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Existence of best simultaneous approximations in $L_p(S, \Sigma, X)$

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Abstract

Let (S, Σ, μ) be a complete positive σ-finite measure space and let X be a Banach space. We are concerned with the proximinality problem for the best simultaneous approximations to two functions in $L_p(S, \Sigma, X)$. Let Σ_0 be a sub-σ-algebra of Σ and Y a nonempty locally weakly compact convex subset of X such that $\overline{\operatorname{span} Y}$ and its dual have the Radon–Nikodym property. We prove that $L_p(S, \Sigma_0, Y)$ is N-simultaneous proximinal in $L_p(S, \Sigma, X)$ (with the additional assumption that (S, Σ, μ) be finite for the case when $p = 1$). Furthermore, for the special case when $\Sigma_0 = \Sigma$, we show that the assumption that the dual of $\overline{\operatorname{span} Y}$ has the Radon–Nikodym property can be removed.

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1. Introduction

Let X be a Banach space with norm $\|\cdot\|$ and (S, Σ, μ) a complete positive σ-finite measure space. Let $1 \le p \le \infty$ and let $L_p(S, \Sigma, X)$ denote the Banach space of all Bochner p-integrable

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(essentially bounded for $p = \infty$) functions defined on S with values in X, endowed with the norm $\|\cdot\|_p$ defined by

$$\|f\|_p = \begin{cases} \left(\int_S \|f(s)\|^p \mathrm{d}\mu\right)^{1/p}, & 1 \le p < \infty \\ \operatorname{ess\,sup} \|f(s)\|, & p = \infty \end{cases} \quad \text{for each } f \in L_p(S, \Sigma, X).$$

Let G be a nonempty subset of $L_p(S, \Sigma, X)$ and let $f \in L_p(S, \Sigma, X)$. Recall that an element g_0 of G is called a best approximation to f from G if

$$\|f - g_0\|_p = d_p(f, G) := \inf\{\|f - g\|_p : g \in G\}.$$

The set of all best approximations to f from G is denoted by $P_G(f)$. Recall also that G is proximinal in $L_p(S, \Sigma, X)$ if $P_G(f) \neq \emptyset$ for each $f \in L_p(S, \Sigma, X)$.

In the case when Y is a closed subspace of X, the problem whether $L_p(S, \Sigma, Y)$ is proximinal in $L_p(S, \Sigma, X)$ has been studied deeply and extensively, see for example [4–6,11,13,15,18]. In particular, in the case when (S, Σ, μ) is a finite measure space, it was proved in [4] that $L_1(S, \Sigma, Y)$ is proximinal in $L_1(S, \Sigma, X)$ if Y is reflexive and in [5] that $L_p(S, \Sigma, Y)$ is proximinal in $L_p(S, \Sigma, X)$ if and only if $L_1(S, \Sigma, Y)$ is proximinal in $L_1(S, \Sigma, X)$. These results have been extended to the case when (S, Σ, μ) is a σ-finite measure space in [13], where it was further proved for a closed separable subspace Y that $L_p(S, \Sigma, Y)$ is proximinal in $L_p(S, \Sigma, X)$ if and only if Y is proximinal in X.

The problem of the best simultaneous approximations in $L_p(S, \Sigma, X)$ was studied in [17]. The setting is as follows. Let m be a positive integer or $m = +\infty$ and let $N(\cdot)$ be a monotonic norm in the space $\mathbb{R}^m$ (where $\mathbb{R}^m$ is a linear space consisting of some real sequences in the case when $m = +\infty$) in the sense that, for each pair of $\mathbf{a} = (a_i), \mathbf{b} = (b_i) \in \mathbb{R}^m$, the condition $|a_i| \le |b_i|$ for each $i = 1, \ldots, m$ implies that $N(\mathbf{a}) \le N(\mathbf{b})$. Let $\{f_i\}_{i=1}^m \subseteq L_p(S, \Sigma, X)$ satisfying $(\|f_i\|_p) \in \mathbb{R}^m$ and $G \subseteq L_p(S, \Sigma, X)$. Then the best simultaneous approximation problem considered here is to find an element $g_0 \in G$ such that

$$N((\|f_i - g_0\|_p)) \le N((\|f_i - g\|_p)) \quad \text{for each } g \in G.$$

Such an element g_0 is called a best N-simultaneous approximation to $\{f_i\}_{i=1}^m$ from G. Characterization results were established in [17] for the case when $m < +\infty$, extensions of which to general Banach spaces and/or to infinitely many elements in Banach spaces are referred to [7–10,12].

Saidi et al. considered in [16] the proximinality problem for the best N-simultaneous approximations in $L_p(S, \Sigma, X)$. Recall that a nonempty subset G of $L_p(S, \Sigma, X)$ is called N-simultaneously proximinal in $L_p(S, \Sigma, X)$ if each $\{f_i\}_{i=1}^m \subseteq L_p(S, \Sigma, X)$ satisfying $(\|f_i\|_p) \in \mathbb{R}^m$ admits a best N-simultaneous approximation from G. In the case when $m < +\infty$, Y is a reflexive subspace and (S, Σ, μ) is a finite measure space, it was proved in [16] that $L_p(S, \Sigma, Y)$ is N-simultaneous proximinal in $L_p(S, \Sigma, X)$ for each $1 \le p < \infty$, which clearly extends the corresponding results in [4, Theorem 1.2], [5, Theorem 1.2] and [15, Theorem 4.2] for best approximation problems.

Recently, Mendoza and Pakhrou considered in [14] the N-simultaneous proximinality problem of $L_1(S, \Sigma_0, X)$ in $L_1(S, \Sigma, X)$ for the case when $m < +\infty$, where Σ_0 is a sub-σ-algebra of Σ. They proved that $L_1(S, \Sigma_0, X)$ is N-simultaneous proximinal in $L_1(S, \Sigma, X)$ if X is reflexive. As shown by Mendoza and Pakhrou, in [14], the N-simultaneous proximinality is strictly stronger than the (ordinary) proximinality. In fact, they showed in [14, Example 2] that there exist a Banach space X and a sub-σ-algebra Σ_0 such that $L_1(S, \Sigma_0, X)$ is proximinal but not N-simultaneously proximal in $L_1(S, \Sigma, X)$.

In the present paper, we will continue to study the N-simultaneous proximinality problem of $L_p(S, \Sigma_0, Y)$ or $L_p(S, \Sigma, Y)$ in $L_p(S, \Sigma, X)$ but in the more general case. More precisely, we assume that $m = 2$ for simplicity and that (S, Σ, μ) is a complete positive σ-finite measure space and Y a locally weakly compact closed convex subset of X. It should be remarked that this problem for the general case is nontrivial and more difficult. We will prove in the present paper that $L_p(S, \Sigma_0, Y)$ is N-simultaneous proximinal in $L_p(S, \Sigma, X)$ for each $1 \le p < +\infty$ (with the additional assumption that (S, Σ, μ) be finite for the case when $p = 1$) if $\overline{\operatorname{span} Y}$ and $\overline{\operatorname{span} Y}^*$ have the Radon–Nikodym property. While for the special case when $\Sigma_0 = \Sigma$, we will show that $L_p(S, \Sigma, Y)$ is N-simultaneous proximinal in $L_p(S, \Sigma, X)$ for each $1 \le p < \infty$ if $\overline{\operatorname{span} Y}$ has the Radon–Nikodym property. These results extend and improve both [16, Theorems 3 and 4] and the main theorem of [14]. Even in the case of $m = 1$, namely, the case of the best approximation, our results in the present paper seem new.

2. Auxiliary lemmas

Let (S, Σ, μ) be a complete positive σ-finite measure space and let X be a Banach space with norm $\|\cdot\|$. Let $1 \le p < \infty$, $Y \subseteq X$, and Σ_0 a sub-σ-algebra of Σ. We use $L_p(S, \Sigma_0, Y)$ to denote the subset of $L_p(S, \Sigma, X)$ defined by

$$L_p(S, \Sigma_0, Y) = \{g \in L_p(S, \Sigma_0, X) : g(s) \in Y \text{ for a.e. } s \in S\}.$$

For a set $A \in \Sigma$, let χ_A stand for the characteristic function of A; that is, $\chi_A(s) = 1$ if $s \in A$ and $\chi_A(s) = 0$ otherwise. For a point $x \in X$ and $r > 0$, we use $\mathbf{B}(x, r)$ to denote the closed ball with center x and radius r. Recall that a subset Y of X is locally weakly compact (resp. boundedly weakly compact) if, for each point $y \in Y$ (resp. for each $r > 0$), there exists $\delta > 0$ such that $\mathbf{B}(y, \delta) \cap Y$ (resp. $\mathbf{B}(0, r) \cap Y$) is weakly compact. To prepare for the proof of the main theorem of this paper, we will verify some useful lemmas in this section. The first one is trivial and its proof is thus omitted.

