. \quad (4.47)$$

Take $x \in S \cap \mathbf{B}(\bar{x}, \varepsilon)$ and $z \in P(x|\bar{S})$, where the number $\varepsilon > 0$ is defined by

$$\varepsilon := \min \left\{ \frac{r}{2}, \frac{\varepsilon_1}{\ell} \right\}.$$

As in the proof of Theorem 4.7, we have the equalities in (4.43), where $\gamma \in \Gamma_{zx}^S$ is a minimizing geodesic connecting z and x . Applying further Proposition 3.4(i) with $D = \bar{S}$ and the equivalent dual description (3.15) of condition (3.4) ensures that $\gamma'(0) \in N_{\bar{S}}(z)$. Noting that $\gamma \subset S$, we get therefore that

$$\gamma'(0) \in T_z S \cap N_{\bar{S}}(z). \quad (4.48)$$

On the other hand, since

$$d(z, \bar{x}) \leq d(z, x) + d(x, \bar{x}) \leq d(x, \bar{x}) + d(x, \bar{x}) \leq 2\varepsilon \leq r,$$

it follows from (4.47) and the choice of $x \in \mathbf{B}(\bar{x}, \varepsilon)$ with $\varepsilon > 0$ defined above that

$$d(z, \bar{x}) \leq \ell d(x, \bar{x}) < \ell \varepsilon \leq \varepsilon_1.$$

Combining the latter with (4.46) and (4.48) yields that

$$f'(z; \gamma'(0)) \geq \alpha \|\gamma'(0)\| = \alpha d_{\bar{S}}(x),$$

where the last equality holds due to (4.43). Thus we arrive at assertion (vii) of the theorem.

Observing finally that the verification of implication (vii) \implies (i) in this theorem is similar to the one in Theorem 4.7, we complete the proof of the result. \square

The following characterizations of the set of *local weak sharp minima* are direct consequences of the corresponding characterizations of local weak sharp minimizers from Theorem 4.9 and Definition 4.1(b).

COROLLARY 4.10. (Characterizations of the set of local weak sharp minima on Riemannian manifolds.) *Let all the assumptions of Theorem 4.6 be satisfied. The following assertions are equivalent:*

- (i) \bar{S} is the set of local weak sharp minima for problem (4.1).
- (ii) For every $\bar{x} \in \bar{S}$ there are $r > 0$ and $\alpha(r) > 0$ such that each of conditions (4.35)–(4.39) holds.
- (iii) For every $\bar{x} \in \bar{S}$ there are $r > 0$ and $\alpha(r) > 0$ such that condition (4.40) holds, where $\gamma \in \Gamma_{zx}^S$ is a minimizing geodesic connecting the points z and x .

Comparing the results of Theorem 4.7 and of Corollary 4.10 and taking into account that the Riemannian manifold M considered in this section is *finite-dimensional*, we conclude by compactness arguments that sets of boundedly weak sharp minima and local weak sharp minima agree under the assumptions made. Corollary 4.11 below is an extension to Riemannian manifolds of the corresponding one [13, Corollary 6.4] for \mathbb{R}^n .

COROLLARY 4.11. (Sets of boundedly weak sharp minima and local weak sharp minima agree on finite-dimensional Riemannian manifolds.) *Under the assumptions made in Theorem 4.6, the solution set \bar{S} for problem (4.1) is the set of boundedly weak sharp minima for problem (4.1) if and only if it is the set of local weak sharp minima for this problem.*

Proof. It immediately follows from the definitions that the set of boundedly weak sharp minima is the set of local ones. To justify the opposite implication, fix $\bar{x} \in \bar{S}$ and $r > 0$. Then for any $y \in \bar{S} \cap \overline{\mathbf{B}(\bar{x}, r)}$ we find by assertions (ii) of Corollary 4.10 numbers $r_y > 0$ and $\alpha(r_y) > 0$ such that

$$\alpha(r_y)\mathbb{B}_z \cap N_{\bar{S}}(z) \subset \text{cl}(\partial f(z) + N_S(z) + T_z\bar{S}) \quad \text{for all } z \in \bar{S} \cap \mathbf{B}(y, r_y).$$

Since $\bar{S} \cap \overline{\mathbf{B}(\bar{x}, r)}$ is a *compact* subset of the finite-dimensional Riemannian manifold M and since

$$\bigcup_{y \in \bar{S} \cap \overline{\mathbf{B}(\bar{x}, r)}} \mathbf{B}(y, r_y) \supset (\bar{S} \cap \overline{\mathbf{B}(\bar{x}, r)}),$$

there exists a *finite covering* of the set $\bar{S} \cap \overline{\mathbf{B}(\bar{x}, r)}$ by the above balls, i.e., a natural number $n \geq 1$ such that

$$\bigcup_{i=1}^n \mathbf{B}(y_i, r_{y_i}) \supset (\bar{S} \cap \overline{\mathbf{B}(\bar{x}, r)}).$$

Letting now $\alpha(r) := \min \{\alpha(r_{y_i}) \mid 1 \leq i \leq n\}$, we have

$$\alpha(r)\mathbb{B}_z \cap N_{\bar{S}}(z) \subset \text{cl}(\partial f(z) + N_S(z) + T_z\bar{S}) \quad \text{for all } z \in \bar{S} \cap \mathbf{B}(\bar{x}, r),$$

which ensures the validity of condition (ii) in Theorem 4.7. The latter justifies that \bar{S} is the set of boundedly weak sharp minima for problem (4.1) and thus completes the proof of the corollary. \square

5. Weak sharp minima on Hadamard manifolds. In this section we obtain new characterizations of all the versions of weak sharp minima under consideration for convex problems on Hadamard manifolds essentially exploiting their special structure; the new conditions obtained do not have appropriate analogs in the general Riemannian case. Recall that a *Hadamard manifold* is a complete simply connected m -dimensional Riemannian manifold with nonpositive sectional curvature. Throughout the whole section we assume that M is a Hadamard manifold. In this case the mapping $\exp_x : T_x M \rightarrow M$ is a *diffeomorphism* for each $x \in M$; see, e.g., [60, p.221, Theorem 4.1]. The latter implies that for any two points $x, y \in M$ there is one and only one geodesic connecting x, y , which is a minimizing geodesic. This means that the notions of strong convexity and weak convexity agree for a subset of a Hadamard manifold, and thus we unify them by using the term “convexity” in the Hadamard setting.

The following well-known result (see, e.g., [60, Proposition 4.5, p.223]) concerns some properties of *geodesic triangle* $\Delta(p_1 p_2 p_3)$ consisting by definition of three points p_1, p_2, p_3 and three minimizing geodesic segments γ_i that join p_i and p_{i+1} with $i = 1, 2, 3 \pmod{3}$.

