Abstract

In this article, we develop the Yoccoz puzzle technique to study a family of rational maps termed McMullen maps. We show that the boundary of the immediate basin of infinity is always a Jordan curve if it is connected. This gives a positive answer to the question of Devaney. Higher regularity of this boundary is obtained in almost all cases. We show that the boundary is a quasi-circle if it contains neither a parabolic point nor a recurrent critical point. For the whole Julia set, we show that the McMullen maps have locally connected Julia sets except in some special cases.

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1. Introduction

The local connectivity of Julia sets for rational maps is a central problem in complex dynamical systems. It is well studied for classical types of rational maps, such as hyperbolic and semi-hyperbolic maps and geometrically finite maps [4,20,29]. The polynomial case is also well known [10,13,15,16,21,23]. For quadratic polynomials, Yoccoz proved that the Julia set is locally connected provided all periodic points are repelling and the map is not infinitely renormalizable [14,21]. Douady exhibited a striking example of an infinitely renormalizable quadratic
polynomial with a non-locally connected Julia set [21]. For a general polynomial with connected 
Julia sets and without irrationally neutral cycles, Kiwi shows in [15] that the local connectivity 
of the Julia set is equivalent to the non-existence of wandering continua.

A powerful tool for studying the local connectivity of Julia sets for polynomials is the so-
called ‘Branner–Hubbard–Yoccoz puzzle’ technique introduced by Branner and Hubbard [2]. 
This technique uses a natural method of construction involving finitely many periodic external 
rays together with an equipotential curve. However, for general rational maps, the situation is 
different, and the construction of the Yoccoz puzzle becomes quite involved, even impossible. 
Until now, the only known rational maps that admit Yoccoz puzzle structures were cubic Newton 
maps, whose Yoccoz puzzles were constructed by Roesch. In [26], Roesch applied Yoccoz puzzle 
techniques to show striking differences between rational maps and polynomials. The method also 
leads to the local connectivity of Julia sets except in some specific cases.

In this article, we present the Yoccoz puzzle structure for another family of rational maps 
known as McMullen maps. These maps are of the form

\[ f_\lambda : z \mapsto z^n + \lambda/z^n, \quad \lambda \in \mathbb{C}\setminus\{0\}, \quad n \geq 3. \]

The dynamics of this family of maps have been studied by Devaney and his group [5–8].

The Yoccoz puzzle differs for cubic Newton maps and McMullen maps in the following way. 
For cubic Newton maps, the Yoccoz puzzle is induced by a periodic Jordan curve that intersects 
the Julia set at countably many points. However, for McMullen maps, the element used to con-
struct the Yoccoz puzzle is a periodic Jordan curve (this curve will be called the ‘cut ray’) 
that intersects the Julia set in a Cantor set. This type of Jordan curve is induced by some particular 
angle and can be viewed as an extension of the corresponding external ray (see Section 3.2).

We denote by \( B_\lambda \) the immediate basin of attraction of \( \infty \). The topology of \( \partial B_\lambda \) 
is of special interest. Based on Yoccoz puzzle techniques and on combinatorial and topological analysis, we 
prove:

**Theorem 1.1.** For any \( n \geq 3 \) and any complex parameter \( \lambda \), if the Julia set \( J(f_\lambda) \) is not a Cantor 
set, then \( \partial B_\lambda \) is a Jordan curve.

This affirmatively answers a question posed by Devaney at the Snowbird Conference on the 
25th Birthday of the Mandelbrot set [7]. For higher regularity of \( \partial B_\lambda \), we show that \( \partial B_\lambda \) is a 
quasi-circle except in two special cases.

**Theorem 1.2.** Suppose that the Julia set \( J(f_\lambda) \) is not a Cantor set; then, \( \partial B_\lambda \) is a quasi-circle if 
it contains neither a parabolic point nor a recurrent critical point.

Here, a recurrent critical point \( c \) on the Julia set of a rational map \( f \) is a critical point such that 
\( c \in \omega(c) \), where \( \omega(c) \) is the \( \omega \)-limit set of \( c \), defined as \( \{ z \in \overline{\mathbb{C}} : \text{there exist } n_k \to \infty \text{ such that } z = \text{lim } f^{n_k}(c) \} \). It follows from Proposition 7.5 that if \( \partial B_\lambda \) contains a parabolic point, then \( \partial B_\lambda \) is 
not a quasi-circle by the Leau–Fatou–Flower Theorem [21]. Whether \( \partial B_\lambda \) is a quasi-circle when 
\( \partial B_\lambda \) contains a recurrent critical point is still unknown.

For the topology of the Julia set, we show

**Theorem 1.3.** Suppose \( f_\lambda \) has no Siegel disk and the Julia set \( J(f_\lambda) \) is connected, then \( J(f_\lambda) \) is 
locally connected in the following cases:
1. The critical orbit does not accumulate on the boundary $\partial B_\lambda$.
2. $f_\lambda$ is neither renormalizable nor $\ast$-renormalizable.
3. The parameter $\lambda$ is real and positive.

See Section 5 for the definitions of renormalization and $\ast$-renormalization. Theorem 1.3 implies that the Julia set is locally connected except in some special cases. In fact, the theorem is stronger than the following statement:

**Theorem 1.4.** Suppose $f_\lambda$ has no Siegel disk and the Julia set $J(f_\lambda)$ is connected, then $J(f_\lambda)$ is locally connected if the critical orbit does not accumulate on the boundary $\partial B_\lambda$.

Theorem 1.4 is an analogue of Roesch’s Theorem [26]:

**Theorem 1.5 (Roesch).** A genuine cubic Newton map without Siegel disks has a locally connected Julia set provided the orbit of the non-fixed critical point does not accumulate on the boundary of any invariant basin of attraction.

We exclude the case $n = 2$ because it is impossible to find a non-degenerate critical annulus for the Yoccoz puzzle constructed in this paper. The existence of a non-degenerate critical annulus is technically necessary in our proof.

The paper is organized as follows:

In Section 2, we present some basic results on McMullen maps.

In Section 3, we construct ‘cut rays’, each of which is a type of Jordan curve that divides the Julia set into two different parts. We first construct a Cantor set of angles on the unit circle which is used to generate ‘cut rays’. We then discuss the construction of ‘cut rays’ based on the work of Devaney [6].

In Section 4, basic knowledge of Yoccoz puzzles, graphs and tableaux are presented. The aim of this section is to find a Yoccoz puzzle with a non-degenerate critical annulus (see Section 4.2). A natural construction of the ‘modified puzzle piece’ is discussed (see Section 4.3).

In Section 5, we discuss the renormalizations of McMullen maps in the context of the puzzle piece.

In Section 6, we present a criterion of local connectivity. We introduce a ‘BD condition’ on the boundary of the immediate basin of attraction. Such a condition can be considered as ‘local semi-hyperbolicity’. We show that existence of the ‘BD condition’ implies good topology.

In Section 7, we study the local connectivity of $\partial B_\lambda$ in all possible cases and show that $\partial B_\lambda$ enjoys higher regularity except in two special cases.

In Section 8, we study the local connectivity of the Julia set $J(f_\lambda)$ based on the ‘Characterization of Local Connectivity’ and the ‘Shrinking Lemma’.

2. Preliminaries and notations

In this section, we present some basic results and notations for the family of rational maps

$$f_\lambda(z) = z^n + \lambda/z^n$$
where $\lambda \in \mathbb{C}^*$ and $n \geq 3$. This type of map is known as a ‘McMullen map’ because it was first studied by McMullen, who proved that when $|\lambda|$ is sufficiently small, and the Julia set is a Cantor set of circles \cite{[18]}

For any $\lambda \in \mathbb{C}^*$, the map $f_\lambda$ has a superattracting fixed point at $\infty$. The immediate basin of $\infty$ is denoted by $B_\lambda$, and the component of $f_\lambda^{-1}(B_\lambda)$ that contains $0$ is denoted by $T_\lambda$. The set of all critical points of $f_\lambda$ is $\{0, \infty\} \cup C_\lambda$, where $C_\lambda = \{\sqrt[2n]{\lambda} \omega; \omega^{2n} = 1\}$. Besides $\infty$, there are only two critical values for $f_\lambda$: $v_+^\lambda = 2\sqrt[2n]{\lambda}$ and $v_-^\lambda = -2\sqrt[2n]{\lambda}$. In fact, there is only one critical orbit (up to a sign). Let $P(\lambda) = \bigcup_{n \geq 1} f_\lambda^n(C_\lambda) \cup \{\infty\}$ be the post-critical set.