Lemma 2.1. *Suppose that Y is a closed convex subset of X. Then the following statements are equivalent.*

(i) *Y is locally weakly compact.*
(ii) *Y is boundedly weakly compact.*
(iii) *There exist a point $y \in Y$ and $\delta > 0$ such that $\mathbf{B}(y, \delta) \cap Y$ is weakly compact.*

Throughout the rest of this paper, we shall always assume that N is a monotonic norm in the space $\mathbb{R}^2$ and that Y is a closed convex subset of X such that $L_p(S, \Sigma_0, Y)$ is nonempty. Thus, without loss of generality, we may assume that $0 \in Y$. In fact, in the case when $\mu(S) = +\infty$, it holds automatically because $L_p(S, \Sigma_0, Y)$ is nonempty; while in the case when $\mu(S) < +\infty$, one can take $y_0 \in Y$ and consider $Y - y_0$ in place of Y if necessary. The following lemma is an extension of [16, Corollary 1].

Lemma 2.2. *Let $f^1, f^2 \in L_p(S, \Sigma, X)$ be a pair of countably valued functions. Then f^1, f^2 admit a best N-simultaneous approximation from $L_p(S, \Sigma, Y)$.*

Proof. Let $k = 1, 2$ and assume that $f^k = \sum_{i=1}^{\infty} x_i^k \chi_{A_i}$ for some sequence $\{A_i\}$ of disjoint measurable sets in S and some sequence $\{x_i^k\} \subseteq X$. Then $\mu(A_i) < \infty$ whenever $x_i^k \neq 0$ because

$$\|f^k\|_p^p = \sum_{i=1}^{\infty} \|x_i^k\|^p \mu(A_i) < \infty.$$

Thus, we may assume that $0 < \mu(A_i) < \infty$ for each $i \in \mathbb{N}$. Set

$$\mathcal{G} = \left\{ g = \sum_{i=1}^{\infty} y_i \chi_{A_i} : g \in L_p(S, \Sigma, Y) \right\}$$

and

$$\delta(f^1, f^2; g) = N(\|f^1 - g\|_p, \|f^2 - g\|_p) \quad \text{for each } g \in \mathcal{G}.$$

We first show that there exists $g_0 \in \mathcal{G}$ such that

$$\delta(f^1, f^2; g_0) = \delta(f^1, f^2) := \inf\{\delta(f^1, f^2; g) : g \in \mathcal{G}\}. \tag{2.1}$$

To do this, let $\{g^n\} \subseteq \mathcal{G}$ be a sequence such that $\delta(f^1, f^2; g^n) \to \delta(f^1, f^2)$. Then there exists some positive number M_1 such that $\delta(f^1, f^2; g^n) \leq M_1$ for all n. Let $n \in \mathbb{N}$ and assume that $g^n = \sum_{i=1}^{\infty} y_i^n \chi_{A_i}$. Then, thanks to the monotonicity of the norm N,

$$\begin{aligned} N(1,1) \left(\sum_{i=1}^{\infty} \|y_i^n\|^p \mu(A_i) \right)^{\frac{1}{p}} &= N(\|g^n\|_p, \|g^n\|_p) \\ &\leq N(\|f^1 - g^n\|_p + \|f^1\|_p, \|f^2 - g^n\|_p + \|f^2\|_p) \\ &\leq N(\|f^1 - g^n\|_p, \|f^2 - g^n\|_p) + N(\|f^1\|_p, \|f^2\|_p) \\ &\leq M_1 + N(\|f^1\|_p, \|f^2\|_p). \end{aligned}$$

This shows that, for each i, $\{y_i^n\}_{n=1}^{\infty}$ is bounded in Y. Since Y is locally weakly compact, it follows from Lemma 2.1 that $\{y_1^n\}$ has a weakly convergent subsequence $\{y_1^{n,1}\}$ (say), with weak limit y_1. Then $y_1 \in Y$ because Y is a closed convex subset of X. Similarly, noting that $\{y_2^{n,1}\}$ is a subsequence of $\{y_2^n\}$, there exists a subsequence $\{y_2^{n,2}\}$ of $\{y_2^{n,1}\}$ such that $\lim_{n\to\infty} y_2^{n,2} = y_2$ weakly for some $y_2 \in Y$. Continuing in this way, one has that, for each i, there exists a subsequence $\{y_{i+1}^{n,i+1}\}$ of $\{y_{i+1}^{n,i}\}$ such that $\{y_{i+1}^{n,i+1}\}$ weakly converges to some element $y_{i+1} \in Y$. Since, for each fixed natural number m and each $i = 1, \ldots, m$, $\{y_i^{n,m}\}$ is a subsequence of $\{y_i^{n,i}\}$, we have that $\lim_n y_i^{n,m} = y_i$ weakly by the choice of $\{y_i^{n,i}\}$. Using the weak lower semicontinuity of the norm of X, we obtain that

$$\|x_i^k - y_i\| \leq \liminf_n \|x_i^k - y_i^{n,m}\| \quad \text{for each } i = 1, \ldots, m.$$

Thus

$$\begin{aligned} & N\left(\left(\sum_{i=1}^{m} \|x_i^1 - y_i\|^p \mu(A_i) \right)^{\frac{1}{p}}, \left(\sum_{i=1}^{m} \|x_i^2 - y_i\|^p \mu(A_i) \right)^{\frac{1}{p}} \right) \\ & \quad \leq N\left(\left(\sum_{i=1}^{m} \liminf_n \|x_i^1 - y_i^{n,m}\|^p \mu(A_i) \right)^{\frac{1}{p}}, \left(\sum_{i=1}^{m} \liminf_n \|x_i^2 - y_i^{n,m}\|^p \mu(A_i) \right)^{\frac{1}{p}} \right) \end{aligned}$$

$$\begin{aligned}
&\le N\left(\liminf_n\left(\sum_{i=1}^m \|x_i^1 - y_i^{n,m}\|^p\mu(A_i)\right)^{\frac{1}{p}}, \liminf_n\left(\sum_{i=1}^m \|x_i^2 - y_i^{n,m}\|^p\mu(A_i)\right)^{\frac{1}{p}}\right)\\
&\le \liminf_n N\left(\left(\sum_{i=1}^m \|x_i^1 - y_i^{n,m}\|^p\mu(A_i)\right)^{\frac{1}{p}}, \left(\sum_{i=1}^m \|x_i^2 - y_i^{n,m}\|^p\mu(A_i)\right)^{\frac{1}{p}}\right)\\
&\le \liminf_n N\left(\left(\sum_{i=1}^\infty \|x_i^1 - y_i^{n,m}\|^p\mu(A_i)\right)^{\frac{1}{p}}, \left(\sum_{i=1}^\infty \|x_i^2 - y_i^{n,m}\|^p\mu(A_i)\right)^{\frac{1}{p}}\right)\\
&= \liminf_n \delta(f^1, f^2; g^{n,m}) = \delta(f^1, f^2),
\end{aligned}$$

where the last equality holds because $\{g^{n,m}\}_{n=1}^\infty$ is a subsequence of $\{g^n\}$. Letting $m \to \infty$ in the above expression, we get that

$$N\left(\left(\sum_{i=1}^\infty \|x_i^1 - y_i\|^p\mu(A_i)\right)^{\frac{1}{p}}, \left(\sum_{i=1}^\infty \|x_i^2 - y_i\|^p\mu(A_i)\right)^{\frac{1}{p}}\right) \le \delta(f^1, f^2). \tag{2.2}$$

It follows that $\sum_{i=1}^\infty \|y_i\|^p\mu(A_i) < \infty$. Define $g_0 = \sum_{i=1}^\infty y_i\chi_{A_i}$. Then $g_0 \in \mathcal{G}$ and

$$N(\|f^1 - g_0\|_p, \|f^2 - g_0\|_p) = \delta(f^1, f^2)$$

thanks to (2.2). Hence (2.1) is proved.