PROPOSITION 5.1. (Comparison result for geodesic triangles.) *Let $\Delta(p_1 p_2 p_3)$ be a geodesic triangle, and let $\gamma_i : [0, 1] \rightarrow M$ be the corresponding geodesic segments joining p_i and p_{i+1} with $i = 1, 2, 3 \pmod{3}$. Denote $l_i := l(\gamma_i)$, $\alpha_i := \angle(\gamma'_i(0), -\gamma'_{i-1}(1))$. Then we have the relationships*

$$\alpha_1 + \alpha_2 + \alpha_3 \leq \pi \quad \text{and} \quad l_i^2 + l_{i+1}^2 - 2l_i l_{i+1} \cos \alpha_{i+1} \leq l_{i-1}^2.$$

The next result reveals a significant *specific feature* of Hadamard manifolds in comparison with the general class of Riemannian manifolds: it shows namely that the distance function for convex subsets of Hadamard manifolds is convex.

LEMMA 5.2. (Convexity of the distance function on Hadamard manifolds). *Let D be a convex subset of a Hadamard manifold M . Then the distance function $d_D(\cdot)$ is convex on M .*

Proof. Take $x, y \in M$ and for any $\varepsilon > 0$ pick elements $c_x, c_y \in D$ satisfying the conditions

$$d(x, c_x) \leq d_D(x) + \varepsilon \quad \text{and} \quad d(y, c_y) \leq d_D(y) + \varepsilon.$$

Let $\gamma_1: [0, 1] \rightarrow M$ be a geodesic connecting x and y , and let $\gamma_2: [0, 1] \rightarrow M$ be another geodesic connecting c_x and c_y . It follows from the *convexity* of the set D that $\gamma_2([0, 1]) \subset D$. Since a function $h: [0, 1] \rightarrow [0, \infty)$ defined by $h(t) := d(\gamma_1(t), \gamma_2(t))$ is *convex* on $[0, 1]$ by [62, p. 105], we have the inequalities

$$\begin{aligned} d_D(\gamma_1(t)) &\leq d(\gamma_1(t), \gamma_2(t)) \leq (1-t)d(x, c_x) + td(y, c_y) \quad \text{for all } t \in [0, 1]. \\ &\leq (1-t)d_D(x) + td_D(y) + \varepsilon \end{aligned}$$

This completes the proof of the lemma due to the arbitrary choice of $\varepsilon > 0$. \square

The following theorem is certainly of independent interest for *convex analysis on Hadamard manifolds*, while it plays a crucial role in deriving the main characterizations of weak sharp minima in the rest of this section. Observe that results of this type relating subgradients of the distance function and normals to the corresponding set are known for various subdifferentials in general nonconvex settings of Banach spaces, being of great importance for many aspects of variational analysis and its applications; see [53, 54], particularly [53, Subsection 1.3.3], with the references and commentaries therein.

THEOREM 5.3. (Subdifferential representation for the distance function of convex sets in Hadamard manifolds.) *Let $D \subset M$ be a closed and convex subset of a Hadamard manifold. Then the subdifferential of the convex distance function $d_D(\cdot)$ is computed by*

$$\partial d_D(x) = \mathbb{B}_x \cap N_D(x) \quad \text{for all } x \in D \tag{5.1}$$

via the normal cone $N_D(\cdot)$ to the set D at the corresponding point.

Proof. The convexity of the distance function $d_D(\cdot)$ on Hadamard manifolds is established in Lemma 5.2. Using the definitions of $d_D(x)$ and $N_D(x)$ on Riemannian manifolds given in Section 3 and recalling that $d_D(\cdot)$ is Lipschitz continuous with constant 1 by Proposition 3.1, we easily conclude that the inclusion “ \subset ” in (5.1) holds. Let us now prove the opposite inclusion in (5.1), i.e.,

$$\mathbb{B}_x \cap N_D(x) \subset \partial d_D(x)$$

for any fixed $x \in D$. To proceed, take $w \in \mathbb{B}_x \cap N_D(x)$ and check that $w \in \partial d_D(x)$. By Proposition 3.8(i) it is sufficient to verify that the inequality

$$d'_D(x; v) \geq \langle w, v \rangle \tag{5.2}$$

holds for all $v \in T_x M$. Since $d'_D(x; v) \geq 0$ and $\langle w, v \rangle \leq 0$ for each $v \in T_x D$, we have that (5.2) is satisfied on $T_x D$. It remains to show that (5.2) also holds on $T_x M \setminus T_x D$. Pick any $v \in T_x M \setminus T_x D$ and verify first that

$$d_D(\exp_x tv) \geq \inf_{u \in \exp_x^{-1} D} \|tv - u\| \quad \text{whenever } t > 0. \tag{5.3}$$

Indeed, take arbitrary $t > 0$ and $u \in \exp_x^{-1} D$ and denote

$$p_1 := \exp_x tv, \quad p_2 := x, \quad \text{and} \quad p_3 := \exp_x u.$$

Let further $\gamma_i: [0, 1] \rightarrow M$ be a geodesic connecting p_i and p_{i+1} for $i = 1, 2, 3 \pmod{3}$. Then

$$l(\gamma_1) = \|tv\|, \quad l(\gamma_2) = \|u\|, \quad \text{and} \quad \theta := \angle(-\gamma_1'(1), \gamma_2'(0)) = \angle(tv, u).$$

Applying Proposition 5.1 to the geodesic triangle $\triangle(p_1 p_2 p_3)$, we get

$$\begin{aligned} d^2(\exp_x tv, \exp_x u) &= d^2(p_1, p_3) \\ &\geq l(\gamma_1)^2 + l(\gamma_2)^2 - 2l(\gamma_1)l(\gamma_2) \cos \theta \\ &= \|tv\|^2 + \|u\|^2 - 2\|tv\| \cdot \|u\| \cos \theta \\ &= \|tv - u\|^2. \end{aligned}$$

This allows us to conclude that

$$d_D(\exp_x tv) = \inf_{u \in \exp_x^{-1} D} d(\exp_x tv, \exp_x u) \geq \inf_{u \in \exp_x^{-1} D} \|tv - u\|,$$

and thus condition (5.3) is verified. Furthermore, it follows from $w \in \mathbb{B}_x \cap N_D(x)$ that

$$\begin{aligned} \inf_{u \in \exp_x^{-1} D} \|tv - u\| &\geq \inf_{u \in \exp_x^{-1} D} \langle w, tv - u \rangle \\ &= \inf_{u \in \exp_x^{-1} D} (\langle w, tv \rangle - \langle w, u \rangle) \\ &\geq \langle w, tv \rangle, \end{aligned} \tag{5.4}$$

where the last inequality holds due to $u \in \exp_x^{-1} D$ and $\langle w, u \rangle \leq 0$. Combining consequently (5.3) and (5.4), we arrive at the relationship

$$d'_D(x; v) = \lim_{t \rightarrow 0^+} \frac{d_D(\exp_x tv) - d_D(x)}{t} \geq \langle w, v \rangle,$$

which justifies (5.2) and thus completes the proof of the theorem. \square

To establish the main characterizations of weak sharp minima on Hadamard manifolds, we need one more auxiliary result involving the distance function of the solution set for problem (4.1).