The Böttcher map $\phi_\lambda$ for $f_\lambda$ is defined in a neighborhood of $\infty$ by $\phi_\lambda(z) = \lim_{k \to \infty} (f_\lambda^k(z))^{n^{-k}}$. The Böttcher map is unique if we require $\phi_\lambda'(\infty) = 1$. It is known that the Böttcher map $\phi_\lambda$ can be extended to a domain $\text{Dom}(\phi_\lambda) \subset B_\lambda$ such that $\phi_\lambda : \text{Dom}(\phi_\lambda) \to \{z \in \mathbb{C} \mid |z| > R\}$ is a conformal isomorphism for some largest number $R \geq 1$. In particular, if $B_\lambda$ contains no critical point other than $\infty$, then $\text{Dom}(\phi_\lambda) = B_\lambda$; if $B_\lambda$ contains a critical point $c \in \{0\} \cup C_\lambda$, then by ‘The Escape Trichotomy’ (Theorem 2.1), the Julia set $J(f_\lambda)$ is a Cantor set.

The Green function $G_\lambda : B_\lambda \to (0, \infty]$ is defined by

$$G_\lambda(z) = \lim_{k \to \infty} n^{-k} \log |f_\lambda^k(z)|.$$  

By definition, $G_\lambda(f_\lambda(z)) = nG_\lambda(z)$ for $z \in B_\lambda$, and $G_\lambda(z) = \log |\phi_\lambda(z)|$ for $z \in \text{Dom}(\phi_\lambda)$. The Green function $G_\lambda$ can be extended to $A_\lambda = \bigcup_{k \geq 0} f_\lambda^{-k}(B_\lambda)$ by defining

$$G_\lambda(z) = n^{-k} G_\lambda(f_\lambda^k(z)) \quad \text{for } z \in f_\lambda^{-k}(B_\lambda).$$

In the following, for a set $E$ in $\mathbb{C}$ and $a \in \mathbb{C}$, let $aE = \{az; \ z \in E\}$, $a + E = \{a + z; \ z \in E\}$, $\bar{E}$ be the closure of $E$ and $\text{int}(E)$ be the interior of $E$.

**Lemma 2.1** (Symmetry of the dynamical plane). Let $\omega$ satisfy $\omega^{2n} = 1$; then,

1. $\omega J(f_\lambda) = J(f_\lambda)$.
2. $G_\lambda(\omega z)$ for $z \in A_\lambda$.
3. $\omega \text{Dom}(\phi_\lambda) = \text{Dom}(\phi_\lambda)$ and $\phi_\lambda(\omega z) = \omega \phi_\lambda(z)$ for $z \in \text{Dom}(\phi_\lambda)$.

**Proof.** For 1, because $A_\lambda = \{z \in \mathbb{C}; \ f_\lambda^k(z) \text{ tends to infinity as } k \to \infty\}$ and $f_\lambda^k(\omega z) = \pm f_\lambda^k(z)$ for $k \geq 1$, $f_\lambda^k(\omega z)$ tends toward infinity if and only if $f_\lambda^k(z)$ tends toward infinity as $k \to \infty$. Thus, $\omega A_\lambda = A_\lambda$. The conclusion follows from the fact that $J(f_\lambda) = \partial A_\lambda$.

2. By the definition of $G_\lambda$.

3. Because $\text{Dom}(\phi_\lambda)$ is the connected component of $\{z \in B_\lambda; \ G_\lambda(z) > \log R\}$ that contains $\infty$, we conclude that $\omega \text{Dom}(\phi_\lambda) = \text{Dom}(\phi_\lambda)$. Note that $\phi_\lambda(\omega z)$ and $\omega \phi_\lambda(z)$ are two Riemann mappings of $\text{Dom}(\phi_\lambda)$ onto $\{z \in \mathbb{C}; \ |z| > R\}$ with the same derivative at $\infty$, we have $\phi_\lambda(\omega z) = \omega \phi_\lambda(z)$ by the uniqueness of the Riemann mapping theorem. \hfill \Box

The non-escape locus of this family is defined by

$$M = \{\lambda \in \mathbb{C}^*; \ f_\lambda^k(v_+^\lambda) \text{ does not tend to infinity as } k \to \infty\}.$$
Lemma 2.2 (Symmetry of the parameter plane). The non-escape locus $M$ satisfies:

1. $M$ is symmetric about the real axis.
2. $\nu M = M$ with $\nu^{n-1} = 1$.
3. For any line $\ell \in \{\epsilon \mathbb{R}; \epsilon^{2n-2} = 1\}$, $M$ is symmetric about $\ell$.

Proof. 1. Because $f_\lambda(\bar{z}) = f_{\bar{\lambda}}(z)$, the Critical orbit of $f_\lambda$ and the critical orbit of $f_{\bar{\lambda}}$ are symmetric under the map $z \mapsto \bar{z}$, they either both remain bounded or both tend to infinity. Thus, $M$ is symmetric about the real axis.

2. Let $\nu = e^{2\pi i/(n-1)}$ and $\varphi(z) = e^{\pi i/(n-1)}z$. For $k \geq 1$,

$$\varphi^{-1} \circ f^{k}_{\nu \lambda} \circ \varphi(z) = \begin{cases} (-1)^k f^{k}_{\lambda}(z), & n \text{ odd}, \\ f^{k}_{\lambda}(z), & n \text{ even}. \end{cases}$$

Thus, the critical orbit of $f_\lambda$ tends toward infinity if and only if the critical orbit of $f_{\nu \lambda}$ tends toward infinity. Equivalently, $\lambda \in M$ if and only if $\nu \lambda \in M$.

3. The conclusion follows from 1 and 2. \qed

From Lemma 2.2, $f_\lambda$ and $f_{\lambda e^{2\pi i/(n-1)}}$ have the same dynamical properties and their Julia sets are identical up to a rotation. Thus, the fundamental domain of the parameter plane is $\{\lambda \in \mathbb{C}^*; \arg \lambda \in (0, 2\pi n-1)\}$.

The following theorem of Devaney, Look and Uminsky gives a classification of Julia sets of different topological types [8].

Theorem 2.1 (Devaney–Look–Uminsky). The Escape Trichotomy.

1. If $\nu^+ \lambda \in B_\lambda$, then $J(f_\lambda)$ is a Cantor set.
2. If $\nu^+ \lambda \in T_\lambda \neq B_\lambda$, then $J(f_\lambda)$ is a Cantor set of circles.
3. If $f^{k}_{\lambda}(\nu^+ \lambda) \in T_\lambda \neq B_\lambda$ for some $k \geq 1$, then $J(f_\lambda)$ is a Sierpiński curve, which is locally connected.

In all other cases, the critical orbits remain bounded and the Julia set $J(f_\lambda)$ is connected.

For $n \geq 3$, it is known that the unbounded component of $\mathbb{C}^* - M$ consists of the parameters for which the Julia set is a Cantor set. This region is called a Cantor set locus (see Fig. 1). The component of $\mathbb{C}^* - M$ that contains a punctured neighborhood of 0 is the region in which the Julia set $J(f_\lambda)$ is a Cantor set of circles; this is referred to as the McMullen domain in honor of McMullen, who first discovered this type of Julia set. The complement of these two regions is the connected locus. The small copies of the quadratic Mandelbrot set correspond to the renormalizable parameters, while the ‘holes’ in the connected locus are always called Sierpiński holes according to Devaney. These regions correspond to the parameters for which the Julia set is a Sierpiński curve.

We will see later that, when the critical orbit tends to $\infty$, the boundary $\partial B_\lambda$ is a quasi-circle if it is connected. Thus, this case is already well studied.

In this paper, we will restrict our attention to the parameters $\lambda \in \mathcal{H} = \{\lambda \in \mathbb{C}^*; \arg \lambda \in (0, 2\pi n-1)\}$ for the most part because of the symmetry of the parameter plane. For these parameters,
we can develop Yoccoz puzzle techniques to study the local connectivity of Julia set. However, for real parameters, Yoccoz puzzle theory cannot be applied because of the absence of critical puzzle pieces. The real positive parameters will be considered separately in Section 7.3.

Therefore, if there is no further assumption, most discussions are based on the following:

**Hypothesis.** \( \lambda \in \mathcal{H} \) and the critical orbits remain bounded, or equivalently, \( C_\lambda \cap A_\lambda = \emptyset \).

### 2.1. Notations

Let \( c_0 = c_0(\lambda) = \frac{2\sqrt{\lambda}}{\lambda} \) be the critical point that lies on \( \mathbb{R}^+ \) when \( \lambda \in \mathbb{R}^+ \) and varies analytically as \( \lambda \) ranges over \( \mathcal{H} \). Let \( c_k = c_0 e^{k\pi i/n} \) for \( 1 \leq k \leq 2n - 1 \). The critical points \( c_k \) with \( k \) even are mapped to \( v_\lambda^+ = 2\sqrt{\lambda} \) while the critical points \( c_k \) with \( k \) odd are mapped to \( v_\lambda^- = -2\sqrt{\lambda} \).

Let \( \ell_k = c_k \mathbb{R}^+ (\mathbb{R}^+ := [0, +\infty)) \) be the real straight line connecting the origin to \( \infty \) and passing through \( c_k \) for \( 0 \leq k \leq 2n - 1 \). We call \( \ell_k \) a critical ray. The closed sector bounded by \( \ell_k \) and \( \ell_{k+1} \) is denoted by \( S_k \) for \( 0 \leq k \leq n \). Define \( S_{-k} = -S_k \) for \( 1 \leq k \leq n - 1 \). Therefore, the sectors are arranged counterclockwise about the origin as \( S_0, S_1, \ldots, S_n, S_{-1}, \ldots, S_{-(n-1)} \) (see Fig. 2).