Next we claim that

$$\delta(f^1, f^2) \le N(\|f^1 - h\|_p, \|f^2 - h\|_p) \quad \text{for each } h \in L_p(S, \Sigma, Y). \tag{2.3}$$

Granting this, one has that g_0 is a best N-simultaneous approximation to $\{f_1, f_2\}$ from $L_p(S, \Sigma, Y)$ and the proof is then complete.

It therefore remains to show (2.3). Let $h \in L_p(S, \Sigma, Y)$ be a countably valued function that has the expression $h = \sum_{j=1}^\infty h_j\chi_{B_j}$ for some sequence $\{B_j\}$ of disjoint measurable sets in S and some sequence $\{h_j\} \subseteq Y$. Then f^k and h can be rewritten respectively as

$$f^k = \sum_{i,j=1}^\infty x_{ij}^k\chi_{A_i\cap B_j} \quad \text{and} \quad h = \sum_{i,j=1}^\infty h_{ij}\chi_{A_i\cap B_j},$$

where

$$x_{ij}^k = x_i^k \quad \text{and} \quad h_{ij} = h_j \quad \text{for each } i, j = 1, 2, \ldots.$$

Then

$$\sum_{j=1}^\infty \mu(A_i \cap B_j)\|h_j\| \le \|h\|_p(\mu(A_i))^{\frac{1}{q}} \quad \text{for each } i \in \mathbb{N}, \tag{2.4}$$

where and in what follows, q satisfies $\frac{1}{p} + \frac{1}{q} = 1$ if $1 < p < \infty$ and $q = \infty$ if $p = 1$. In fact, (2.4) is clear in the case when $p = 1$. In the case when $1 < p < \infty$, one can apply Hölder's inequality to conclude that

$$\begin{aligned}\sum_{j=1}^{\infty}\mu(A_i\cap B_j)\|h_j\| &\le \left(\sum_{j=1}^{\infty}\left(\frac{\mu(A_i\cap B_j)}{\mu(B_j)^{\frac{1}{p}}}\right)^q\right)^{\frac{1}{q}}\left(\sum_{j=1}^{\infty}\mu(B_j)\|h_j\|^p\right)^{\frac{1}{p}}\\ &= \|h\|_p\left(\sum_{j=1}^{\infty}\left(\frac{\mu(A_i\cap B_j)}{\mu(B_j)}\right)^q\mu(B_j)\right)^{\frac{1}{q}}\\ &\le \|h\|_p\left(\sum_{j=1}^{\infty}\frac{\mu(A_i\cap B_j)}{\mu(B_j)}\mu(B_j)\right)^{\frac{1}{q}}\\ &= \|h\|_p(\mu(A_i))^{\frac{1}{q}}.\end{aligned}$$

Hence (2.4) holds. Set

$$y_i' = \frac{\sum\limits_{j=1}^{\infty}\mu(A_i\cap B_j)h_j}{\mu(A_i)} \quad \text{for each } i\in\mathbb{N}.$$

Then $y_i'\in Y$ because $\sum_{j=1}^{\infty}[\mu(A_i\cap B_j)]/\mu(A_i)=1$ and Y is a closed convex subset. Define $g'=\sum_{i=1}^{\infty}y_i'\chi_{A_i}$. Then $g'\in L_p(S,\Sigma,Y)$ and

$$\begin{aligned}\|f^k-h\|_p^p &= \sum_{i,j}^{\infty}\|x_{ij}^k-h_{ij}\|^p\mu(A_i\cap B_j)\\ &= \sum_{i=1}^{\infty}\mu(A_i)\sum_{j=1}^{\infty}\frac{\mu(A_i\cap B_j)}{\mu(A_i)}\|x_i^k-h_j\|^p\\ &\ge \sum_{i=1}^{\infty}\mu(A_i)\left(\sum_{j=1}^{\infty}\frac{\mu(A_i\cap B_j)}{\mu(A_i)}\|x_i^k-h_j\|\right)^p\\ &\ge \sum_{i=1}^{\infty}\|x_i^k-y_i'\|^p\mu(A_i)\\ &= \|f^k-g'\|_p^p,\end{aligned}$$

where the inequality holds because the function $t\mapsto t^p$ is convex on $[0,+\infty)$. Now from (2.1) and the monotonicity of the norm N, it follows that

$$N(\|f^1-h\|_p,\|f^2-h\|_p)\ge N(\|f^1-g'\|_p,\|f^2-g'\|_p)\ge\delta(f^1,f^2).$$

As the set of all countably valued functions in $L_p(S,\Sigma,Y)$ is dense in $L_p(S,\Sigma,Y)$, (2.3) is seen to hold and the proof is complete. □

The following lemma is useful and taken from [16], where it was proved for the case when Y is a subspace.

Lemma 2.3. *Let g be a best N-simultaneous approximation to $f^1, f^2\in L_1(S,\Sigma,X)$ from $L_1(S,\Sigma,Y)$. Then,*

$$\int_A\|g(s)\|\mathrm{d}\mu\le 2\max\left\{\int_A\|f^1(s)\|\mathrm{d}\mu,\int_A\|f^2(s)\|\mathrm{d}\mu\right\}\quad \textit{for each } A\in\Sigma.$$

The following lemma shows that the set $L_p(S, \Sigma_0, Y)$ is closed.

Lemma 2.4. *$L_p(S, \Sigma_0, Y)$ is a closed convex subset of $L_p(S, \Sigma, X)$.*

Proof. The convexity of $L_p(S, \Sigma_0, Y)$ is clear since Y is convex. Thus we only need to show that $L_p(S, \Sigma_0, Y)$ is closed. To this purpose, let $f \in L_p(S, \Sigma, X)$ and $\{f_n\} \subseteq L_p(S, \Sigma_0, Y)$ be a sequence satisfying $\lim_n \|f_n - f\|_p = 0$. Then by Fatou's lemma, we obtain that

$$\int_S \liminf_n \|f_n(s) - f(s)\|^p \mathrm{d}\mu \le \liminf_n \int_S \|f_n(s) - f(s)\|^p \mathrm{d}\mu = 0.$$

It follows that $\liminf_n \|f_n(s) - f(s)\| = 0$ for a.e. $s \in S$. This implies that $f(s) \in Y$ for almost all $s \in S$ since Y is closed in X, which completes the proof. □

Lemma 2.5. *Let $f \in L_p(S, \Sigma, X)$ be a countably valued function. Then there exists $h^* \in L_q(S, \Sigma, X^*)$ such that $\|h^*\|_q \le 1$ and $\langle f, h^* \rangle = \|f\|_p$.*

Proof. Let $f = \sum_{i=1}^{\infty} x_i \chi_{A_i}$, where $\{A_i\}$ is a sequence of disjoint measurable sets with $\mu(A_i) > 0$ for each $i \in \mathbb{N}$. Without loss of generality, we may assume that $x_i \neq 0$ for each $i \in \mathbb{N}$. For each $i \in \mathbb{N}$, let $x_i^* \in X^*$ be such that $\|x_i^*\| = (\|x_i\|/\|f\|_p)^{p-1}$ and $x_i^*(x_i) = \|x_i^*\|\|x_i\|$. Define $h^* = \sum_{i=1}^{\infty} x_i^* \chi_{A_i}$. Then $h^* \in L_q(S, \Sigma, X^*)$ and $\|h^*\|_q = 1$ because $\sup_i \|x_i^*\| = 1$ if $q = \infty$ and

$$\sum_{i=1}^{\infty} \|x_i^*\|^q \mu(A_i) = \frac{\sum_{i=1}^{\infty} \|x_i\|^p \mu(A_i)}{\|f\|_p^p} = 1 \quad \text{if } q < \infty.$$

On the other hand,

$$\langle f, h^* \rangle = \sum_{i=1}^{\infty} x_i^*(x_i)\mu(A_i) = \sum_{i=1}^{\infty} \|x_i^*\|\|x_i\|\mu(A_i) = \sum_{i=1}^{\infty} \frac{\|x_i\|^p}{\|f\|_p^{p-1}} \mu(A_i) = \|f\|_p,$$

which completes the proof. □

Lemma 2.6. *Suppose that (S, Σ, μ) is a finite measure space. Let $f \in L_p(S, \Sigma, X)$ satisfy $\langle f, h^* \rangle = 0$ for each $h^* \in L_q(S, \Sigma, X^*)$. Then $f = 0$.*

Proof. Let $\{f_n\}$ be a sequence of countably valued functions in $L_p(S, \Sigma, X)$ such that $\|(f_n - f)(s)\| \le \frac{1}{n}$ for a.e. $s \in S$ and each n. Then

$$\|f_n - f\|_p \le \frac{1}{n}(\mu(S))^{\frac{1}{n}} \quad \text{for each } n. \tag{2.5}$$

By Lemma 2.5, for each $n \in \mathbb{N}$, there is $h_n^* \in L_q(S, \Sigma, X^*)$ such that $\|h_n^*\|_q \le 1$ and $\langle f_n, h_n^* \rangle = \|f_n\|_p$. Thus

$$\|f_n\|_p = \langle f_n, h_n^* \rangle = \langle f_n - f, h_n^* \rangle + \langle f, h_n^* \rangle \le \|f_n - f\|_p \quad \text{for each } n.$$

This together with (2.5) implies that

$$\|f\|_p \le \lim_n \|f_n\|_p + \lim_n \|f_n - f\|_p \le 2 \lim_n \|f_n - f\|_p = 0$$

and the proof is complete. □

The following lemma is taken from [14, Lemma 2], which was proved for the case when $Y = X$.