LEMMA 5.4. (Some relationships involving the distance function of the solution set on Hadamard manifolds.) *Let \bar{S} be the solution set for problem (4.1), and let $\bar{x} \in \bar{S}$. In addition to the standing assumptions made, suppose that the MQC (4.6) holds at \bar{x} . Consider the following relationships, where the function f_0 is defined in (4.4):*

$$\alpha \mathbb{B}_{\bar{x}} \cap N_{\bar{S}}(\bar{x}) \subset \text{cl}(\partial f_0(\bar{x})) = \text{cl}(\partial f(\bar{x}) + N_S(\bar{x})); \tag{5.5}$$

$$f'_0(\bar{x}; v) \geq \alpha d'_{\bar{S}}(\bar{x}; v) \text{ for all } v \in T_{\bar{x}}M; \tag{5.6}$$

$$f'(\bar{x}; v) \geq \alpha d_{T_{\bar{x}}\bar{S}}(v) \text{ for all } v \in T_{\bar{x}}S; \tag{5.7}$$

$$f'(\bar{x}; v) \geq \alpha \|v\| \text{ for all } v \in T_{\bar{x}}S \cap N_{\bar{S}}(\bar{x}). \tag{5.8}$$

Then we have (5.5) \iff (5.6) \implies (5.7) \implies (5.8).

Proof. First we observe the validity of the following equalities:

$$d'_{\bar{S}}(\bar{x}; v) = \sigma_{\partial d_{\bar{S}}(\bar{x})}(v) = \sigma_{\mathbb{B}_{\bar{x}} \cap N_{\bar{S}}(\bar{x})}(v) = d_{T_{\bar{x}}\bar{S}}(v) \text{ for all } v \in T_{\bar{x}}M. \tag{5.9}$$

Indeed, it follows from the condition $\text{dom } d_{\bar{S}} = M$ that the function $d'_{\bar{S}}(\bar{x}; \cdot)$ is sublinear and thus continuous on $T_{\bar{x}}M$. Then applying to $d'_{\bar{S}}(\bar{x}; \cdot)$ equality (3.19) from Proposition 3.8 together with Proposition 2.1(ii) and the subdifferential representation (5.1) from Theorem 5.3, we see that all the equalities in (5.9) hold.

Returning now to the proof of this lemma, let us verify equivalence (5.5) \iff (5.6). Since $d'_{\overline{S}}(\overline{x}; \cdot)$ is continuous on $T_{\overline{x}}M$ as noted, we have that condition (5.6) is equivalent to

$$\text{cl } f'_0(\overline{x}; v) \geq \alpha d'_{\overline{S}}(\overline{x}; v) \quad \text{for all } v \in T_{\overline{x}}M. \quad (5.10)$$

Thus applying (3.19) to f_0 and then using (5.9) allow us to conclude that (5.10) is equivalent to

$$\sigma_{\partial f_0(\overline{x})}(v) \geq \sigma_{\partial(\alpha d_{\overline{S}})(\overline{x})}(v) = \sigma_{\alpha \mathbb{B}_{\overline{x}} \cap N_{\overline{S}}(\overline{x})}(v) \quad \text{for all } v \in T_{\overline{x}}M,$$

which is equivalent in turn to (5.5) by Proposition 2.1 and the qualification condition (4.6).

Assuming further that (5.5) holds and taking into account relationships (3.19) and (5.9), to verify (5.7) it is sufficient to show that

$$\sigma_{\partial f(\overline{x})}(v) \geq \sigma_{\alpha \mathbb{B}_{\overline{x}} \cap N_{\overline{S}}(\overline{x})}(v) \quad \text{for all } v \in T_{\overline{x}}S. \quad (5.11)$$

Note that by (5.5) we clearly have

$$\sigma_{\alpha \mathbb{B}_{\overline{x}} \cap N_{\overline{S}}(\overline{x})}(v) \leq \sigma_{\text{cl}(\partial f(\overline{x}) + N_S(\overline{x}))}(v) = \sigma_{(\partial f(\overline{x}) + N_S(\overline{x}))}(v) \quad \text{for all } v \in T_{\overline{x}}M. \quad (5.12)$$

Pick $v \in T_{\overline{x}}S$ and $w \in \partial f(\overline{x}) + N_S(\overline{x})$ and find $w_1 \in \partial f(\overline{x})$ and $w_2 \in N_S(\overline{x}) = (T_{\overline{x}}S)^\circ$ such that

$$w = w_1 + w_2 \quad \text{and} \quad \langle w, v \rangle = \langle w_1 + w_2, v \rangle \leq \langle w_1, v \rangle. \quad (5.13)$$

Since w is arbitrary, (5.11) follows from (5.13) and (5.12); thus implication (5.5) \implies (5.7) holds.

It remains to prove implication (5.7) \implies (5.8). Taking $v \in T_{\overline{x}}S \cap N_{\overline{S}}(\overline{x})$ and applying the equalities in (4.9) at the point $z = \overline{x}$, we have

$$\sigma_{\mathbb{B}_{\overline{x}} \cap N_{\overline{S}}(\overline{x})}(v) = \|v\|.$$

The latter implies together with (5.9) that $d_{T_{\overline{x}}\overline{S}}(v) = \|v\|$, which justifies the claimed implication and completes the proof of the lemma. \square

Now we are ready to establish characterizations of all the types of weak sharp minima from Definition 4.1 in the case of Hadamard manifolds. We start with *global weak sharp minima* deriving new characterizations in addition to those in Theorem 4.6. Note that, in the case of finite-dimensional Euclidean spaces, the equivalences between assertions (i), (viii), and (ix) in the next theorem follows from [13, Theorem 2.3].

THEOREM 5.5. (Characterizations of the set of global weak sharp minima on Hadamard manifolds.) *Suppose that all the assumptions of Theorem 4.6 are satisfied and that in addition the manifold M is Hadamard. Then, given a number $\alpha > 0$, assertions (i)–(vii) of Theorem 4.6 are equivalent to the following ones:*

(viii) *For each $\overline{x} \in \overline{S}$ we have the inclusion*

$$\alpha \mathbb{B}_{\overline{x}} \cap N_{\overline{S}}(\overline{x}) \subset \text{cl}(\partial f(\overline{x}) + N_S(\overline{x})).$$

(ix) *For each $\overline{x} \in \overline{S}$ and each $v \in T_{\overline{x}}S$ we have the estimate*

$$f'(\overline{x}; v) \geq \alpha d_{T_{\overline{x}}\overline{S}}(v).$$

Proof. Assume that \bar{S} is the set of global weak sharp minima for problem (4.1) and let $\bar{x} \in \bar{S}$. Then for all $t > 0$ and all $v \in T_{\bar{x}}M$ we have the inequality

$$f_0(\exp_{\bar{x}} tv) - f_0(\bar{x}) \geq \alpha d_{\bar{S}}(\exp_{\bar{x}} tv),$$

which implies in turn that

$$\frac{f_0(\exp_{\bar{x}} tv) - f_0(\bar{x})}{t} \geq \alpha \frac{d_{\bar{S}}(\exp_{\bar{x}} tv) - d_{\bar{S}}(\bar{x})}{t}. \quad (5.14)$$

Taking limits as $t \downarrow 0$ on both sides in (5.14), we get the estimate

$$f'_0(\bar{x}; v) \geq \alpha d'_{\bar{S}}(\bar{x}; v).$$

Since $v \in T_{\bar{x}}M$ was chosen arbitrarily, condition (5.6) of Lemma 5.4 holds. The latter yields together with equivalence (5.5) \iff (5.6) in Lemma 5.4 that condition (viii) is satisfied. Hence assertion (i) of Theorem 4.6 implies that of (viii) in this theorem.