The critical value \( v_\lambda^+ \) always lies in \( S_0 \) because \( \arg c_0 < \arg v_\lambda^+ < \arg c_1 \) for all \( \lambda \in \mathcal{H} \). Correspondingly, the critical value \( v_\lambda^- \) lies in \( S_n \). It is easy to confirm that the image of \( \ell_k \) under \( f_\lambda \) is a straight ray connecting one of the critical values to \( \infty \); this ray is called a critical value ray. As a consequence, \( f_\lambda \) maps the interior of each of the sectors of \( \{ S_{\pm 1}, \ldots, S_{\pm (n-1)} \} \) univalently onto a region \( \Upsilon_\lambda \), which can be identified as the complex sphere \( \bar{\mathbb{C}} \) minus two critical value rays.

Let \( \mathcal{P} \) denote the set of all components of \( \bigcup_{k \geq 0} f_\lambda^{-k} (B_\lambda) \). For \( U \in \mathcal{P} \) and \( v > 0 \), let \( \mathbf{e}(U, v) = \{ z \in U; \ G_\lambda(z) = v \} \) be the equipotential curve. The annulus bounded by \( \mathbf{e}(B_\lambda, v) \) and \( \mathbf{e}(T_\lambda, v) \) is denoted by \( \mathcal{Q}_v \). We may choose a \( v \) large enough that \( \partial \mathcal{Q}_v \) intersects with every critical ray at exactly two points (to see this, notice that the Böttcher map \( \phi_\lambda : B_\lambda \to \bar{\mathbb{C}} - \mathbb{D} \) acts like the identity map near \( \infty \); thus, \( \mathbf{e}(B_\lambda, v) \) looks like a circle when \( v \) is large. The curve \( \mathbf{e}(T_\lambda, v) \) also

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Fig. 1. Parameter plane for McMullen maps when \( n = 3 \).
looks like a circle because \( f_\lambda(e(T_\lambda, v)) = e(B_\lambda, nv) \) and \( f_\lambda \) acts like \( z \mapsto \lambda/z^n \) near zero. The bounded and unbounded components of \( \mathbb{C} \setminus e(B_\lambda, v) \) are denoted by \( \mathbf{V}(v) \) and \( \mathbf{U}(v) \), respectively.

Now, we define radial rays of \( U \) for every \( U \in \mathcal{P} \setminus \{B_\lambda, T_\lambda\} \). In the Hypothesis section, we see that there is a unique Riemann mapping \( \phi_{T_\lambda} : T_\lambda \to \mathbb{D} \), such that

\[
\phi_{T_\lambda}(z) = \phi_\lambda(f_\lambda(z)), \quad z \in T_\lambda, \quad \phi_{T_\lambda}'(0) = 1/\sqrt{\lambda}.
\]

The radial ray \( R_{T_\lambda}(\theta) \) of angle \( \theta \) is defined as \( \phi_{T_\lambda}^{-1}((0, 1)e^{2\pi i \theta}) \). For \( U \in \mathcal{P} \setminus \{B_\lambda, T_\lambda\} \), there is a smallest integer \( k \geq 1 \), such that \( f_\lambda^k : U \to T_\lambda \) is a conformal map. The radial ray \( R_U(\theta) \) is defined as the pullback of \( R_{T_\lambda}(\theta) \) under \( f_\lambda^k \).

Let \( \mathbb{I} = \{0, n, \pm 1, \ldots, \pm(n-1)\} \) be an index set. \( S_k^v = \overline{O_v} \cap S_k \) for \( k \in \mathbb{I} \) and \( S^v = \bigcup_{k \in \mathbb{I} \cap \{0, n\}} S_k^v \). The set of all points with orbits that remain in \( S^v \) under all iterations of \( f_\lambda \) is denoted by \( \Lambda_\lambda \). Obviously, \( \Lambda_\lambda = \bigcap_{k \geq 0} f_\lambda^{-k}(S^v) \).

For any \( k \in \mathbb{I} \setminus \{0, n\} \), the map \( f_\lambda^k : \text{int}(S_k) \to \gamma_\lambda \) is a conformal map; its inverse is denoted by \( h_k : \gamma_\lambda \to \text{int}(S_k) \).

Given a point \( z \in \Lambda_\lambda \), suppose \( f_\lambda^k(z) \in S_k \) for \( k \geq 0 \) and define the itinerary of \( z \) as \( s_\lambda(z) = (s_0, s_1, s_2, \ldots) \). The itinerary is always well defined in the set \( \Lambda_\lambda \) because if some iteration \( f_\lambda^k(z) \) lies on the boundary of two adjacent sectors, then the next iteration \( f_\lambda^{k+1}(z) \) will lie inside \( S_0 \cup S_n \).

Let \( \Sigma = \{s = (s_0, s_1, s_2, \ldots); \ s_k \in \mathbb{I} \setminus \{0, n\} \text{ for every } k \geq 0\} \) be the space of one-sided sequences of the symbols \( \pm 1, \ldots, \pm(n-1) \). For \( s = (s_0, s_1, s_2, \ldots) \in \Sigma \), and the shift map \( \sigma : \Sigma \to \Sigma \) is defined by \( \sigma(s) = (s_1, s_2, \ldots) \). If there is an integer \( p > 0 \) such that \( s_k + p = s_k \) for all \( k \geq 0 \), we say the itinerary \( s \) is periodic and the least integer \( p \) is called the period of \( s \). In this case, \( s \) is also denoted by \( (s_0, \ldots, s_{p-1}) \).

It is obvious that \( s_\lambda(f_\lambda(z)) = \sigma(s_\lambda(z)) \) for \( z \in \Lambda_\lambda \).
Lemma 2.3. The set $\Lambda_\lambda$ is a Cantor set, and the itinerary map $s_\lambda : \Lambda_\lambda \to \Sigma$ is bijective. Moreover, $\Lambda_\lambda \subset J(f_\lambda)$.

Proof. First, note that for any $\lambda \in \mathcal{H}$, $S^u$ is a compact subset of $T_\lambda$. With respect to the hyperbolic metric of $T_\lambda$ and by the Schwarz Lemma, there is a number $\delta \in (0, 1)$ such that for any $s = (s_0, s_1, s_2, \ldots) \in \Sigma$ and any $m \geq 0$,

$$\text{Hyper} \cdot \text{diam}(h_{s_0} \circ \cdots \circ h_{s_m}(S^u)) \leq \text{Hyper} \cdot \text{diam}(S^u) \cdot \delta^m.$$

Thus, $\bigcap_{k \geq 0} h_{s_0} \circ \cdots \circ h_{s_k}(S^u)$ consists of a single point, say $z_s$. Therefore, $\Lambda_\lambda$ is a Cantor set, and the map $s_\lambda : \Lambda_\lambda \to \Sigma$ defined by $s_\lambda(z_s) = s$ is bijective.

When $s = (s_0, \ldots, s_{m-1}) \in \Sigma$ is a periodic itinerary of period $m$, then $z_s$ is a fixed point of $h = h_{s_0} \circ \cdots \circ h_{s_{m-1}}$. Because $h : \text{int}(S^u) \to h(\text{int}(S^u)) \subset \text{int}(S^u) \subset \text{int}(S^u)$ is strictly contractive, it follows by the Schwarz Lemma that the fixed point $z_s$ is attracting. Therefore, $z_s$ is a repelling periodic point of $f_\lambda$.

To show $\Lambda_\lambda \subset J(f_\lambda)$, it suffices to prove that any point of $\Lambda_\lambda$ can be approximated by a sequence of repelling periodic points in $\Lambda_\lambda$. Suppose $z \in \Lambda_\lambda$. For any $\varepsilon > 0$, there is an integer $m > 0$ such that $\text{Hyper} \cdot \text{diam}(S^u) \cdot \delta^m < \varepsilon$. Take a periodic itinerary $s \in \Sigma$ with first $m$ symbols that are the same as those of $s_\lambda(z)$. (Notice that such an itinerary always exists.) Because the map $s_\lambda$ is bijective, there is a unique point $w \in \Lambda_\lambda$ with $s_\lambda(w) = s$. The hyperbolic distance between $z$ and $w$ is smaller than $\varepsilon$. The previous argument implies that $w$ is periodic and repelling. \(\square\)

3. Cut rays in the dynamical plane

In this section, we will construct the ‘cut ray’, a type of Jordan curve that cuts the Julia set into two different parts. The construction is due to R. Devaney [6]. We give some additional properties that will be used in our paper.

We first construct a Cantor set of angles on the unit circle and use these angles to generate ‘cut rays’ as in [6]. These angles can be considered as a combinatorial invariant when the parameter $\lambda$ ranges over $\mathcal{H}$.