Lemma 2.7. *Let $f^1, f^2 \in L_p(S, \Sigma, X)$ and let $\{g_n\} \subseteq L_p(S, \Sigma_0, Y)$ be a minimizing sequence for best N-simultaneous approximation to f^1, f^2 from $L_p(S, \Sigma_0, Y)$. If $\{A_n\} \subseteq \Sigma_0$ is such that $\lim_{n\to\infty} \mu(A_n) = 0$, then $\{g_n \chi_{A_n^c}\}$ is a minimizing sequence for best N-simultaneous approximation to f^1, f^2 from $L_p(S, \Sigma_0, Y)$.*

Finally we still need the following lemma, see for example [2, Lemma 2.1.3]. Recall that $L_1(\mu)$ is the space of integrable real-valued function on (S, Σ, μ); that is, $L_1(\mu) = L_1(S, \Sigma, \mathbb{R})$.

Lemma 2.8. *Suppose that (S, Σ, μ) is a finite measure space. Let $\{f_m\}$ be a bounded sequence in $L_1(\mu)$. Then there exist a subsequence $\{f_{m_k}\}$ of $\{f_m\}$ and a sequence $\{A_k\}$ of pairwise disjoint measurable sets such that $\{f_{m_k} \chi_{A_k^c}\}$ is uniformly integrable.*

3. Main results

We begin with the well known notion of the Radon–Nikodym property, see for example [3]. A Banach space X is said to have the Radon–Nikodym property with respect to a finite measure space (S, Σ, μ) if, for each μ-continuous vector measure $G : \Sigma \to X$ of bounded variation, there exists $g \in L_1(S, \Sigma, X)$ such that $G(E) = \int_E g \mathrm{d}\mu$ for each $E \in \Sigma$. Whereas X is said to have the Radon–Nikodym property if X has the Radon–Nikodym property with respect to each finite measure space. It is well known that any reflexive Banach space has the Radon–Nikodym property.

Before proving the main theorem of the present paper, we need an extension of Dunford Theorem [3, Theorem IV.2.1]. Recall that q satisfies $\frac{1}{p} + \frac{1}{q} = 1$ if $1 < p < \infty$ and $q = \infty$ if $p = 1$.

Lemma 3.1. *Let (S, Σ_1, μ) be a σ-finite measure space with Σ_1 generated by a countable algebra. Suppose that X has the Radon–Nikodym property. Let $1 \le p < \infty$ and let $\{g_n\}$ be a sequence in $L_p(S, \Sigma_1, X)$ satisfying the following conditions:*

(a) *$\{g_n\}$ is bounded in $L_p(S, \Sigma_1, X)$,*
(b) *$\{g_n\}$ is uniformly integrable,*
(c) *for each $E \in \Sigma_1$ with $\mu(E) < \infty$, $\{\int_E g_n \mathrm{d}\mu\}$ is relatively weakly compact in X.*

Then there exist a subsequence $\{g_{n_k}\}$ of $\{g_n\}$ and $g_0 \in L_p(S, \Sigma_1, X)$ such that for each $E \in \Sigma_1$ with $\mu(E) < \infty$,

$$\lim_k \langle g_{n_k} - g_0, h^* \chi_E \rangle = 0 \quad \text{for each } h^* \in L_q(S, \Sigma_1, X^*). \tag{3.1}$$

Furthermore, in the case when $1 < p < \infty$, (3.1) can be improved to the following assertion

$$\lim_k \langle g_{n_k} - g_0, h^* \rangle = 0 \quad \text{for each } h^* \in L_q(S, \Sigma_1, X^*). \tag{3.2}$$

Proof. We divide the proof into two steps.

Step One: The conclusion of the lemma holds in the case when $\mu(S) < \infty$.

Suppose that $\mu(S) < \infty$. Then (3.1) is equivalent to (3.2). Let $\{E_m\}$ be a countable algebra which generates the σ-algebra Σ_1. Then by (c), a Cantor diagonalization will produce a subsequence $\{g_{n_k}\}$ of $\{g_n\}$ such that for each $E \in \{E_m\}$

$$\text{weak-}\lim_k \int_E g_{n_k} \mathrm{d}\mu \quad \text{exists.} \tag{3.3}$$

We claim that (3.3) holds for each $E \in \Sigma_1$. To do this, let $E \in \Sigma_1$. Since the weak closure of $\{\int_E g_{n_k} \mathrm{d}\mu\}$ is weakly compact by (c), it suffices to verify that $\{\int_E g_{n_k} \mathrm{d}\mu\}$ is a weak Cauchy sequence in X. Let $\epsilon > 0$ and $x^* \in X^* \setminus \{0\}$. Then by (b), there is $\delta > 0$ such that $\int_A \|g_n\| \mathrm{d}\mu < \frac{\epsilon}{4\|x^*\|}$ for each $n \in \mathbb{N}$ whenever $A \in \Sigma_1$ and $\mu(A) < \delta$. By [1, Theorem 1.3.11], there exists E_{k_0} such that $\mu(E \triangle E_{k_0}) < \delta$. It turns out that

$$\int_{E \triangle E_{k_0}} \|g_{n_k}\| \mathrm{d}\mu < \frac{\epsilon}{4\|x^*\|} \quad \text{for each } k \in \mathbb{N}.$$

Since (3.3) holds for $E = E_{k_0}$, there is a positive number N such that

$$\left| x^* \int_{E_{k_0}} g_{n_k} \mathrm{d}\mu - x^* \int_{E_{k_0}} g_{n_m} \mathrm{d}\mu \right| < \frac{\epsilon}{2} \quad \text{for all } k, m > N.$$

Consequently, for all $k, m > N$,

$$\begin{aligned} \left| x^* \int_E g_{n_k} \mathrm{d}\mu - x^* \int_E g_{n_m} \mathrm{d}\mu \right| &= \left| \int_{E_{k_0}} x^*(g_{n_k} - g_{n_m}) \mathrm{d}\mu + \int_{E \setminus E_{k_0}} x^*(g_{n_k} - g_{n_m}) \mathrm{d}\mu \right. \\ &\quad \left. - \int_{E_{k_0} \setminus E} x^*(g_{n_k} - g_{n_m}) \mathrm{d}\mu \right| \\ &\le \frac{\epsilon}{2} + \int_{E \triangle E_{k_0}} \|x^*\| (\|g_{n_k}\| + \|g_{n_m}\|) \mathrm{d}\mu < \epsilon \end{aligned}$$

and the claim stands. Thus, we may define a vector measure $G : \Sigma_1 \to X$ by

$$G(E) = \text{weak-}\lim_k \int_E g_{n_k} \mathrm{d}\mu \quad \text{for each } E \in \Sigma_1. \tag{3.4}$$

For each $x^* \in X^*$, as (ii) guarantees that $\lim_{\mu(E) \to 0} \int_E x^* g_{n_k} \mathrm{d}\mu = 0$ uniformly in k, it follows that $\lim_{\mu(E) \to 0} x^* G(E) = 0$. This shows that G is weakly countably additive and so norm countably additive by the Orlicz–Pettis Theorem (see [3, Corollary I.4.4]). Since G vanishes on sets of μ-zero in Σ_1, G is μ-continuous on Σ_1. Let

$$C = \sup_{n \ge 1} \|g_n\|_p. \tag{3.5}$$

Then, for any partition $\{E_i : i = 1, 2, \dots, m\}$ of S with each $E_i \in \Sigma_1$, one has that

$$\sum_{i=1}^m \|G(E_i)\|_1 \le \liminf_k \int_S \|g_{n_k}(t)\| \mathrm{d}\mu \le \liminf_k \|g_{n_k}\|_p \mu(S)^{\frac{1}{q}} \le C \mu(S)^{\frac{1}{q}}.$$