It easily follows from implications (5.5) \implies (5.7) \implies (5.8) in Lemma 5.4 that condition (viii) of this theorem implies that of (ix) and the latter implies in turn condition (vi) of Theorem 4.6. Thus all the aforementioned conditions (i)–(ix) are equivalent, and the proof of this theorem is complete. \square

Observe that, in contrast to the corresponding characterization in Theorem 4.6(iii), condition (viii) of Theorem 5.5 does not contain the term $T_{\bar{x}}\bar{S}$. This is due to the convexity of the distance function $d_D(\cdot)$ and representation (5.1) of its subdifferential, which distinguish Hadamard manifolds in the general class of Riemannian ones. The aforementioned facts are strongly used in the proof of Lemma 5.4, which is crucial for deriving the new characterizations of Theorem 5.5 in the Hadamard case.

Next we establish new characterizations of *boundedly weak sharp minima* for constrained problem of convex optimization on Hadamard manifolds. Note that for finite-dimensional Euclidean spaces the equivalence between assertions (i) and (viii) in the following theorem was shown in [13, Theorem 6.3].

THEOREM 5.6. (Characterizations of boundedly weak sharp minima on Hadamard manifolds.) *Let all the assumptions of Theorem 5.5 be satisfied. Then assertions (i)–(vii) in Theorem 4.7 and the following new assertions are equivalent:*

(viii) *For every $\bar{x} \in \bar{S}$ and every $r > 0$ there is $\alpha(r) > 0$ such that*

$$\alpha(r)\mathbb{B}_z \cap N_{\bar{S}}(z) \subset \text{cl}(\partial f(z) + N_S(z)) \quad \text{whenever } z \in \bar{S} \cap \mathbf{B}(\bar{x}, r). \quad (5.15)$$

(ix) *For every $\bar{x} \in \bar{S}$ and every $r > 0$ there is $\alpha(r) > 0$ such that*

$$f'(z; v) \geq \alpha(r)d_{T_z\bar{S}}(v) \quad \text{whenever } z \in \bar{S} \cap \mathbf{B}(\bar{x}, r) \text{ and } v \in T_z\bar{S}. \quad (5.16)$$

Proof. Let us assume according to assertion (i) of Theorem 4.7 that \bar{S} is the set of boundedly weak sharp minima. Pick any $\bar{x} \in \bar{S}$ and $r > 0$, we find by Definition 4.1(c) such $\alpha(r) > 0$ that

$$f(x) \geq f(\bar{x}) + \alpha(r)d_{\bar{S}}(x) \quad \text{for all } x \in S \cap \mathbf{B}(\bar{x}, r). \quad (5.17)$$

Take further $z \in \bar{S} \cap \mathbf{B}(\bar{x}, r)$ and pick $v \in T_z M$ with $\|v\| \neq 0$. Then for any t satisfying $0 < t < \frac{r - d(z, \bar{x})}{\|v\|}$, we get from (5.17) and construction (4.4) of f_0 that

$$f_0(\exp_z tv) - f_0(z) = f_0(\exp_z tv) - f_0(\bar{x}) = f_0(\exp_z tv) - f(\bar{x}) \geq \alpha(r) d_S(\exp_z tv).$$

The latter implies by the definition of the directional derivative that

$$f'_0(z; v) = \lim_{t \rightarrow 0^+} \frac{f_0(\exp_z tv) - f_0(z)}{t} \geq \alpha(r) d'_S(z; v),$$

which gives assertion (viii) of the theorem by equivalence (5.5) \iff (5.6) in Lemma 5.4.

The remaining implications (viii) \implies (ix) \implies [(vi) in Theorem 4.7] follow from those

$$(5.5) \implies (5.7) \implies (5.8)$$

in Lemma 5.4. This completes the proof of the theorem. \square

The next theorem provides new characterizations of *local weak sharp minimizers* in Hadamard manifolds in addition to those obtained in Theorem 4.9 for the general Riemannian case. In the standard Euclidean space setting the equivalence between assertions (i) and (viii) of Theorem 5.7 follows from [13, Theorem 5.2].

THEOREM 5.7. (Characterizations of local weak sharp minimizers on Hadamard manifolds.)
In addition to all the assumptions of Theorem 4.9 suppose that M is a Hadamard manifold. Then, given a number $\alpha > 0$, assertions (i)–(vii) of Theorem 4.9 are equivalent to the following:

(viii) *There exists a positive number ε such that*

$$\alpha \mathbb{B}_z \cap N_{\bar{S}}(z) \subset \text{cl}(\partial f(z) + N_S(z)) \quad \text{whenever } z \in \bar{S} \cap \mathbf{B}(\bar{x}, \varepsilon).$$

(ix) *There exists a positive number ε such that*

$$f'(z; v) \geq \alpha d_{T_z \bar{S}}(v) \quad \text{whenever } z \in \bar{S} \cap \mathbf{B}(\bar{x}, \varepsilon) \text{ and } v \in T_z S.$$

Proof. Take an arbitrary local weak sharp minimizer \bar{x} for problem (4.1). By Definition 4.1(a) there is $\varepsilon > 0$ such that we have the underlying weak sharp inequality

$$f(x) \geq f(\bar{x}) + \alpha d_{\bar{S}}(x) \quad \text{for all } x \in S \cap \mathbf{B}(\bar{x}, \varepsilon).$$

Similar to the proof of Theorem 5.6, pick any $z \in \bar{S} \cap \mathbf{B}(\bar{x}, \varepsilon)$ and $v \in T_z M$ with $\|v\| \neq 0$ and observe that

$$f_0(\exp_z tv) - f_0(z) = f_0(\exp_z tv) - f_0(\bar{x}) \geq \alpha d_S(\exp_z tv) \quad \text{whenever } 0 < t < \frac{\varepsilon - d(z, \bar{x})}{\|v\|},$$

where the function f_0 is defined in (4.4). The latter inequality gives

$$f'_0(z; v) = \lim_{t \rightarrow 0^+} \frac{f_0(\exp_z tv) - f_0(z)}{t} \geq \alpha d'_S(z; v),$$

which in turn ensures the validity of assertion (viii) by equivalence (5.5) \iff (5.6) in Lemma 5.4.

To conclude the proof of the theorem, observe that implications (viii) \implies (ix) \implies [(vi) in Theorem 4.9] follow from the corresponding ones (5.5) \implies (5.7) \implies (5.8) in Lemma 5.4. \square

Finally in this section, we provide new characterizations of the set of *local weak sharp minima* on Hadamard manifolds. They follow directly from Theorem 5.7 by Definition 4.1(b).