To begin, we identify the unit circle $S = \mathbb{R}/\mathbb{Z}$ with $(0, 1]$. We say that three angles satisfy $t_1 \leq t_2 \leq t_3$ on $S$ if $t_1, t_2, t_3$ are in counterclockwise order.

3.1. A Cantor set on the unit circle

In the following, we construct a subset $\Theta$ of $(0, 1]$. The set $\Theta$ is a Cantor set and is used to generate ‘cut rays’ in the next section.

First, define a map $\tau : (0, 1] \to (0, 1]$ by $\tau(\theta) = n\theta \mod 1$. Let $\Theta_k = (k \cdot 2^{-n}, (k+1) \cdot 2^{-n}]$ for $0 \leq k \leq n$ and $\Theta_{-k} = \Theta_k + \frac{1}{2}$ for $1 \leq k \leq n - 1$. Obviously, $(0, 1] = \bigcup_{k \in \mathbb{N}} \Theta_k$.

Define a map $\chi : \mathbb{I} \to \mathbb{N}$ by

$$\chi(k) = \begin{cases} k, & \text{if } 0 \leq k \leq n, \\ n - k, & \text{if } -(n - 1) \leq k \leq -1. \end{cases}$$

For $k \in \mathbb{I}$, we have...
First, we show Proof. Lemma 3.1. \( \Sigma \) can easily verify that \( \tau(\Theta) \) consists of a single point. In the following, we first construct an inverse map for \( \Sigma \) (Lemma 3.1).

Let \( s = (s_0, s_1, s_2, \ldots) \in \Sigma \). We define a map \( \kappa : \Sigma \to (0, 1] \) by

\[
\kappa(s) = \frac{1}{2} \left( \frac{\chi(s_0)}{n} + \sum_{k \geq 1} \frac{|s_k|}{n^{k+1}} \right).
\]

**Lemma 3.1.** \( \kappa(\Sigma) = \Theta \) and \( \kappa(s(\theta)) = \theta \) for all \( \theta \in \Theta \).

**Proof.** First, we show \( \kappa(s(\theta)) = \theta \) for \( \theta \in \Theta \). Let \( s(\theta) = (s_0, s_1, s_2, \ldots) \) and \( \hat{\theta} = \kappa(s(\theta)) \). Because \( s : \Theta \to \Sigma_0 \) is bijective, it suffices to show that \( s(\hat{\theta}) = s(\theta) \).

It follows that \( \hat{\theta} \in \Theta_{s_0} \) because

\[
\frac{\chi(s_0)}{2n} < \hat{\theta} \leq \frac{1}{2} \left( \frac{\chi(s_0)}{n} + \sum_{k \geq 1} \frac{n-1}{n^{k+1}} \right) = \frac{\chi(s_0)}{2n} + \frac{1}{2n}.
\]

For \( k \geq 1 \),

\[
\tau^k(\hat{\theta}) = \begin{cases} \frac{1}{2}(\chi(s_0) + |s_1| + \cdots + |s_{k-1}|) + \frac{1}{2} \sum_{j \geq k} \frac{|s_j|}{n^{j+k+1}}, & \text{if } n \text{ is odd,} \\ \frac{|s_{k-1}|}{2} + \frac{1}{2} \sum_{j \geq k} \frac{|s_j|}{n^{j+k+1}}, & \text{if } n \text{ is even.} \end{cases}
\]

Because \( s(\theta) = (s_0, s_1, s_2, \ldots) \in \Sigma_0 \), we have for \( j \geq 1 \),

\[
\frac{|s_j|}{2} = \begin{cases} \frac{1}{2}(\chi(s_j) - \chi(s_{j-1})) \mod 1, & \text{if } n \text{ is odd,} \\ \frac{1}{2} \chi(s_j) \mod 1, & \text{if } n \text{ is even}, \end{cases}
\]

and

\[
\frac{\chi(s_{j-1})}{2} + \frac{|s_j|}{2} = \frac{\chi(s_j)}{2} \mod 1.
\]

Thus, we have
\[
\tau^k(\hat{\theta}) = \frac{\chi(s_k)}{2} + \frac{1}{2} \sum_{j \geq k} \frac{|s_j|}{n^j - k + 1} = \frac{\chi(s_k)}{2n^k} + \frac{1}{2} \sum_{j \geq k+1} \frac{|s_j|}{n^j - k + 1}.
\]

This means \(\tau^k(\hat{\theta}) \in \Theta_{s_k}\) for \(k \geq 1\). Therefore, \(\theta\) and \(\hat{\theta}\) have the same itinerary.

In the following, we show \(\kappa(\Sigma) = \Theta\). First, by the previous argument, \(\Theta = \kappa(\Sigma_0) \subset \kappa(\Sigma)\). Conversely, for any \(s = (s_0, s_1, s_2, \ldots) \in \Sigma\), there is a unique sequence of symbols \(\epsilon_1, \epsilon_2, \ldots \in \{\pm 1\}\), such that \(s^* = (s_0, \epsilon_1 s_1, \epsilon_2 s_2, \ldots) \in \Sigma_0\). Thus, \(\kappa(s) = \kappa(s^*) \in \Theta\). \(\square\)

**Remark 3.1.** For any \(s = (s_0, s_1, s_2, \ldots) \in \Sigma\), one can verify that

\[
\kappa^{-1}(\kappa(s)) = \{(s_0, \pm s_1, \pm s_2, \ldots)\}.
\]

**Lemma 3.2.** The set \(\Theta\) satisfies:

1. \(\tau(\Theta) = \Theta\).
2. \(\Theta + \frac{1}{2} = \Theta\).
3. Periodic angles are dense in \(\Theta\).

**Proof.** 1. It is obvious that \(\tau(\Theta) \subset \Theta\). \(\tau\) is surjective because \(\tau^{-1}(\theta) \cap \mathcal{E} \neq \emptyset\) for all \(\theta \in \Theta\).

2. First note that \(\mathcal{E} + \frac{1}{2} = \mathcal{E} \mod 1\). For \(k \geq 1\), because \(\tau^k(\theta + \frac{1}{2}) = \tau^k(\theta)\) when \(n\) is even and \(\tau^k(\theta + \frac{1}{2}) = \tau^k(\theta) + \frac{1}{2}\) when \(n\) is odd, we have \(\tau^k(\theta + \frac{1}{2}) \in \mathcal{E}\) if and only if \(\tau^k(\theta) \in \mathcal{E}\). Thus, \(\theta \in \Theta\) if and only if \(\theta + \frac{1}{2} \in \Theta\).

3. Let \(\theta \in \Theta\) with itinerary \(s(\theta) = (s_0, s_1, s_2, \ldots)\). For any \(k \geq 1\), either \((s_0, \ldots, s_k) \in \Sigma_0\), or there is a symbol \(s_{k+1}^* \in \{\pm 1, \ldots, \pm (n-1)\}\) such that \((s_0, \ldots, s_k, s_{k+1}^*) \in \Sigma_0\). If \((s_0, \ldots, s_k) \in \Sigma_0\), let \(\theta_k = \kappa((s_0, \ldots, s_k, s_k^*))\). Else, let \(\theta_k = \kappa((s_0, \ldots, s_k, s_{k+1}^*))\). It’s obvious that \(\theta_k\) is periodic. By Lemma 3.1, \(\theta_k \in \Theta\) and

\[
|\theta - \theta_k| \leq C(n)n^{-k} (\rightarrow 0 \text{ as } k \rightarrow \infty),
\]

where \(C(n)\) is a constant, depending only on \(n\), which implies that periodic angles are dense in \(\Theta\). \(\square\)

**Remark 3.2.** The Hausdorff dimension of \(\Theta\) is \(\frac{\log(n-1)}{\log n}\).

For \(\lambda \in \mathcal{H}\) and \(k \in \mathbb{I}\), let \(\Theta_k^\lambda = \Theta_k + \frac{\arg c_0(\lambda)}{2\pi} = \Theta_k + \frac{\arg \lambda}{4n\pi} \mod 1\). Recall that for \(\lambda \in \mathcal{H}\), \(\arg \lambda \in (0, \frac{2\pi}{n-1})\). It is easy to check that

\[
\tau(\Theta_k^\lambda) \supset \begin{cases} \bigcup_{j=1}^{n-1} \Theta_j^\lambda, & \text{if } \chi(k) \text{ is even,} \\ \bigcup_{j=1}^{n-1} \Theta_j^\lambda, & \text{if } \chi(k) \text{ is odd.} \end{cases}
\]

Again, we define \(\Theta^\lambda\) as the set of all angles in \((0, 1]\) whose orbits remain in \(\mathcal{E}^\lambda = \bigcup_{k=1}^{n-1}(\Theta_k^\lambda \cup \Theta_k^\lambda)\) under all iterations of \(\tau\). Thus, \(\Theta^\lambda = \bigcap_{k \geq 0} \tau^{-k}(\mathcal{E}^\lambda)\). For \(\theta \in (0, 1]\), suppose \(\tau^k(\theta) \in \Theta_{s_k}^\lambda\) for \(k \geq 0\) and define the itinerary of \(\theta\) by \(s^k(\theta) = (s_0, s_1, s_2, \ldots)\). It is easy to show that the itinerary map \(s^k : \Theta^\lambda \rightarrow \Sigma_0\) is bijective.
Lemma 3.3. $\Theta^\lambda = \Theta$ and for any $\theta \in \Theta$, $s^\lambda(\theta) = s(\theta)$.