Hence, G is of bounded variation. Thus by the assumed Radon–Nikodym property, there exists $g_0 \in L_1(S, \Sigma_1, X)$ such that $G(E) = \int_E g_0 \mathrm{d}\mu$ for each $E \in \Sigma_1$. This together with (3.4) implies that

$$\lim_k \langle g_{n_k} - g_0, h^* \rangle = 0 \quad \text{for each } h^* \in SL(S, \Sigma_1, X^*), \tag{3.6}$$

where $SL(S, \Sigma_1, X^*)$ denotes the set of all simple measurable functions on S. Below we shall verify that $g_0 \in L_p(S, \Sigma_1, X)$. Granting this, (3.2) follows from (3.6) because $SL(S, \Sigma_1, X^*)$ is dense in $L_q(S, \Sigma_1, X^*)$ and $\{g_{n_k}\}$ is bounded in $L_p(S, \Sigma_1, X)$. Thus (3.1) is proved in the case when $\mu(S) < \infty$.

To show that $g_0 \in L_p(S, \Sigma_1, X)$, let $\{A_i\}$ be an expanding sequence in Σ_1 with $\cup_i A_i = S$ such that g_0 is bounded on each A_i. Then, for each $i \in \mathbb{N}$, $g_0\chi_{A_i} \in L_p(S, \Sigma_1, X)$. Since $\langle g_0\chi_{A_i}, h^*\rangle = \langle g_0, h^*\chi_{A_i}\rangle$ for each $h^* \in SL(S, \Sigma_1, X^*)$, it follows from (3.6) that

$$|\langle g_0\chi_{A_i}, h^*\rangle| = \lim_k |\langle g_{n_k}, h^*\chi_{A_i}\rangle| \le \liminf_k \|g_{n_k}\|_p \|h^*\chi_{A_i}\|_q \le C\|h^*\|_q \tag{3.7}$$

holds for each $h^* \in SL(S, \Sigma_1, X^*)$. Note that $L_p(S, \Sigma_1, X) \subseteq L_p(S, \Sigma_1, X)^{**} \subseteq L_q(S, \Sigma_1, X^*)^*$ and that $SL(S, \Sigma_1, X^*)$ is dense in $L_q(S, \Sigma_1, X^*)$. One has that

$$\begin{aligned}\|g_0\chi_{A_i}\|_p &= \sup\left\{|\langle g_0\chi_{A_i}, h^*\rangle| : h^* \in L_q(S, \Sigma_1, X^*),\ \|h^*\|_q \le 1\right\}\\ &= \sup\left\{|\langle g_0\chi_{A_i}, h^*\rangle| : h^* \in SL(S, \Sigma_1, X^*),\ \|h^*\|_q \le 1\right\}\\ &\le C\end{aligned}$$

thanks to (3.7). Consequently, Fatou's lemma guarantees that

$$\int_S \|g_0\|^p \mathrm{d}\mu = \int_S \lim_i \|g_0\chi_{A_i}\|^p \mathrm{d}\mu \le \liminf_i \int_S \|g_0\chi_{A_i}\|^p \mathrm{d}\mu \le C^p,$$

which means that $g_0 \in L_p(S, \Sigma_1, X)$.

Step Two. Consider the case when (S, Σ_1, μ) is σ-finite. Take an expanding sequence $\{S_i\}$ in Σ_1 such that $0 < \mu(S_i) < \infty$ for each i and $S = \cup_{i=1}^{\infty} S_i$. For a measurable subset A of S, let $\Sigma_1|_A$ denote the σ-algebra defined by

$$\Sigma_1|_A = \{E \cap A : E \in \Sigma_1\}. \tag{3.8}$$

Also, for a measurable function f on S, we use $f|_A$ to stand for the restriction to A of f. Since $\mu(S_1) < \infty$, we have by *Step One* that the sequence $\{g_n|_{S_1}\}$ yields a subsequence $\{g_{n,1}\}$ of $\{g_n\}$ and $g'_1 \in L_p(S_1, \Sigma_1|_{S_1}, X)$ such that

$$\lim_n \langle g_{n,1}|_{S_1} - g'_1, h^*\rangle = 0 \quad \text{for each } h^* \in L_q(S_1, \Sigma_1|_{S_1}, X^*). \tag{3.9}$$

Similarly, there exist a subsequence $\{g_{n,2}\}$ of $\{g_{n,1}\}$ and $g'_2 \in L_p(S_2, \Sigma_1|_{S_2}, X)$ such that

$$\lim_n \langle g_{n,2}|_{S_2} - g'_2, h^*\rangle = 0 \quad \text{for each } h^* \in L_q(S_2, \Sigma_1|_{S_2}, X^*). \tag{3.10}$$

Since $S_1 \subseteq S_2$, it follows from (3.9) and (3.10) that $\langle g'_1 - g'_2|_{S_1}, h^*\rangle = 0$ for each $h^* \in L_q(S_1, \Sigma_1|_{S_1}, X^*)$, which together with Lemma 2.6 implies $g'_2|_{S_1} = g'_1$. Continuing in this way, we have that, for each $i \in \mathbb{N}$, there exist a subsequence $\{g_{n,i+1}\}$ of $\{g_{n,i}\}$ and $g'_{i+1} \in L_p(S_{i+1}, \Sigma_1|_{S_{i+1}}, X)$ such that

$$g'_{i+1}|_{S_i} = g'_i \tag{3.11}$$

and

$$\lim_n \langle g_{n,i+1}|_{S_{i+1}} - g'_{i+1}, h^*\rangle = 0 \quad \text{for each } h^* \in L_q(S_{i+1}, \Sigma_1|_{S_{i+1}}, X^*). \tag{3.12}$$

Define the function g_0 on S by $g_0(s) = g'_i(s)$ if $s \in S_i$ for some $i \in \mathbb{N}$. Then g_0 is well-defined by (3.11) and

$$g_0\chi_{S_i} = g'_i\chi_{S_i} \quad \text{for each } i \in \mathbb{N}. \tag{3.13}$$

Below we will show that the subsequence $\{g_{n,n}\}$ and g_0 are as desired. For each $h^* \in L_q(S, \Sigma_1, X^*)$, by (3.12) and (3.13), one has that, for each $i \in \mathbb{N}$,

$$|\langle g_0\chi_{S_i}, h^*\rangle| = |\langle g_i'\chi_{S_i}, h^*\rangle| = \lim_n |\langle g_{n,i}|_{S_i}, h^*|_{S_i}\rangle| \le \liminf_n \|g_{n,i}\|_p\|h^*\|_q \le C\|h^*\|_q.$$

Hence, for each $i \in \mathbb{N}$,

$$\left(\int_S \|g_0\chi_{S_i}\|^p \mathrm{d}\mu\right)^{\frac{1}{p}} = \sup\{|\langle g_0\chi_{S_i}, h^*\rangle| : h^* \in L_q(S, \Sigma_1|_{S_i}, X^*), \|h^*\|_q \le 1\} \le C.$$

Consequently, Fatou's lemma guarantees that

$$\int_S \|g_0\|^p \mathrm{d}\mu = \int_S \lim_i \|g_0\chi_{S_i}\|^p \mathrm{d}\mu \le \liminf_i \int_S \|g_0\chi_{A_i}\|^p \mathrm{d}\mu \le C^p, \tag{3.14}$$

which shows that $g_0 \in L_p(S, \Sigma_1, X)$. It remains to show that (3.1) holds for each $E \in \Sigma_1$ with $\mu(E) < \infty$. For this purpose, let $h^* \in L_q(S, \Sigma_1, X^*)$ and let $E \in \Sigma_1$ be such that $\mu(E) < \infty$. Then the Cantor diagonalization yields

$$\lim_n \langle g_{n,n} - g_0, h^*\chi_{(S_i\cap E)}\rangle = \lim_n \langle g_{n,n}|_{S_i} - g_0|_{S_i}, h^*\chi_{(S_i\cap E)}\rangle = 0 \quad \text{for each } i \in \mathbb{N}. \tag{3.15}$$