COROLLARY 5.8. (Characterizations of the set of local weak sharp minima on Hadamard manifolds.) *Under the assumptions of Theorem 5.5 we have the equivalence of the assertions of Corollary 4.10 with the following conditions:*

- (iv) *For every $\bar{x} \in \bar{S}$ there exist $r > 0$ and $\alpha(r) > 0$ such that inclusion (5.15) is satisfied.*
- (v) *For every $\bar{x} \in \bar{S}$ there exist $r > 0$ and $\alpha(r) > 0$ such that estimate (5.16) is satisfied.*

6. Examples and applications. In this section we first present two examples of their independent interest, which illustrate how our major characterizations of weak sharp minima work in particular nontrivial settings of Riemannian manifolds. Our final result gives an application of the obtained characterizations to stability/well-posedness issues for the class of optimization problems under consideration.

EXAMPLE 6.1. (Optimization problems on Riemannian unit spheres). Let

$$M = \mathbb{S}^2 := \{(y_1, y_2, y_3) \in \mathbb{R}^3 \mid y_1^2 + y_2^2 + y_3^2 = 1\}$$

be the 2-dimensional unit sphere; see [62, p. 84] for more details. Denoting $\mathbf{x} := (0, 0, 1)$ and $\mathbf{y} := (0, 0, -1)$, observe that the manifold $\mathbb{S}^2 \setminus \{\mathbf{x}, \mathbf{y}\}$ can be parameterized by $\Phi: (0, \pi) \times [0, 2\pi] \subset \mathbb{R}^2 \rightarrow \mathbb{S}^2 \setminus \{\mathbf{x}, \mathbf{y}\}$ defined as $\Phi(\theta, \varphi) := (y_1, y_2, y_3)^T$ for each $\theta \in (0, \pi)$ and $\varphi \in [0, 2\pi]$ with

$$\begin{cases} y_1 : &= \sin \theta \cos \varphi, \\ y_2 : &= \sin \theta \sin \varphi, \\ y_3 : &= \cos \theta. \end{cases}$$

It is easy to see from the above that $(\mathbb{S}^2 \setminus \{\mathbf{x}, \mathbf{y}\}, \Phi^{-1})$ is a system of coordinates around x whenever $x \in \mathbb{S}^2 \setminus \{\mathbf{x}, \mathbf{y}\}$. Then the Riemannian metric on $\mathbb{S}^2 \setminus \{\mathbf{x}, \mathbf{y}\}$ is given by

$$g_{11} = 1, \quad g_{12} = 0, \quad g_{22} = \sin^2 \theta \quad \text{for each } \theta \in (0, \pi) \text{ and } \varphi \in [0, 2\pi].$$

The geodesics of $\mathbb{S}^2 \setminus \{\mathbf{x}, \mathbf{y}\}$ are great circles or semicircles. Taking $\bar{x} := (\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \mathbb{S}^2 \setminus \{\mathbf{x}, \mathbf{y}\}$, we have

$$T_{\bar{x}}\mathbb{S}^2 = \bar{x}^\perp := \{(y_1, y_2, y_3) \in \mathbb{R}^3 \mid \sum_{i=1}^3 \bar{y}_i y_i = 0\}.$$

It follows from the definition of the Riemannian metric on \mathbb{S}^2 that

$$\langle u, v \rangle_{\bar{x}} = \langle u, v \rangle \quad \text{for any pair } (u, v) \in T_{\bar{x}}\mathbb{S}^2 \times T_{\bar{x}}\mathbb{S}^2, \quad (6.1)$$

where $\langle \cdot, \cdot \rangle_{\bar{x}}$ and $\langle \cdot, \cdot \rangle$ denote the inner products in $T_{\bar{x}}\mathbb{S}^2$ and \mathbb{R}^3 , respectively.

Letting $x_1 := (0, \frac{\sqrt{3}}{3}, \frac{\sqrt{6}}{3})$ and $x_2 := (0, \frac{\sqrt{6}}{3}, \frac{\sqrt{3}}{3})$, define a cost function $f: M \rightarrow \bar{\mathbb{R}}$ in problem (4.1) by

$$f(x) := \begin{cases} d(x_1, x) + d(x_2, x) & \text{for each } x \in \mathbf{B}(x_1, \frac{\pi}{4}) \cap \mathbf{B}(x_2, \frac{\pi}{4}); \\ \infty & \text{otherwise.} \end{cases} \quad (6.2)$$

Since each distance function $d(x_i, \cdot)$ in (6.2) is convex on $\mathbf{B}(x_i, \frac{\pi}{4})$ as $i = 1, 2$, the cost function f is proper and convex. Furthermore, for each $\bar{x} \in \mathbf{B}(x_i, \frac{\pi}{4})$, $i = 1, 2$, the subdifferential of $d(x_i, \cdot)$ at \bar{x} is computed by

$$\partial d(x_i, \cdot)(\bar{x}) = \begin{cases} \left\{ -\frac{\exp_{\bar{x}}^{-1} x_i}{d(x_i, \bar{x})} \right\} & \text{for } \bar{x} \neq x_i; \\ \mathbb{B}_{\bar{x}} & \text{for } \bar{x} = x_i. \end{cases} \quad (6.3)$$

This follows from Proposition 3.3 when $\bar{x} \neq x_i$ (by observing that $r_{\bar{x}} \geq \frac{\pi}{4}$ for the totally normal radius) and directly from definition when $\bar{x} = x_i$.

We illustrate the application of our results to two versions of problem (4.1) with the same cost function (6.2) while with different constraint sets S . Consider first the case when S in problem (4.1) is given by

$$S := \{(y_1, y_2, y_3) \in \mathbb{S}^2 \mid y_1 \geq 0, y_2 \geq 0, y_3 \geq 0\}. \quad (6.4)$$

The S in (6.4) is strongly convex. It is not hard to check that the solution set for problem (4.1) with f and S given by (6.2) and (6.4) is

$$\bar{S} = \left\{ (y_1, y_2, y_3) \in \mathbb{S}^2 \mid y_1 = 0, y_2, y_3 \in \left[\frac{\sqrt{3}}{3}, \frac{\sqrt{6}}{3} \right] \right\}. \quad (6.5)$$

We clearly have $\bar{S} \subset \text{int}(\text{dom } f)$ and thus conclude by Proposition 4.3 that $\{f, S\}$ satisfies the MQC (4.6) at every point $x \in \bar{S}$. This allows us to use characterizations of weak sharp minima for problem (4.1). To proceed, take $\bar{x} = (0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) \in \bar{S}$ and get by (6.1) and the definitions that

$$T_{\bar{x}}S = \{(y_1, y_2, y_3) \in \mathbb{R}^3 \mid y_2 + y_3 = 0, y_1 \geq 0\},$$

$$N_S(\bar{x}) = \{(y_1, y_2, y_3) \in \mathbb{R}^3 \mid y_2 = y_3 = 0, y_1 \leq 0\},$$

$$T_{\bar{x}}\bar{S} = \{(y_1, y_2, y_3) \in \mathbb{R}^3 \mid y_2 + y_3 = 0, y_1 = 0\}.$$

This yields therefore the representation

$$T_{\bar{x}}\bar{S} + N_S(\bar{x}) = \{(y_1, y_2, y_3) \in \mathbb{R}^3 \mid y_2 + y_3 = 0, y_1 \leq 0\}. \quad (6.6)$$

Then it follows from (6.3) and the subdifferential sum formula (4.7) that $\partial f(\bar{x}) = \{0\}$. Combining the latter with (6.6) ensures the condition

$$0 \notin \text{int}(\partial f(\bar{x}) + N_S(\bar{x}) + T_{\bar{x}}\bar{S}).$$

Employing the equivalence (i) \iff (ii) of Theorem 4.6 allows us to conclude that \bar{S} in (6.7) is not the set of global weak sharp minima for problem (4.1) with the cost function f and the constraint set S defined by (6.2) and (6.4), respectively.