Proof. It suffices to show that if $s^\lambda(\alpha) = s(\beta)$ for $\alpha \in \Theta^\lambda$ and $\beta \in \Theta$, then $\alpha = \beta$.

First, note that $\Theta^\lambda \cap \Theta \neq \emptyset$ for any $k \in \mathbb{N}$. Suppose $s^\lambda(\alpha) = s(\beta) = (s_0, s_1, s_2, \ldots)$, and let $A_m = \bigcap_{0 \leq k \leq m} \tau^{-k}(\Theta^\lambda) \cap \Theta^\lambda$ for $m \geq 0$. By induction, we see that $A_m$ is a connected interval of the form $(a_m, b_m]$ with $a_{m+1} > a_m, b_{m+1} < b_m$ and $n(b_{m+1} - a_{m+1}) = b_m - a_m$ for $m \geq 0$. Thus, $A_{m+1} \subset A_m \subset A_m$ and $\bigcap_{k \geq 0} A_m = \bigcap_{k \geq 0} \overline{A_m}$ consists of a single point, say $\theta$. On the other hand,

$$\{ \theta \} = \bigcap_{k \geq 0} A_m = \left( \bigcap_{k \geq 0} \tau^{-k}(\Theta^\lambda) \right) \cap \left( \bigcap_{k \geq 0} \tau^{-k}(\Theta_s^\lambda) \right) = \{ \alpha \} \cap \{ \beta \}.$$

Thus, we have $\alpha = \beta = \theta$. \[\Box\]

3.2. Cut rays

In this section, for any $\lambda \in \mathcal{H}$ and any $\theta \in \Theta$, we will construct a Jordan curve, say $\Omega^\theta$, that cuts the dynamical plane of $f_\lambda$ into two parts. The curve will meet the Julia set $J(f_\lambda)$ in a Cantor set of points. This kind of Jordan curve $\Omega^\theta$ will be called a ‘cut ray’ of angle $\theta$. In the following, we construct such rays following a slightly different presentation from Devaney’s in [6].

Recall that the itinerary map $s_\lambda : A_\lambda \rightarrow \Sigma$ from a Cantor set onto a symbolic space is bijective. We first extend the definition of $s_\lambda$ to a larger set. Let $E_\lambda = \bigcap_{k \geq 0} f^{-k}_\lambda \left( \bigcup_{j \in \mathbb{N} \setminus \{0, n\}} S_j \right)$ be the set of all points in the dynamical plane with orbits that remain in $\bigcup_{j \in \mathbb{N} \setminus \{0, n\}} S_j$ under all iterations of $f_\lambda$. By definition, $E_\lambda$ is a compact subset of $\mathbb{C}$ containing 0 and $\infty$. The assumption $\lambda \in \mathcal{H}$ implies that $E_\lambda$ contains no critical points other than 0 and $\infty$.

Let $O_\lambda = \bigcup_{k \geq 0} f^{-k}_\lambda(\infty)$ be the grand orbit of $\infty$. The map $s_\lambda : A_\lambda \rightarrow \Sigma$ can be extended to $s_\lambda : E_\lambda \setminus O_\lambda \rightarrow \Sigma$ as follows: for any $z \in E_\lambda \setminus O_\lambda$, suppose $f^{-k}_\lambda(z) \in S_{s_k}$ for $k \geq 0$; the itinerary of $z$ is then defined by $s_\lambda(z) = (s_0, s_1, s_2, \ldots)$. One can see that the map $s_\lambda : E_\lambda \setminus O_\lambda \rightarrow \Sigma$ is well defined. (In fact, if $f^{n+1}_\lambda(z)$ lies on the intersection of two sectors, then $f^{n+1}_\lambda(z)$ will land on the critical value $r$.)

Given an angle $\theta \in \Theta$ with itinerary $s(\theta) = (s_0, s_1, s_2, \ldots)$, it is easy to check that when $n$ is odd, $s(\theta + 1/2) = (-s_0, -s_1, -s_2, \ldots) = -s(\theta)$ and that when $n$ is even, $s(\theta + 1/2) = (-s_0, s_1, s_2, \ldots)$. We consider the set of all points in $E_\lambda \setminus O_\lambda$ with itineraries that take the form $(s_0, \pm s_1, \pm s_2, \ldots)$. The closure of this set is denoted by $\omega^\theta_\lambda$:

$$\omega^\theta_\lambda = \left\{ z \in E_\lambda \setminus O_\lambda ; s_\lambda(z) = (s_0, \pm s_1, \pm s_2, \ldots) \right\} = \left\{ z \in E_\lambda \setminus O_\lambda ; k(s_\lambda(z)) = \theta \right\}.$$

According to Devaney, the set $\omega^\theta_\lambda$ is called a ‘full ray’ of angle $\theta$. Let $\Omega^\theta_\lambda = \omega^\theta_\lambda \cup \omega^{\theta + 1/2}_\lambda$; we call the set $\Omega^\theta_\lambda$ a ‘cut ray’ of angle $\theta$ (or $\theta + 1/2$). One may verify that

$$\Omega^\theta_\lambda = \left\{ z \in E_\lambda \setminus O_\lambda ; s_\lambda(z) = (\pm s_0, \pm s_1, \pm s_2, \ldots) \right\} = \bigcup_{k \geq 0} f^{-k}_\lambda(S_{s_k} \cup S_{-s_k}).$$

We first give an intuitive description of the cut ray $\Omega^\theta_\lambda$. For $m \geq 0$, let
\[ \Omega_{\theta, m}^\theta = \bigcap_{0 \leq k \leq m} f_{\lambda}^{-k} (S_{sk} \cup S_{-sk}). \]

Note that the set \( \Omega_{\theta, 0}^\theta \) is a union of the two closed sectors \( S_0 \) and \( S_{-0} \). \( \Omega_{\theta, 1}^\theta \) is a string of four closed disks that lie inside \( \Omega_{\theta, 0}^\theta \). Inductively, \( \Omega_{\theta, m}^\theta \) is a string of \( 2^{m+1} \) closed disks that are contained in \( \Omega_{\theta, m-1}^\theta \), and each of these disks meets exactly two others at the preimages of \( \infty \). Hence, \( \Omega_{\theta, m}^\theta \) is a connected and compact set. One can show that \( \Omega_{\theta, m}^\theta \) converges to \( \Omega_{\theta}^\theta = \bigcap_{k \geq 0} \Omega_{\theta, k}^\theta \) in Hausdorff topology as \( m \to \infty \) (because a shrinking sequence of compact sets always converges in Hausdorff topology). Roughly, the set \( \Omega_{\theta, m}^\theta \) becomes thinner when \( m \) becomes larger and \( \Omega_{\theta, m}^\theta \) finally shrinks to \( \Omega_{\theta}^\theta \). It is therefore conjectured that \( \Omega_{\theta}^\theta \) is a Jordan curve. (A rigorous proof of this fact will be given in Proposition 3.3.)

By construction, the cut ray satisfies:

- \( \Omega_{\theta}^\theta = -\Omega_{\lambda}^\theta \).
- \( \Omega_{\theta}^\theta \setminus \{0, \infty\} \) is contained in the interior of \( S_0 \cup S_{-0} \).
- \( f_\lambda : \Omega_{\theta}^\theta \to \Omega_{\theta}^{t(\theta)} \) is a two-to-one map.
- \( \bigcup_{\theta \in \Theta} \Omega_{\theta}^\theta = E_\lambda \).

**Lemma 3.4.** Let \( \lambda \in \mathcal{H} \); then, there is a constant \( v > 0 \) such that for any \( \theta \in \Theta \),

\[ \overline{R_\lambda(\theta)} \cap U(v) = \{ z \in E_\lambda \cap U(v); s_\lambda(z) = s(\theta) \}. \]

**Proof.** For any small number \( \varepsilon > 0 \), we define \( \Theta_{k, \varepsilon}^\lambda = [\frac{\lambda(k)}{2\pi} + \frac{\lambda}{4\pi} + \varepsilon, \frac{\lambda(k)+1}{2\pi} + \frac{\lambda}{4\pi} - \varepsilon] \), \( S_{k, \varepsilon} = \{ z \in S_k \setminus \{0, \infty\}; \arg z \in \Theta_{k, \varepsilon}^\lambda \} \cup \{0, \infty\} \) for \( k \in \mathbb{N} \). \( \{0, \infty\} \) is a closed subset of \( S_k \). One can verify that there is an \( \varepsilon > 0 \) such that \( \Theta^\lambda = \bigcap_{j \geq 0} \tau^{-j} (\bigcup_{k \in \mathbb{N}} (\Theta_{k, \varepsilon}^\lambda \cup \Theta_{-k, \varepsilon}^\lambda)) \) and \( E_\lambda = \bigcap_{k \geq 0} f_\lambda^{-k}(S_{k, \varepsilon} \cup S_{-k, \varepsilon}) \). Thus, for any \( \theta \in \Theta \) with \( s(\theta) = (s_0, s_1, \ldots) \), the cut ray \( \Omega_{\theta}^\theta = \bigcap_{k \geq 0} f_\lambda^{-k}(S_{k, \varepsilon} \cup S_{-k, \varepsilon}) \). We fix such \( \varepsilon \) (notice that \( \varepsilon \) is independent of \( \theta \)).