By (ii), there is $\delta > 0$ such that

$$\max\left\{\int_A \|g_0\|^p \mathrm{d}\mu, \sup_n \int_A \|g_{n,n}\|^p \mathrm{d}\mu\right\} < \left(\frac{\epsilon}{4\|h^*\|_q}\right)^p$$
$$\text{for each } A \in \Sigma_1 \text{ with } \mu(A) < \delta. \tag{3.16}$$

Since $\{E \cap S_i\}$ is monotone, $E = \cup_i(E \cap S_i)$ and $\mu(E) < \infty$, it follows that

$$\lim_i \mu(E \setminus (E \cap S_i)) = 0.$$

Combining this with (3.16) shows that there exists a positive integer K such that

$$\max\left\{\int_{E\setminus(E\cap S_K)} \|g_0\|^p \mathrm{d}\mu, \sup_n \int_{E\setminus(E\cap S_K)} \|g_{n,n}\|^p \mathrm{d}\mu\right\} < \left(\frac{\epsilon}{4\|h^*\|_q}\right)^p. \tag{3.17}$$

By (3.15), there exists $n_0 \in \mathbb{N}$ such that

$$\left|\int_{E\cap S_K} \langle (g_{n,n} - g_0)(s), h^*(s)\rangle \mathrm{d}\mu\right| = |\langle g_{n,n} - g_0, h^*\chi_{E\cap S_K}\rangle| < \frac{\epsilon}{2} \quad \text{for all } n > n_0. \tag{3.18}$$

Hence, by (3.17) and (3.18), we have that, for all $n > n_0$,

$$\begin{aligned}|\langle g_{n,n} - g_0, h^*\chi_E\rangle| &\le \left|\int_{E\cap S_K} \langle (g_{n,n} - g_0)(s), h^*(s)\rangle \mathrm{d}\mu\right| \\ &\quad + \left|\int_{E\setminus(E\cap S_K)} \langle (g_{n,n} - g_0)(s), h^*(s)\rangle \mathrm{d}\mu\right| \\ &< \frac{\epsilon}{2} + \|h^*\|_q \left[\left(\int_{E\setminus(E\cap S_K)} \|g_{n,n}\|^p \mathrm{d}\mu\right)^{\frac{1}{p}} \right. \\ &\quad \left. + \left(\int_{E\setminus(E\cap S_K)} \|g_0\|^p \mathrm{d}\mu\right)^{\frac{1}{p}}\right] < \epsilon\end{aligned}$$

and (3.1) is seen to hold for each $E \in \Sigma_1$ with $\mu(E) < \infty$.

Below we prove (3.2) for the case when $1 < p < \infty$. Let $h^* \in L_q(S, \Sigma_1, X^*)$ and let $\epsilon > 0$. Then there exists $E \in \Sigma_1$ with $\mu(E) < \infty$ such that

$$\|h^* \chi_{S\setminus E}\|_q < \frac{\epsilon}{4C}, \tag{3.19}$$

where C is defined by (3.5). Furthermore, by (3.1), there is $k_0 \in \mathbb{N}$ such that

$$|\langle g_{n_k} - g_0, h^* \chi_E\rangle| < \epsilon/2 \quad \text{for each } k > k_0.$$

This together with (3.5), (3.14) and (3.19) implies that

$$\begin{aligned} |\langle g_{n_k} - g_0, h^*\rangle| &\le |\langle g_{n_k} - g_0, h^* \chi_E\rangle| + |\langle g_{n_k} - g_0, h^* \chi_{S\setminus E}\rangle| \\ &< \frac{\epsilon}{2} + (\|g_{n_k}\|_p + \|g_0\|_p)\|h^* \chi_{S\setminus E}\|_q = \epsilon, \end{aligned}$$

which completes the proof of (3.2). □

Let $\{g_n\}$ be a bounded sequence in $L_p(S, \Sigma, \overline{\operatorname{span} Y})$. Then for each $A \in \Sigma$, we get by Hölder's inequality and (3.5) that

$$\int_A \|g_n(s)\| \mathrm{d}\mu \le \left(\int_A \|g_n(s)\|^p \mathrm{d}\mu\right)^{\frac{1}{p}} \left(\int_A \mathrm{d}\mu\right)^{\frac{1}{q}} \le C(\mu(A))^{\frac{1}{q}}. \tag{3.20}$$

Now we are ready to give the first main theorem of this paper.

Theorem 3.1. *Let Y be a locally weakly compact closed convex subset of X such that both $\overline{\operatorname{span} Y}$ and $\overline{\operatorname{span} Y^*}$ have the Radon–Nikodym property. Suppose that either $p > 1$ or $p = 1$ and (S, Σ, μ) is finite. Then $L_p(S, \Sigma_0, Y)$ is N-simultaneously proximinal in $L_p(S, \Sigma, X)$.*

Proof. Let $f^1, f^2 \in L_p(S, \Sigma, X)$ and let $\{g_n\} \subseteq L_p(S, \Sigma_0, Y)$ be a minimizing sequence for best N-simultaneous approximation to f^1, f^2 from $L_p(S, \Sigma_0, Y)$. Then $\{N(\|f^1 - g_n\|_p, \|f^2 - g_n\|_p)\}$ is bounded. Since

$$\begin{aligned} \|g_n\|_p N(1,1) &= N(\|g_n\|_p, \|g_n\|_p) \\ &\le N(\|f^1 - g_n\|_p + \|f^1\|_p, \|f^2 - g_n\|_p + \|f^2\|_p) \\ &\le N(\|f^1 - g_n\|_p, \|f^2 - g_n\|_p) + N(\|f^1\|_p, \|f^2\|_p), \end{aligned}$$

it follows that $\{g_n\}$ is bounded. Now, Let $\Sigma_1 \subseteq \Sigma_0$ be a σ-algebra generated by a countable algebra such that each g_n is measurable with respect to (S, Σ_1, μ); hence $\{g_n\} \subseteq L_p(S, \Sigma_1, Y)$.

By Lemma 2.8, there exist a subsequence of $\{g_n\}$, still denoted as $\{g_n\}$ itself, and a sequence $\{A_n\}$ of pairwise disjoint measurable sets in Σ_1 such that $\{g_n \chi_{A_n^c}\}$ is uniformly integrable in $L_1(S, \Sigma_1, \overline{\operatorname{span} Y})$. Define

$$\hat{g}_n = \begin{cases} g_n \chi_{A_n^c}, & p = 1, \\ g_n, & 1 < p < +\infty. \end{cases}$$

Then $\{\hat{g}_n\}$ is a minimizing sequence for a best N-simultaneous approximation to f^1, f^2 from $L_p(S, \Sigma_0, Y)$, which is trivial for $p > 1$, and is because of Lemma 2.7 for $p = 1$ as $\lim_n \mu(A_n) = 0$. Below we only prove that there exist a subsequence of $\{\hat{g}_n\}$, again denoted by $\{\hat{g}_n\}$ itself, and $g_0 \in L_p(S, \Sigma_0, Y)$ such that $\{\hat{g}_n\}$ converges weakly to g_0 in $L_p(S, \Sigma, X)$. Granting this, g_0 is a best N-simultaneous approximation to f_1, f_2 from $L_p(S, \Sigma_0, Y)$ (cf. [14, Lemma 1]). To do this, we first check that conditions (a)–(c) in Lemma 3.1 are satisfied by the sequence $\{\hat{g}_n\}$. By the definition of $\{\hat{g}_n\}$, conditions (a) and (b) are satisfied and it remains to check condition (c). To this end, let $A \in \Sigma_1$ with $0 < \mu(A) < \infty$. Note that $\{\int_A \hat{g}_n(s)\mathrm{d}\mu\}$ is bounded, and

$$\frac{1}{\mu(A)} \int_A \hat{g}_n(s)\mathrm{d}\mu \in \overline{\mathrm{co}((\hat{g}_n)(A))} \subseteq Y \quad \text{for each } n \in \mathbb{N} \tag{3.21}$$

thanks to [3, Corollary II.2.8]. We have that $\{\int_A \hat{g}_n(s)\mathrm{d}\mu\}$ is relatively weakly compact in $\overline{\mathrm{span}Y}$, which shows that condition (c) in Lemma 3.1 is true. Thus, Lemma 3.1 guarantees that there exist a subsequence of $\{\hat{g}_n\}$, still denoted as $\{\hat{g}_n\}$ and $g_0 \in L_p(S, \Sigma_1, \overline{\mathrm{span}Y})$ such that

$$\lim_n \langle \hat{g}_n - g_0, h^* \rangle = 0 \quad \text{for each } h^* \in L_q(S, \Sigma_1, \overline{\mathrm{span}Y}^*). \tag{3.22}$$