Next we consider the optimization problem (4.1) with the same cost function f defined in (6.2) but with another constraint set S given now by

$$S = \{(y_1, y_2, y_3) \in \mathbb{S}^2 \mid y_1 = 0, y_2 > 0, y_3 > 0\}. \quad (6.7)$$

It is easy to check that the set S in (6.7) is strongly convex and the solution set for problem (4.1) under consideration is equal to the same solution set \bar{S} given in (6.5), and so $\bar{S} \subset \text{int}(\text{dom } f)$ as noted earlier.

It follows from Proposition 4.3 that the pair $\{f, S\}$ satisfies the MQC (4.6) at every point $x \in \bar{S}$. To apply Theorem 4.6 to this problem, take any $\bar{x} \in \bar{S}$ and observe that if $\bar{x} \neq x_1$ and $\bar{x} \neq x_2$, then $T_{\bar{x}}S = T_{\bar{x}}\bar{S}$. This gives $N_S(\bar{x}) + T_{\bar{x}}\bar{S} = T_{\bar{x}}M$, and hence for any $\alpha > 0$ we have the inclusion

$$\alpha\mathbb{B}_{\bar{x}} \subset T_{\bar{x}}M = \partial f(\bar{x}) + N_S(\bar{x}) + T_{\bar{x}}\bar{S} \quad \text{whenever } \bar{x} \in \bar{S} \setminus \{x_1, x_2\}, \quad (6.8)$$

which justifies the fulfillment of assertion (ii) of Theorem 4.6. It remains to consider the case when either $\bar{x} = x_1$ or $\bar{x} = x_2$. For definiteness, suppose that $\bar{x} = x_1$. Then we get

$$T_{\bar{x}}S = \{(0, y_2, y_3) \in \mathbb{R}^3 \mid y_2 + \sqrt{2}y_3 = 0\},$$

$$T_{\bar{x}}\bar{S} = \{(0, y_2, y_3) \in \mathbb{R}^3 \mid y_2 + \sqrt{2}y_3 = 0, y_3 \leq 0\},$$

$$N_S(\bar{x}) = \{(y_1, 0, 0) \mid y_1 \in \mathbb{R}\}.$$

Therefore we arrive at the representation

$$N_S(\bar{x}) + T_{\bar{x}}\bar{S} = \{(y_1, y_2, y_3) \in \mathbb{R}^3 \mid y_2 + \sqrt{2}y_3 = 0, y_3 \leq 0\}.$$

Using further the subdifferential formula (6.3) and the sum rule (4.7) gives us

$$\partial f(\bar{x}) = \mathbb{B}_{\bar{x}} + \left\{ -\frac{\exp_{\bar{x}}^{-1} x_2}{d(\bar{x}, x_2)} \right\} = \mathbb{B}_{\bar{x}} + \frac{1}{\sqrt{3}}(0, -\sqrt{2}, 1),$$

which implies in turn that

$$\partial f(\bar{x}) + N_S(\bar{x}) + T_{\bar{x}}\bar{S} \supset \frac{2}{\sqrt{3}}(0, -\sqrt{2}, 1) + N_S(\bar{x}) + T_{\bar{x}}\bar{S}$$

and

$$2\mathbb{B}_{\bar{x}} \subset \text{cl}(\partial f(\bar{x}) + N_S(\bar{x}) + T_{\bar{x}}\bar{S}).$$

Combining the latter with and (6.8) yields the inclusion

$$2\mathbb{B}_{\bar{x}} \subset \text{cl}(\partial f(\bar{x}) + N_S(\bar{x}) + T_{\bar{x}}\bar{S}) \quad \text{for any } \bar{x} \in \bar{S},$$

which shows that \bar{S} in (6.5) is the set of weak sharp minima for problem (4.1) with the cost function f in (6.2) and the constraint set S defined in (6.7) by equivalence (i) \iff (ii) of Theorem 4.6.

The next example illustrates the application of the obtained characterizations of weak sharp minima in an essentially different framework in comparison with that considered in Example 6.1.

EXAMPLE 6.2. (Optimization problems on Poincaré planes). Let

$$M = H := \{(y_1, y_2) \in \mathbb{R}^2 \mid y_2 > 0\}$$

be the *Poincaré plane* endowed with the Riemannian metric defined by

$$g_{11} = g_{22} = \frac{1}{y_2}, \quad g_{12} = 0 \quad \text{for each } (y_1, y_2) \in H;$$

(cf. [62, p. 86]). Taking $\bar{x} = (y_1, y_2) \in H$, we get $T_{\bar{x}}H = \mathbb{R}^2$ and

$$\langle u, v \rangle_{\bar{x}} = \frac{1}{y_2^2} \langle u, v \rangle \quad \text{for any pair } (u, v) \in T_{\bar{x}}H \times T_{\bar{x}}H, \quad (6.9)$$

where $\langle \cdot, \cdot \rangle_{\bar{x}}$ and $\langle \cdot, \cdot \rangle$ denote the inner products on $T_{\bar{x}}H$ and \mathbb{R}^2 , respectively, as in Example 6.1. Hence,

$$\mathbb{B}_{\bar{x}} = \{v \in T_{\bar{x}}H \mid \|v\|_{\bar{x}} \leq 1\} = \{v \in T_{\bar{x}}H \mid \|v\| \leq y_2\}.$$

The geodesics of the Poincaré plane are the semilines $C_a : x = a, y > 0$ and the semicircles $C_{b,r} : (x - b)^2 + y^2 = r^2, y > 0$. Let the constraint set S in problem (4.1) be given by

$$S := \{(y_1, y_2) \in \mathbb{R}^2 \mid (y_1 - 1)^2 + y_2^2 \leq 5, (y_1 + 1)^2 + y_2^2 \leq 5, y_2 \geq 1\}, \quad (6.10)$$

which is obviously convex. Define $f : M \rightarrow \mathbb{R}$ by

$$f(y_1, y_2) := \frac{1}{y_2} \quad \text{for each } (y_1, y_2) \in M \quad (6.11)$$

and observe that it is convex on M ; see [62, p. 86]. Given any $\lambda \in (0, 1)$, construct the parametric cost function $f_\lambda : M \rightarrow \mathbb{R}$ as

$$f_\lambda(y_1, y_2) := \max \{f(y_1, y_2), \lambda\} \quad \text{for each } (y_1, y_2) \in M, \quad (6.12)$$

which is convex on M . Furthermore, the solution set for problem (4.1) with the cost function f_λ is computed by

$$\bar{S}_\lambda = \begin{cases} \{(0, 2)\} & \text{if } \lambda \leq \frac{1}{2}; \\ S \cap \{(y_1, y_2) \in M \mid y_2 \geq \frac{1}{\lambda}\} & \text{if } \lambda \geq \frac{1}{2}. \end{cases} \quad (6.13)$$

Since $\bar{S}_\lambda \subset \text{int}(\text{dom } f_\lambda)$, it follows by Proposition 4.3 that the pair $\{f_\lambda, S\}$ satisfies the MQC (4.6) at every point $x \in \bar{S}_\lambda$. To apply Theorem 4.6, we split our consideration into two cases regarding the parameter $\lambda \in (0, 1)$: **(a)** $0 < \lambda \leq \frac{1}{2}$ and **(b)** $\frac{1}{2} < \lambda < 1$.