Because \( \phi_{\lambda}(\infty) = 1 \), we may choose \( v = v(\varepsilon) \) large enough such that \( |\arg z - \arg \phi_{s}(z)| < \varepsilon \) for all \( z \in U(v) \). We define a map \( \xi : U(v) \to S \) by \( \xi(z) = \frac{\arg \phi_{s}(z)}{2\pi} \). The map \( \xi \) satisfies \( \xi \circ f_\lambda = \tau \circ \xi \).

If \( z \in \overline{R_\lambda(\theta)} \cap U(v) \) and \( z \neq \infty \), then for any \( k \geq 0 \), \arg \phi_{s}(f_\lambda^k(z)) \in \Theta_{k, \varepsilon}^\lambda \). We conclude that \arg \phi_{s}(f_\lambda^k(z)) \in \Theta_{s_k, \varepsilon}^\lambda \). Or, equivalently, \( f_\lambda^k(z) \in S_{s_k} \) for all \( k \geq 0 \). Thus, \( s_\lambda(z) = s(\theta) \).

On the other hand, for any \( z \neq \infty \), \( z \in \overline{E_\lambda} \cap U(v) \) with \( s_\lambda(z) = s(\theta) \), we know from the above that \( f_\lambda^k(z) \in S_{s_k} \) for all \( k \geq 0 \), thus \arg \phi_{s}(f_\lambda^k(z)) \in \Theta_{s_k, \varepsilon}^\lambda \). It turns out that \arg \phi_{s}(f_\lambda^k(z)) = \tau^k(\xi(z)) \in \Theta_{s_k, \varepsilon}^\lambda \). By Lemma 3.3, \( \xi^k(\xi(z)) = \xi(z) = \xi(\theta) \). Thus, we have \( \xi(z) = \theta \); this means \( z \in \overline{R_\lambda(\theta)} \cap U(v) \).

**Proposition 3.1.** For any \( \lambda \in \mathcal{H} \) and any \( \theta \in \Theta \), the external ray \( R_\lambda(\theta) \) lands at a unique point \( p_\lambda(\theta) \in \partial B_\lambda \) and \( \overline{R_\lambda(\theta)} = \{ z \in E_\lambda \setminus O_\lambda; s_\lambda(z) = s(\theta) \} \cup \{ \infty \} = \{ z \in (E_\lambda \setminus O_\lambda) \cap B_\lambda; s_\lambda(z) = s(\theta) \} \cup \{ p_\lambda(\theta) \} \cup \{ \infty \} \).

**Proof.** Suppose \( s(\theta) = (s_0, s_1, s_2, \ldots) \). Let \( \ell_\lambda(v, \theta) = \{ z \in R_\lambda(\theta); v \leq G_\lambda(z) \leq nv \} \) be the portion of \( R_\lambda(\theta) \) that lies between two equipotential curves \( e(B_\lambda, v) \) and \( e(B_\lambda, nv) \). Based on Lemma 3.4, we can assume \( v \) large enough such that for any \( \beta \in \Theta \), \( \overline{R_\lambda(\beta)} \cap U(v) = \)
\[
\{ z \in E_\lambda \cap U(v); \ s_\lambda(z) = s(\beta) \}. \]
By pulling back \( \ell_\lambda(v, \tau(\theta)) \) by \( f_\lambda^{-1} \) to \( S_{\theta_0} \), we can extend the portion of \( R_\lambda(\theta) \), say \( \gamma_0 = \overline{R_\lambda(\theta)} \cap U(v) \), to a longer one \( \gamma_1 = h_{\theta_0}(\ell_\lambda(v, \tau(\theta))) \cup \gamma_0 \). Obviously, \( \gamma_1 \subset S_{\theta_0} \cap \overline{R_\lambda(\theta)} \). Continuing inductively, suppose we have already constructed a portion \( \gamma_k \) of \( \overline{R_\lambda(\theta)} \); we then add a segment \( h_{\theta_0} \circ \cdots \circ h_{\theta_k}(\ell_\lambda(v, \tau_1(\theta))) \) to \( \gamma_k \) and obtain \( \gamma_{k+1} = \gamma_k \cup h_{\theta_0} \circ \cdots \circ h_{\theta_k}(\ell_\lambda(v, \tau_{k+1}(\theta))) \). By construction, one can confirm that \( h_{\theta_0} \circ \cdots \circ h_{\theta_k}(\ell_\lambda(v, \tau_{k+1}(\theta))) \subset S_{\theta_0} \cap \overline{R_\lambda(\theta)} \), and that for any \( z \in h_{\theta_0} \circ \cdots \circ h_{\theta_k}(\ell_\lambda(v, \tau_{k+1}(\theta))), s_\lambda(z) = (s_0, s_1, s_2, \ldots). \) It turns out that
\[
R_\lambda(\theta) \setminus \gamma_0 = \bigcup_{k \geq 0} h_{\theta_0} \circ \cdots \circ h_{\theta_k}(\ell_\lambda(v, \tau_{k+1}(\theta))).
\]

In the following, we show that the external ray \( R_\lambda(\theta) \) lands at \( \partial B_\lambda \). Because \( h_k : \Gamma_\lambda \to \Gamma_\lambda \) contracts the hyperbolic metric \( \rho_\lambda \) of \( \Gamma_\lambda \), for any \( k \in \mathbb{N} \setminus \{0, n\} \), there is a constant \( \delta \in (0, 1) \) such that
\[
\rho_\lambda(h_k(x), h_k(y)) \leq \delta \rho_\lambda(x, y), \quad \forall x, y \in \overline{\Gamma(v)} \cap \left( \bigcup_{j \in \mathbb{N} \setminus \{0, n\}} S_j \right), \quad \forall k \in \mathbb{N} \setminus \{0, n\}.
\]
Notice that \( \bigcup_{\alpha \in \theta} \ell_\lambda(v, \alpha) = E_\lambda \cap \{ z \in B_\lambda; \ v \leq G_\lambda(z) \leq \pi v \} \) is a compact subset of \( \Gamma_\lambda \), with respect to the hyperbolic metric of \( \Gamma_\lambda \) we have
\[
\text{Hyper.length}(h_{\theta_0} \circ \cdots \circ h_{\theta_k}(\ell_\lambda(v, \tau_{k+1}(\theta)))) = O(\delta^k).
\]
This implies that \( R_\lambda(\theta) \setminus \gamma_0 \) has finite hyperbolic length in \( \Gamma_\lambda \); thus, the external ray \( R_\lambda(\theta) \) lands at \( \partial B_\lambda \). Let \( p_\lambda(\theta) \) be the landing point. It is easy to confirm that \( s_\lambda(p_\lambda(\theta)) = s(\theta) \) and \( p_\lambda(\theta) \in \partial B_\lambda \cap \Lambda_\lambda \). Thus, we have
\[
\overline{R_\lambda(\theta)} \subset \{ z \in (E_\lambda \setminus O_\lambda) \cap B_\lambda; \ s_\lambda(z) = s(\theta) \} \cup \{ p_\lambda(\theta) \} \cup \{ \infty \}
\subset \{ z \in E_\lambda \setminus O_\lambda; \ s_\lambda(z) = s(\theta) \} \cup \{ \infty \}.
\]
Finally, we show \( \overline{R_\lambda(\theta)} \supset \{ z \in E_\lambda \setminus O_\lambda; \ s_\lambda(z) = s(\theta) \} \cup \{ \infty \} \). For any \( x \in \{ z \in E_\lambda \setminus O_\lambda; \ s_\lambda(z) = s(\theta) \} \), we consider the orbit of \( x \).

If the orbit of \( x \) remains bounded, then based on Lemma 2.3, we have \( x \in \Lambda_\lambda \). Because \( s_\lambda|_{\Lambda_\lambda} : \Lambda_\lambda \to \Sigma \) is bijective and \( s_\lambda(x) = s_\lambda(p_\lambda(\theta)) = s(\theta) \), we conclude \( x = p_\lambda(\theta) \in \overline{R_\lambda(\theta)} \).