Since $\overline{\mathrm{span}Y}^*$ has the Radon–Nikodym property, it follows from [3, Theorem IV. 1.1] that

$$(L_p(S, \Sigma_1, \overline{\mathrm{span}Y}))^* = L_q(S, \Sigma_1, \overline{\mathrm{span}Y}^*).$$

This together with (3.22) implies that $\{\hat{g}_n\}$ converges weakly to g_0 in $L_p(S, \Sigma_1, \overline{\mathrm{span}Y})$, and hence $\lim_n \hat{g}_n = g_0$ weakly in $L_p(S, \Sigma, X)$ because $L_p(S, \Sigma_1, \overline{\mathrm{span}Y})$ is a subspace of $L_p(S, \Sigma, X)$. Furthermore, Noting that $L_p(S, \Sigma_1, Y)$ is closed convex by Lemma 2.4, we have that $g_0 \in L_p(S, \Sigma_1, Y)$. Hence $g_0 \in L_p(S, \Sigma_0, Y)$ as $\Sigma_1 \subseteq \Sigma_0$. The proof is complete. □

The following corollary, which is a direct consequence of Theorem 3.1, extends [14, Main theorem] that was proved for the case when $p = 1$ and $Y = X$.

Corollary 3.1. *Let Y be a closed convex subset of X such that $\overline{\mathrm{span}\,Y}$ is reflexive. Suppose that $1 < p < +\infty$, or that $p = 1$ and (S, Σ, μ) is finite. Then $L_p(S, \Sigma_0, Y)$ is N-simultaneously proximinal in $L_p(S, \Sigma, X)$.*

We observe that Theorem 3.1 requires the assumption that both $\overline{\mathrm{span}\,Y}$ and $\overline{\mathrm{span}\,Y}^*$ have the Radon–Nikodym property. We do not know if this assumption can be removed. However, in the case when $\Sigma_0 = \Sigma$, the following theorem shows that the assumption that $\overline{\mathrm{span}\,Y}^*$ have the Radon–Nikodym property can actually be removed.

Theorem 3.2. *Let Y be a locally weakly compact closed convex subset of X such that $\overline{\mathrm{span}\,Y}$ has the Radon–Nikodym property. Then $L_p(S, \Sigma, Y)$ is N-simultaneously proximinal in $L_p(S, \Sigma, X)$ for each $1 \le p < \infty$.*

Proof. Let $f^1, f^2 \in L_p(S, \Sigma, X)$. We have to show that there exists $g_0 \in L_p(S, \Sigma, Y)$ such that

$$N(\|f^1 - g_0\|_p, \|f^2 - g_0\|_p) \le N(\|f^1 - h\|_p, \|f^2 - h\|_p) \quad \text{for each } h \in L_p(S, \Sigma, Y).$$

For each $k = 1, 2$, let $\{f_n^k\}$ be a sequence of countably valued measurable functions in $L_p(S, \Sigma, X)$ such that

$$\lim_n \|f_n^k - f^k\|_p = 0 \quad \text{and} \quad \lim_n \|f_n^k(s) - f^k(s)\| = 0 \quad \text{for a.e. } s \in S. \tag{3.23}$$

By Lemma 2.2, for each n, there exists $g_n \in L_p(S, \Sigma, Y)$ such that g_n is a best N-simultaneous approximation to f_n^1, f_n^2 from $L_p(S, \Sigma, Y)$. Let Σ_1 be a σ-algebra generated by a countable algebra such that each f_n^k and g_n are measurable with respect to (S, Σ_1, μ). Thus, f^1 and f^2 are measurable with respect to (S, Σ_1, μ). Consequently, $\{f^1, f^2\} \subseteq L_p(S, \Sigma_1, X)$, $\{f_n^k\} \subseteq$

$L_p(S, \Sigma_1, X)$ and $\{g_n\} \subseteq L_p(S, \Sigma_1, X)$. We claim that there exist a subsequence of $\{g_n\}$, still denoted as $\{g_n\}$, and $g_0 \in L_p(S, \Sigma_1, \overline{\mathrm{span} Y})$ such that, for each $E \in \Sigma_1$ with $\mu(E) < \infty$,

$$\lim_n \langle g_n - g_0, h^* \chi_E \rangle = 0 \quad \text{for each } h^* \in L_q(S, \Sigma_1, \overline{\mathrm{span} Y}^*). \tag{3.24}$$

By Lemma 3.1, to show the claim, it suffices to verify that $\{g_n\}$ satisfies the following conditions:

(a) $\{g_n\}$ is bounded in $L_p(S, \Sigma_1, \overline{\mathrm{span} Y})$,
(b) $\{g_n\}$ is uniformly integrable in $(S, \Sigma_1, \overline{\mathrm{span} Y})$,
(c) for each $A \in \Sigma_1$ with $\mu(A) < \infty$, $\{\int_A g_n(s) \mathrm{d}\mu\}$ is relatively weakly compact in $\overline{\mathrm{span} Y}$.

Since, for each n, g_n is a best N-simultaneous approximation to f_n^1, f_n^2 from $L_p(S, \Sigma, Y)$ and $0 \in L_p(S, \Sigma, Y)$, we have that

$$\begin{aligned} \|g_n\|_p N(1,1) &= N(\|g_n\|_p, \|g_n\|_p) \\ &\le N(\|f_n^1 - g_n\|_p + \|f_n^1\|_p, \|f_n^2 - g_n\|_p + \|f_n^2\|_p) \\ &\le N(\|f_n^1 - g_n\|_p, \|f_n^2 - g_n\|_p) + N(\|f_n^1\|_p, \|f_n^2\|_p) \\ &\le 2N(\|f_n^1\|_p, \|f_n^2\|_p). \end{aligned}$$

Thus (a) is seen to hold because $\{f_n^1\}$ and $\{f_n^2\}$ are bounded by (3.23). To prove assertion (b), we first consider the case of $p = 1$. Since $\lim_n \|f_n^k - f^k\|_1 = 0$ by (3.23), $\{f_n^k\}$ is uniformly integrable for each $k = 1, 2$. This and Lemma 2.3 imply (b) because $\{g_n\}$ is a best N-simultaneous approximation to f_n^1, f_n^2. For the case when $1 < p < \infty$, it follows from (3.20). Finally, let $A \in \Sigma_1$ with $0 < \mu(A) < \infty$. Note that $\{\int_A g_n(s) \mathrm{d}\mu\}$ is bounded by (3.20), and

$$\frac{1}{\mu(A)} \int_A g_n(s) \mathrm{d}\mu \in \overline{\mathrm{co}(g_n(A))} \subseteq Y \quad \text{for each } n \in \mathbb{N} \tag{3.25}$$

thanks to [3, Corollary II.2.8]. Hence assertion (c) follows and the claim holds.

Next we claim that $g_0 \in L_p(S, \Sigma, Y)$. Since $g_0 \in L_p(S, \Sigma_1, \overline{\mathrm{span}\, Y}) \subseteq L_p(S, \Sigma, \overline{\mathrm{span}\, Y})$, it suffices to prove that, for each $E \in \Sigma_1$ with $0 < \mu(E) < \infty$, $g_0(s) \in Y$ for a.e. $s \in E$. To do this, let $E \in \Sigma_1$ with $0 < \mu(E) < \infty$. By [3, Corollary II.1.3], there exists a sequence $\{\tilde{g}_n\}$ of countably valued measurable functions on E such that

$$\|(\tilde{g}_n - g_0)(s)\| < \frac{1}{n} \quad \text{for a.e. } s \in E \text{ and each } n \in \mathbb{N}. \tag{3.26}$$

Assume that

$$\tilde{g}_n = \sum_{i=1}^{\infty} x_{ni} \chi_{E_{ni}} \quad \text{for each } n \in \mathbb{N},$$

where, for each $n \in \mathbb{N}$, $\{x_{ni}\}_{i=1}^{\infty} \subseteq X$ and $\{E_{ni}\}_{i=1}^{\infty} \subseteq \Sigma_1$ is a countable partition of E. Without loss of generality, we may assume that $\mu(E_{ni}) > 0$ for each $n, i \in \mathbb{N}$. Let

$$y_{ni} = \frac{1}{\mu(E_{ni})} \int_{E_{ni}} g_0 \mathrm{d}\mu \quad \text{for each } n, i \in \mathbb{N}.$$