(a) Let $0 < \lambda \leq \frac{1}{2}$. By (6.13) the only choice for $\bar{x} \in \bar{S}_\lambda$ in this case is $\bar{x} = (0, 2)$. Using then relationship (6.9), we compute $T_{\bar{x}}\bar{S}_\lambda = \{0\}$,

$$T_{\bar{x}}S = \{t_1(-1, -\frac{1}{2}) + t_2(1, -\frac{1}{2}) \mid t_1 \geq 0, t_2 \geq 0\},$$

$$N_S(\bar{x}) = \{t_1(-1, 2) + t_2(1, 2) \mid t_1 \geq 0, t_2 \geq 0\}.$$

It follows furthermore that

$$\partial f_\lambda(\bar{x}) = \begin{cases} \{(0, -\frac{1}{2})\} & 0 < \lambda < \frac{1}{2}; \\ \{t(0, -\frac{1}{2}) \mid 0 \leq t \leq 1\} & \lambda = \frac{1}{2}. \end{cases}$$

The latter readily implies the inclusion

$$\frac{1}{5\sqrt{5}}\mathbb{B}_{\bar{x}} \subset \text{cl}\left(\left(0, -\frac{1}{2}\right) + N_S(\bar{x}) + T_{\bar{x}}\bar{S}_\lambda\right) \subset \text{cl}(\partial f_\lambda(\bar{x}) + N_S(\bar{x}) + T_{\bar{x}}\bar{S}_\lambda),$$

which ensures by the equivalence (i) \iff (ii) of Theorem 4.6 that \bar{S}_λ in (6.13) is the set of global weak sharp minima for problem (4.1) with the cost function f_λ and the constraint set S defined in (6.12) and (6.10), respectively.

(b) Let $\frac{1}{2} < \lambda < 1$. In this case we have from (6.10) and (6.13) that

$$\bar{S}_\lambda = \left\{ (y_1, y_2) \in \mathbb{R}^2 \mid (y_1 - 1)^2 + y_2^2 \leq 5, (y_1 + 1)^2 + y_2^2 \leq 5, y_2 \geq \frac{1}{\lambda} \right\}.$$

Denote $\bar{x}_{-1} := (1 - \frac{\sqrt{5\lambda^2 - 1}}{\lambda}, \frac{1}{\lambda})$, $\bar{x}_1 := (-1 + \frac{\sqrt{5\lambda^2 - 1}}{\lambda}, \frac{1}{\lambda})$,

$$\bar{S}_\lambda^0 := \left\{ (y_1, y_2) \in \bar{S}_\lambda \mid y_2 = \frac{1}{\lambda} \right\}$$

and pick $\bar{x} \in \bar{S}_\lambda$. If $\bar{x} \in \bar{S}_\lambda \setminus \bar{S}_\lambda^0$, we get $T_{\bar{x}}S = T_{\bar{x}}\bar{S}_\lambda$, and thus $N_S(\bar{x}) + T_{\bar{x}}\bar{S}_\lambda = T_{\bar{x}}M$. Then

$$\alpha \mathbb{B}_{\bar{x}} \subset \text{cl}(\partial f_\lambda(\bar{x}) + N_S(\bar{x}) + T_{\bar{x}}\bar{S}_\lambda) \quad \text{for any } \alpha > 0, \quad (6.14)$$

which is the assertion of Theorem 4.6(ii) for $\bar{x} \in \bar{S}_\lambda \setminus \bar{S}_\lambda^0$. It remains to consider the case when $\bar{x} \in \bar{S}_\lambda^0$. In this case we compute the subdifferential of f_λ at \bar{x} by

$$\partial f_\lambda(\bar{x}) = \{t(0, -2\lambda^2) \mid 0 \leq t \leq 1\}. \quad (6.15)$$

Suppose first that $\bar{x} \neq \bar{x}_1$ and $\bar{x} \neq \bar{x}_{-1}$. Then we get $T_{\bar{x}}S = T_{\bar{x}}M = \mathbb{R}^2$, $N_S(\bar{x}) = \{0\}$, and

$$T_{\bar{x}}\bar{S}_\lambda = \{(u_1, u_2) \in \mathbb{R}^2 \mid u_2 \geq 0\}.$$

This implies together with (6.15) that

$$\lambda^2 \mathbb{B}_{\bar{x}} \subset \text{cl}(\partial f_\lambda(\bar{x}) + N_S(\bar{x}) + T_{\bar{x}}\bar{S}_\lambda), \quad (6.16)$$

which again gives the inclusion in Theorem 4.6(ii). Finally, we need to check the latter inclusion when either $\bar{x} = \bar{x}_1$ or $\bar{x} = \bar{x}_{-1}$. Assuming without loss of generality that $\bar{x} = \bar{x}_{-1}$ gives us the relationships

$$T_{\bar{x}}\bar{S}_\lambda = \{t_1(1, \sqrt{5\lambda^2 - 1}) + t_2(1, 0) \mid t_1 > 0, t_2 > 0\},$$

$$T_{\bar{x}}S = \left\{ (u_1, u_2) \in \mathbb{R}^2 \mid -u_1 + \frac{1}{\sqrt{5\lambda^2 - 1}}u_2 \leq 0 \right\},$$

$$N_S(\bar{x}) = \left\{ t(-1, \frac{1}{\sqrt{5\lambda^2 - 1}}) \mid t \geq 0 \right\}.$$

Hence it follows (6.15) that

$$\left(\lambda \sqrt{\lambda^2 - \frac{1}{5}} \right) \mathbb{B}_{\bar{x}} \subset \text{cl}(\partial f_\lambda(\bar{x}) + N_S(\bar{x}) + T_{\bar{x}}\bar{S}_\lambda)$$

in this case. Combining the latter inclusion with those in (6.14) and (6.16) shows that the assertion of Theorem 4.6(ii) holds whenever $\bar{x} \in \bar{S}_\lambda$. Thus the set \bar{S}_λ in (6.13) is the set of global weak sharp minima for problem (4.1) with the cost function f_λ in (6.12) and the constraint set S in (6.10).

Our final result shows that the obtained characterizations of weak sharp minimizers allow us to establish a certain stability of constrained optimization problems under data perturbations. Namely, we find verifiable conditions ensuring that optimal solutions to appropriate perturbations solve also the original problem (4.1).