If the orbit of \( x \) tends toward \( \infty \), then by Lemma 3.4, there is an integer \( M \geq 1 \) such that \( f_\lambda^M(x) \in R_\lambda(\tau^M(\theta)) \). Note that for any \( j \geq 0 \), the above argument implies \( R_\lambda(\tau^j(\theta)) \subset S_{\theta_j} \). Because \( f_\lambda(\tau^j(\theta)) = R_\lambda(\tau^j(\theta)) \) and \( h_{\theta_{j+1}} \) is the inverse branch of \( f : \text{int}(S_{\theta_{j+1}}) \to \Gamma_\lambda \), we conclude that for all \( k \geq 1, h_{\theta_{j+1}}(R_\lambda(\tau^k(\theta))) = R_\lambda(\tau^{k-1}(\theta)) \) and \( h_{\theta_{j+1}}(f_\lambda^k(x)) = f_\lambda^{k-1}(x) \). It turns out that \( x \in R_\lambda(\theta) \) and \( \overline{R_\lambda(\theta)} \supset \{ z \in E_\lambda \setminus O_\lambda; \ s_\lambda(z) = s(\theta) \} \cup \{ \infty \} \). \( \square \)

**Proposition 3.2.** For any \( \lambda \in \mathcal{H} \) and any \( \theta \in \Theta \) with itinerary \( s(\theta) = (s_0, s_1, s_2, \ldots) \), the cut ray \( \Omega^\theta_\lambda \) satisfies:

1. \( \Omega^\theta_\lambda \) meets the Julia set \( J(f_\lambda) \) in a Cantor set of points. More precisely, \( \Omega^\theta_\lambda \cap J(f_\lambda) = (\kappa \circ s_\lambda|_{\Lambda_\lambda})^{-1}((\{ \theta, \theta + \frac{1}{2} \})). \)
\[ \Omega_{\lambda}^\theta \text{ meets the Fatou set } F(f_{\lambda}) \text{ in a countable union of external rays and radial rays together with the preimages of } \infty \text{ that lie in the closure of these rays. More precisely,} \\
\Omega_{\lambda}^\theta \cap B_\lambda = R_{\lambda}(\theta) \cup R_{\lambda}\left(\theta + \frac{1}{2}\right) \cup \{\infty\}, \text{ if } n \text{ is odd,} \\
\Omega_{\lambda}^\theta \cap T_\lambda = \begin{cases} 
\{h-s_0(R_{\lambda}(\tau(\theta))) \cup h_s_0(R_{\lambda}(\tau(\theta) + \frac{1}{2})) \cup \{0\}, \text{ if } n \text{ is odd,} \\
\{h_s_0(R_{\lambda}(\tau(\theta)) + \frac{1}{2})) \cup h_s_0(R_{\lambda}(\tau(\theta) + \frac{1}{2})) \cup \{0\}, \text{ if } n \text{ is even.}
\end{cases}
\]

For any \( U \in P \setminus \{B_\lambda, T_\lambda\} \) with \( U \cap \Omega_{\lambda}^\theta \neq \emptyset \), \( U \) is of the form \( h_{b_0} \circ \cdots \circ h_{b_{k-1}}(T_\lambda) \), where \( k \geq 1 \) and \( (b_0, \ldots, b_{k-1}) \in \{(\pm s_0, \ldots, \pm s_{k-1})\} \). Moreover,

\[ \Omega_{\lambda}^\theta \cap U = h_{b_0} \circ \cdots \circ h_{b_{k-1}}(\Omega_{\lambda}^{\tau^k(\theta)} \cap T_\lambda) \]

\[ = \begin{cases} 
\{h_{b_0} \circ \cdots \circ h_{b_{k-1}}(h_{-s_0}(R_{\lambda}(\tau^{k+1}(\theta))) \cup h_s_0(R_{\lambda}(\tau^{k+1}(\theta) + \frac{1}{2})) \cup \{0\}), \text{ if } n \text{ is odd,} \\
\{h_{b_0} \circ \cdots \circ h_{b_{k-1}}(h_{-s_0}(R_{\lambda}(\tau^{k+1}(\theta) + \frac{1}{2})) \cup h_s_0(R_{\lambda}(\tau^{k+1}(\theta) + \frac{1}{2}) \cup \{0\}), \text{ if } n \text{ is even.}
\end{cases}
\]

See Fig. 3 for the combinatorial structure of a part of a cut ray.

**Proof.**

1. For \( z \in \Omega_{\lambda}^\theta \), first note that \( z \in \Omega_{\lambda}^\theta \cap J(f_{\lambda}) \) if and only if the orbit of \( z \) remains bounded, if and only if \( z \in A_\lambda \) and \( s_\lambda(z) \in \{(\pm s_0, \pm s_1, \pm s_2, \ldots)\} = \kappa^{-1}(\{\theta, \theta + \frac{1}{2}\}) \). Thus, we have \( \Omega_{\lambda}^\theta \cap J(f_{\lambda}) = (\kappa \circ s_\lambda|_{A_\lambda})^{-1}(\{\theta, \theta + \frac{1}{2}\}) \).

2. Let \( U \) be a Fatou component such that \( U \cap \Omega_{\lambda}^\theta \neq \emptyset \). Then, by 1, \( U \) is eventually mapped onto \( B_\lambda \).
Case 1. $U = B_\lambda$. By Proposition 3.1, $\Omega^\theta_\lambda \cap B_\lambda \supset R_\lambda(\theta) \cup R_\lambda(\theta + \frac{1}{2}) \cup \{\infty\}$. On the other hand, for any $z \in (\Omega^\theta_\lambda \cap B_\lambda) \setminus \{\infty\}$, there is an integer $M \geq 1$ such that $f^M_\lambda(z) \in U(v)$, where $v$ is a positive constant chosen by Lemma 3.4. Because $s_\lambda(f^M_\lambda(z)) \in \{(s_M, \pm s_{M+1}, \pm s_{M+2}, \ldots)\}$, we conclude that the itinerary of $f^M_\lambda(z)$ must be the same as that of some angle $\beta \in \Theta$. Thus,

$$s_\lambda(f^M_\lambda(z)) = \begin{cases} (s_M, s_{M+1}, s_{M+2}, \ldots) \text{ or } (-s_M, -s_{M+1}, -s_{M+2}, \ldots), & \text{if } n \text{ is odd}, \\ (s_M, s_{M+1}, s_{M+2}, \ldots), & \text{if } n \text{ is even}. \end{cases}$$

Case 1.1. $n$ is odd. By Proposition 3.1, $f^M_\lambda(z) \in R_\lambda(\tau(M)(\theta)) \cup R_\lambda(\tau(M(\theta) + \frac{1}{2}))$. Note that $f^{-1}_\lambda(R_\lambda(\tau(M(\theta))) \cap (s_{s_{M-1}} \cup s_{s_{M-1}}) \cap B_\lambda = R_\lambda(\tau(M(\theta))) \cap (s_{s_{M-1}} \cup s_{s_{M-1}}) \cap B_\lambda = R_\lambda(\tau(M(\theta))) \cup R_\lambda(\tau(M(\theta) + \frac{1}{2}))$. If $M = 1$, then $z \in R_\lambda(\theta) \cup R_\lambda(\theta + \frac{1}{2})$, and the proof is done. If $M > 1$, then we claim $f^{-1}_\lambda(z) \in R_\lambda(\tau(M(\theta))) \cup (s_{s_{M-2}} \cup S_{s_{M-2}}) \cap B_\lambda = \emptyset$. Again, by induction, we have $z \in R_\lambda(\theta) \cup R_\lambda(\theta + \frac{1}{2})$ in this case.

Case 1.2. $n$ is even. By Proposition 3.1, $f^M_\lambda(z) \in R_\lambda(\tau(M(\theta)))$. Because $f^{-1}_\lambda(R_\lambda(\tau(M(\theta))) \cap (s_{s_{M-1}} \cup s_{s_{M-1}}) \cap B_\lambda = R_\lambda(\tau(M(\theta))) \cap (s_{s_{M-1}} \cup s_{s_{M-1}}) \cap B_\lambda = R_\lambda(\tau(M(\theta))) \cup R_\lambda(\tau(M(\theta) + \frac{1}{2}))$. If $M = 1$, then $z \in R_\lambda(\theta) \cup R_\lambda(\theta + \frac{1}{2})$, and the proof is done. If $M > 1$, then we claim $f^{-1}_\lambda(z) \in R_\lambda(\tau(M(\theta))) \cup (s_{s_{M-2}} \cup S_{s_{M-2}}) \cap B_\lambda = \emptyset$. Again, by induction, we have $z \in R_\lambda(\theta) \cup R_\lambda(\theta + \frac{1}{2})$ in this case.