Then, $\{y_{ni} : n, i \in \mathbb{N}\} \subseteq Y$. In fact, let $n, i \in \mathbb{N}$ and write $A = E_{ni}$. Then, by (3.24),

$$\text{weakly-}\lim_k \frac{1}{\mu(A)} \int_A g_k(s) \mathrm{d}\mu = \frac{1}{\mu(A)} \int_A g_0(s) \mathrm{d}\mu.$$

Hence $y_{ni} = \frac{1}{\mu(A)} \int_A g_0(s)\mathrm{d}\mu \in Y$ by (3.25). Recall that $\Sigma_1|_E$ is defined by (3.8) and write

$$\bar{g}_n = \sum_{i=1}^{\infty} y_{ni} \chi_{E_{ni}} \quad \text{for each } n \in \mathbb{N}.$$

Then $\{\bar{g}_n\} \subseteq L_p(E, \Sigma_1|_E, Y)$. Furthermore, by (3.26),

$$\|x_{ni} - y_{ni}\| = \left\| \frac{1}{\mu(E_{ni})} \int_{E_{ni}} (\tilde{g}_n - g_0)\mathrm{d}\mu \right\| \leq \frac{1}{n} \quad \text{for each } n, i \in \mathbb{N}.$$

Hence $\|(\bar{g}_n - \tilde{g}_n)(s)\| \leq \frac{1}{n}$ on E and $\|(\bar{g}_n - g_0)(s)\| < \frac{2}{n}$ for a.e. $s \in E$ and each $n \in \mathbb{N}$. Thus Lemma 2.4 is applicable and we conclude that $g_0\chi_E \in L_p(E, \Sigma_1|_E, Y)$; consequently, $g_0(s) \in Y$ for a.e. $s \in E$ and so $g_0 \in L_p(S, \Sigma_1, Y)$.

Thus, it remains to show that g_0 is a best N-simultaneous approximation to f^1, f^2 from $L_p(S, \Sigma, Y)$. For this purpose, let $\epsilon > 0$ and $k = 1, 2$. Then there exists $\{\hat{f}_\epsilon^k\} \in L_p(S, \Sigma_1, X)$ with countable values such that

$$\|\hat{f}_\epsilon^k - (f^k - g_0)\|_p < \epsilon. \tag{3.27}$$

By Lemma 2.5, we may take $h_{k,\epsilon}^* \in L_q(S, \Sigma_1, X^*)$ such that $\|h_{k,\epsilon}^*\|_q \leq 1$ and $\langle \hat{f}_\epsilon^k, h_{k,\epsilon}^* \rangle = \|\hat{f}_\epsilon^k\|_p$. It follows from (3.27) that

$$\|\hat{f}_\epsilon^k\|_p \leq |\langle f^k - g_0, h_{k,\epsilon}^* \rangle| + \|f^k - g_0 - \hat{f}_\epsilon^k\|_p \leq |\langle f^k - g_0, h_{k,\epsilon}^* \rangle| + \epsilon. \tag{3.28}$$

On the other hand, there exists $E \in \Sigma_1$ with $\mu(E) < \infty$ such that $\|(f^k - g_0)\chi_{S\setminus E}\|_p < \epsilon$ for each $k = 1, 2$. This together with (3.28) implies that

$$\begin{aligned} \|\hat{f}_\epsilon^k\|_p \leq {} & |\langle f^k - g_0, h_{k,\epsilon}^*\chi_E \rangle| + \left|\langle f^k - g_0, h_{k,\epsilon}^*\chi_{S\setminus E} \rangle\right| \\ & + \epsilon < |\langle f^k - g_0, h_{k,\epsilon}^*\chi_E \rangle| + 2\epsilon. \end{aligned} \tag{3.29}$$

By (3.23) and (3.24), one has that

$$\lim_n \langle f_n^k - g_n, h_{k,\epsilon}^*\chi_E \rangle = \langle f^k - g_0, h_{k,\epsilon}^*\chi_E \rangle.$$

Let $g \in L_p(S, \Sigma, Y)$. Then

$$\begin{aligned} & N(|\langle f^1 - g_0, h_{1,\epsilon}^*\chi_E \rangle|, |\langle f^1 - g_0, h_{2,\epsilon}^*\chi_E \rangle|) \\ & \quad = \lim_n N(|\langle f_n^1 - g_n, h_{1,\epsilon}^*\chi_E \rangle|, |\langle f_n^2 - g_n, h_{2,\epsilon}^*\chi_E \rangle|) \\ & \quad \leq \lim_n N(\|f_n^1 - g_n\|_p, \|f_n^2 - g_n\|_p) \\ & \quad \leq \lim_n N(\|f_n^1 - g\|_p, \|f_n^2 - g\|_p) \\ & \quad = N(\|f^1 - g\|_p, \|f^2 - g\|_p), \end{aligned}$$

where the last equality holds because of (3.23). Furthermore, by (3.28) and (3.29), we have that

$$\begin{aligned} N(\|\hat{f}_\epsilon^1\|_p, \|\hat{f}_\epsilon^2\|_p) & \leq N(|\langle f^1 - g_0, h_{1,\epsilon}^*\chi_E \rangle| + 2\epsilon, |\langle f^2 - g_0, h_{2,\epsilon}^*\chi_E \rangle| + 2\epsilon) \\ & \leq N(\|f^1 - g\|_p, \|f^2 - g\|_p) + 2\epsilon N(1, 1). \end{aligned}$$

Combining this with (3.27) gives that

$$N(\|f^1 - g_0\|_p, \|f^2 - g_0\|_p) \leq N(\|f^1 - g\|_p, \|f^2 - g\|_p) + 3\epsilon N(1, 1).$$

Since $g \in L_p(S, \Sigma, Y)$ and $\epsilon > 0$ are arbitrary, g_0 is a best N-simultaneous approximation to f^1, f^2 from $L_p(S, \Sigma, Y)$ and the proof is complete. □

The following corollary, which extends [16, Theorems 3 and 4] and which was proved for the case when Y is a reflexive subspace, is a direct consequence of Theorem 3.2.

Corollary 3.2. *Let Y be a closed convex subset of X such that $\overline{\operatorname{span} Y}$ is reflexive. Then $L_p(S, \Sigma, Y)$ is N-simultaneously proximinal in $L_p(S, \Sigma, X)$ for each $1 \le p < \infty$.*

Remark 3.1. We do not know if the assumption that $\overline{\operatorname{span} Y}$ have the Radon–Nikodym property can be dropped from Theorem 3.2.

4. Concluding remark

We have established the N-simultaneous proximinality of $L_p(S, \Sigma_0, Y)$ and $L_p(S, \Sigma, Y)$ in $L_p(S, \Sigma, X)$, where N is any monotone norm of $\mathbb{R}^2$. It is not hard to extend our results (i.e., Theorems 3.1 and 3.2) to the case where N is any monotone norm of $\mathbb{R}^m$, with m a finite positive integer.

It is of interest to know (as one of the referees pointed out) if our approach in this paper works to handle the case where N is a monotone norm of $\mathbb{R}^\infty$ (i.e., the case $m = \infty$).

It however looks that our approach may fail to treat the case of $m = \infty$ since in many places, our argument depends heavily on the finiteness of m and we are not sure if the barrier of the finiteness of m can be broken through. For instance, first of all, we do not know if Lemma 2.2 can still hold in the case of $m = \infty$ since our proof cannot be extended to the case of $m = \infty$. As a matter of fact, since in our Lemma 2.2, we have finitely many functions $\{f^k\}_{k=1}^m$ (indeed $m = 2$), we can find countably many disjoint measurable sets $\{A_i\}_{i=1}^\infty$ in Σ such that each f^k can be written as a (possibly infinite) linear combination of the characteristic functions of $\{A_i\}_{i=1}^\infty$. However, we do not know if this is true for infinitely many functions $\{f^k\}_{k=1}^\infty$. Moreover, we do not know if a monotone norm N on $\mathbb{R}^\infty$ is lower semicontinuous with respect to componentwise convergence in $\mathbb{R}^\infty$. This lower semicontinuity of N, which holds in the case of $m < \infty$, is required in the proof of Lemma 2.2.

Secondly, we have also required that m be finite in the proof of Lemma 2.7 (i.e., [14, Lemma 2]). It is thus unclear if Lemma 2.7 holds for the case of $m = \infty$.

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