For a nonempty closed subset S of a Riemannian manifold M , denote by \sum_S the collection of all proper convex functions $f : M \rightarrow \overline{\mathbb{R}}$ such that the pair $\{f, S\}$ satisfies the MQC (4.2) on S . For any $f \in \sum_S$, the symbol \overline{S}_f signifies in what follows the set of optimal solutions for problem (4.1) with the cost function f and the constraint set S .

THEOREM 6.3. (Solutions to perturbed optimization problems.) *Given $\alpha > 0$ and $f \in \sum_S$, suppose that the set \overline{S}_f of optimal solutions for the original problem (4.1) is the set of global weak sharp minima for this problem with modulus $\alpha > 0$. Assume also that for each $x \in \overline{S}_f$ the subdifferential $\partial f(x)$ of f at x is a singleton $\{\nabla f(x)\}$. Then for any $\beta \in (0, \alpha)$ and $g \in \sum_S$ satisfying the estimate*

$$\sup_{x \in \overline{S}_f} \inf_{w \in \partial g(x)} \|w - \nabla f(x)\| < \beta \quad (6.17)$$

we have the inclusion $\overline{S}_g \subset \overline{S}_f$, i.e., any solution to the perturbed optimization problem in form (4.1) with the cost function $g : M \rightarrow \overline{\mathbb{R}}$ and the constraint set S is an optimal solution to the original problem (4.1) with the cost function $f : M \rightarrow \overline{\mathbb{R}}$ and the same constraint set.

Proof. Pick any $\bar{x} \in \overline{S}_f$. Since \overline{S}_f is the set of global weak sharp minima with modulus $\alpha > 0$ for the original problem (4.1), it follows from Theorem 4.6(ii) that

$$\alpha \mathbb{B}_{\bar{x}} \subset \text{cl}(\nabla f(\bar{x}) + N_S(\bar{x}) + T_{\bar{x}}\overline{S}_f).$$

Fix $\beta \in (0, \alpha)$ and take any $\gamma \in (\beta, \alpha)$. Then we get

$$-\nabla f(\bar{x}) + \gamma \mathbb{B}_{\bar{x}} \subset (N_S(\bar{x}) + T_{\bar{x}}\overline{S}_f).$$

Combining the latter with (6.17) for the corresponding function $g \in \sum_S$ gives us $w \in \partial g(\bar{x})$ satisfying

$$-w + \mu \mathbb{B}_{\bar{x}} \subset (N_S(\bar{x}) + T_{\bar{x}}\overline{S}_f),$$

where $\mu := \frac{\gamma - \beta}{2}$. It follows therefore that

$$\mu \mathbb{B}_{\bar{x}} \subset (w + N_S(\bar{x}) + T_{\bar{x}}\overline{S}_f) \subset (\partial g(\bar{x}) + N_S(\bar{x}) + T_{\bar{x}}\overline{S}_f).$$

This implies, together with the implication in (4.33), that

$$\mu d_{\overline{S}_f}(x) \leq g(x) - g(\bar{x}) \quad \text{whenever } x \in S \text{ and } \bar{x} \in P(x|\overline{S}_f). \quad (6.18)$$

Take now arbitrary elements $x \in \overline{S}_g$ and $\bar{x} \in P(x|\overline{S}_f)$. Then $g(x) \leq g(\bar{x})$, and by (6.18) we have

$$\mu d_{\overline{S}_f}(x) \leq g(x) - g(\bar{x}) \leq 0.$$

The latter yields that $x \in \overline{S}_f$. Thus $\overline{S}_g \subset \overline{S}_f$, which completes the proof of the theorem. \square

It is worth observing that the perturbation condition (6.17) is ensured by

$$\sup_{x \in S \cap \text{dom } f} \inf_{w \in \partial g(x)} \|w - \nabla f(x)\| < \beta, \quad (6.19)$$

which does not involve the solution set \overline{S}_f for the original problem (4.1).

Observe finally that Example 6.2 presents an optimization problem satisfying all the assumptions of Theorem 6.3. Indeed, for the function f defined by (6.11) we have $f \in \sum_S$ and $\text{dom } f = M$. Since $f = f_{\frac{1}{4}}$ on S , it follows that $\overline{S}_f = \{(0, 2)\}$ is the set of global weak sharp minima with modulus $\alpha = \frac{1}{5\sqrt{5}}$ for the original problem (4.1). Given further $x_0 = (0, 1)$ and $0 < \beta < \frac{1}{5\sqrt{5}}$, define a function $g_\beta: M \rightarrow \mathbb{R}$ by $g_\beta(\cdot) := f(\cdot) + \beta d(x_0, \cdot)$. Then $g_\beta \in \sum_S$ and

$$\partial g(x) = \nabla f(x) + \beta \partial d(x_0, \cdot)(x) \quad \text{for each } x \in M.$$

Thus it is easy to see that condition (6.17) is satisfied; in fact, condition (6.19) is satisfied since $d(x_0, \cdot)$ is global Lipschitzian on M with constant $\ell = 1$ by Proposition 3.1. Hence all the assumptions of Theorem 6.3 are satisfied in the setting of Example 6.2.

7. Concluding remarks. To the best of our knowledge, this paper is *the first one* in the literature dealing with weak sharp minima for constrained optimization problems on *Riemannian* and *Hadamard manifolds*. The main characterizations of global, boundedly, and local sharp minima for convex problems on Riemannian manifolds in assertions (ii), (iii) and (v) of Theorems 4.6, 4.7, 4.9 and Corollary 4.10 are *new* even for the case of *finite-dimensional Euclidean spaces*. The other characterizations, including those for Hadamard manifolds from Section 5, are extensions of the corresponding results by Burke and Ferris [11] and Burke and Deng [13] obtained in the case of spaces with linear structures. Observe that to proceed with no linearity, we need to develop new methods and results of variational analysis on Riemannian and Hadamard manifolds including, in particular, the *characterization of projections* on convex subsets of Riemannian manifolds given in Proposition 3.4 and the normal cone representation for the *subdifferential* of the *distance function* for convex subsets of Hadamard manifolds derived in Theorem 5.3. These results seem to be of independent interest for various aspects of analysis on Riemannian and Hadamard manifolds regardless of their particular applications to the study of weak sharp minima.

Note that this paper mainly concerns *convex* problems of constrained optimization on Riemannian and Hadamard manifolds while some of our methods and results can be used for the further analysis of weak sharp minima in *nonconvex* optimization problems on spaces with no linear structure. In this respect we mention, in particular, the possibility of direct extending to the case of Hadamard manifolds necessary optimality conditions for weak sharp minima obtained in [11, 56] for nonconvex optimization problems on Banach spaces in terms of Clarke subgradients, Fréchet subgradients and Mordukhovich basic/limiting and singular subgradients of l.s.c. functions and the corresponding normals to closed subsets of Banach spaces. The latter constructions have been partly explored for other purposes in the recent study [41] in the case of smooth nonlinear manifolds. Among the key ingredients of the aforementioned developments in [56] is the subdifferential formula of type (5.1) for the distance function, which admits appropriate extensions to nonconvex frameworks in spaces with no linearity.

Similar to the conventional setting of finite-dimensional Euclidean spaces, there are strong connections between theoretical and numerical aspects of weak sharp minima on Riemannian and Hadamard manifolds, which we intend to largely explore in our future research.

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