Case 2. $U = T_\lambda$. In this case, if $n$ is odd, then $f_{\lambda}(\Omega^\theta_\lambda \cap T_\lambda \cap S_{s_0}) = \Omega^\tau_{\lambda(\theta)} \cap B_\lambda \cap S_{s_1} = R_\lambda(\theta + \frac{1}{2}) \cup \{\infty\}$ and $f_{\lambda}(\Omega^\theta_\lambda \cap T_\lambda \cap S_{s_0}) = \Omega^\tau_{\lambda(\theta)} \cap B_\lambda \cap S_{s_1} = R_\lambda(\theta) \cup \{\infty\}$. So $\Omega^\theta_{\lambda} \cap T_\lambda = h_{s_0}(R_\lambda(\tau(\theta))) \cup h_{s_0}(R_\lambda(\tau(\theta) + \frac{1}{2})) \cup \{0\}$; if $n$ is even, then $f_{\lambda}(\Omega^\theta_\lambda \cap T_\lambda \cap S_{s_0}) = f_{\lambda}(\Omega^\theta_{\lambda} \cap T_\lambda \cap S_{s_0}) = \Omega^\tau_{\lambda(\theta)} \cap B_\lambda \cap S_{s_1} = R_\lambda(\tau(\theta) + \frac{1}{2}) \cup \{\infty\}$. So $\Omega^\theta_{\lambda} \cap T_\lambda = h_{s_0}(R_\lambda(\tau(\theta))) \cup \{0\}$. The cut ray $\Omega^\theta_{\lambda}$ is a Jordan curve (see Fig. 4).

**Proof.** Suppose $s(\theta) = (s_0, s_1, s_2, \ldots)$. For $k \geq 0$, define

$$\hat{\Omega}^\tau_{\lambda, 0} = \Omega^\tau_{\lambda} \cup S^{u}_s \cup S^{w}_s, \quad \hat{\Omega}^\theta_{\lambda, k} = \bigcap_{0 \leq j < k} f^{-j}_\lambda(\hat{\Omega}^\tau_{\lambda, 0}).$$

The set $\hat{\Omega}^\theta_{\lambda, k}$ is connected and compact, and it contains $\Omega^\theta_{\lambda}$. It is easy to check that $\hat{\Omega}^\theta_{\lambda, k} \supset \hat{\Omega}^\theta_{\lambda, k+1}$ and $\bigcap_{k \geq 0} \hat{\Omega}^\theta_{\lambda, k} = \Omega^\theta_{\lambda}$. In the following discussion, we can assume $k$ is sufficiently large that $\hat{\Omega}^\theta_{\lambda, k}$ avoids the critical values $v_\lambda^\pm$. Let $D^\pm_k$ be the component of $\mathbb{C} \setminus \hat{\Omega}^\theta_{\lambda, k}$ that contains $v_\lambda^\pm$ and
Fig. 4. Cut rays with angles $1/4$, $1/3$, $1/2$ when $n = 3$.

Fig. 5. The cut ray union with the shadow regions is $\hat{\Omega}_{\theta, \lambda, k}$ (resp. $\hat{\Omega}_{\theta, \lambda, k+1}$). The two components of $\hat{C} - \hat{\Omega}_{\theta, \lambda, k}$ (resp. $\hat{C} - \hat{\Omega}_{\theta, \lambda, k+1}$) are $D^+_k$ and $D^-_k$ (resp. $D^+_k + 1$ and $D^-_k + 1$).

$D^-_k$ be the component of $\hat{C} \setminus \hat{\Omega}_{\theta, \lambda, k}$ that contains $v^-_\lambda$. Let $D^+_\infty = \bigcup_{k \geq 0} D^+_k$ and $D^-_\infty = \bigcup_{k \geq 0} D^-_k$; then, $D^+_\infty \cup D^-_\infty \cup \Omega^\theta_{\lambda, k} = \hat{C}$ (Fig. 5).

We first construct a Cantor set on $S = \mathbb{R}/\mathbb{Z}$. Let $E_1 = (5/24, 13/24)$, $E_2 = (17/24, 25/24)$ be two open intervals on $S$ and $\xi$ be the map $t \mapsto 3t \mod Z$. By definition, $\xi(E_i) \supset E_1 \cup E_2$. Let $T_k = \bigcap_{0 \leq j \leq k} \xi^{-j}(E_1 \cup E_2)$. Then, $T_k \supset T_{k+1}$ and $T_k$ has $2^{k+1}$ components. The intersection
\( \bigcap_{k \geq 0} T_k \) is denoted by \( T_\infty \). Because \( T_\infty = \bigcap_{k \geq 0} \xi^{-k}(E_1 \cup E_2) = \bigcap_{k \geq 0} \xi^{-k}(E_1 \cup E_2) \), we conclude that \( T_\infty \) is a Cantor set.

Now, we define two sequences of Jordan curves \( \{y^+_k : S \rightarrow \partial D^+_k\}, \{y^-_k : S \rightarrow \partial D^-_k\} \) in the following manner: for \( k \) large enough,

1. \( y^+_{k+1} \mid S \setminus T_k = y^+_k \mid S \setminus T_k = y^-_k \mid S \setminus T_k = y^-_{k+1} \mid S \setminus T_k \).
2. \( y^+_k(S \setminus T_k) = \Omega^0_k \cap \partial D^+_k = \Omega^0_k \cap \partial D^-_k = y^-_k(S \setminus T_k) \).
3. \( y^+_k(T_k) = \partial D^+_k \setminus \Omega^0_k, y^-_k(T_k) = \partial D^-_k \setminus \Omega^0_k \).

In the following, we show that each sequence of maps \( \{y^+_k : S \rightarrow \partial D^+_k\}, \{y^-_k : S \rightarrow \partial D^-_k\} \) converges in the spherical metric. By construction, \( y^+_{k+1} \mid S \setminus T_k = y^+_k \mid S \setminus T_k \), and for any component \( W \) of \( T_k \), \( y^+_{k+1}(W) \) and \( y^+_k(W) \) are contained in the same component of \( \bigcap_{0 \leq j \leq k} \mathcal{f}^{-j}_\lambda(S^u_{s_j} \cup S^-_{-s_j}) \). Because the spherical metric and the hyperbolic metric are comparable in any compact subset of \( T_\lambda \), we conclude by Lemma 2.3 that

\[
\max_{t \in S} (\delta^+(y^+_k(t), y^+_k(t))) = O(\delta^k),
\]

where \( \delta^+ \) is the spherical metric and \( \delta \in (0, 1) \) is a constant. Thus, the sequence \( \{y^+_k\} \) has a limit map \( y^+_\infty : S \rightarrow \partial D^+_\infty \) that is continuous and surjective. Similarly, the sequence \( \{y^-_k\} \) also has a limit map \( y^-_\infty : S \rightarrow \partial D^-_\infty \) that is continuous and surjective. The limit maps \( y^+_\infty \) and \( y^-_\infty \) satisfy \( y^+_\infty(S \setminus T_\infty) = y^-_\infty(S \setminus T_\infty) \). By continuity, \( y^+_\infty \) and \( y^-_\infty \) are identical on \( S \). This implies that \( \partial D^+_\infty = \partial D^-_\infty = \Omega^0_\lambda \) and \( \Omega^0_\lambda \) is locally connected.

To finish, we show that \( \Omega^0_\lambda \) is a Jordan curve following the idea in [24]. Let \( \Phi : \mathbb{D} \rightarrow D^+_\infty \) be a Riemann mapping. Because \( \partial D^+_\infty \) is locally connected, \( \Phi \) has an extension from \( \overline{D} \) to \( D^+_\infty \). If two distinct radial segments \( \Phi((0, 1)e^{2\pi i \theta}) \) and \( \Phi((0, 1)e^{2\pi i \theta_2}) \) converge on the same point \( p \), then the Jordan curve \( \Phi((0, 1)e^{2\pi i \theta}) \cup \Phi((0, 1)e^{2\pi i \theta_2}) \cup \{\Phi(0), p\} \) separates a section of the boundary \( \partial D^+_\infty \) from \( D^-\infty \). But this is a contradiction because \( D^+_\infty \) and \( D^-\infty \) share a common boundary. \( \square \)

**Proposition 3.4.** For \( \lambda \in \mathcal{H} \) and \( \theta \in \Theta \), all periodic points on \( \Omega^0_\lambda \cap J(f_\lambda) \) are repulsive.

**Proof.** Suppose \( s(\theta) = (s_0, s_1, s_2, \ldots) \). Let \( z \in \Omega^0_\lambda \cap J(f_\lambda) \) be a periodic point with period \( p \). The itinerary of \( z \) is then of the form \((a_0, a_1, \ldots, a_{p-1})\), where \( a_j \in \{\pm s_j\} \) for \( 0 \leq j \leq p - 1 \). Let \( a_k = a_k \mod p \) for \( k \geq 0 \) and \( S_{a_0 \ldots a_k}^u = \bigcap_{0 \leq j \leq k} f_\lambda^{-j}(S^u_{a_k}) \). By Lemma 2.3, the hyperbolic diameter of \( S_{a_0 \ldots a_k}^u \) is \( O(\delta^k) \) when \( s \) is large. We can therefore choose an \( N \) sufficiently large that \( f_\lambda^k : \text{int}(S_{a_0 \ldots a_N}^u) \rightarrow \text{int}(S_{a_0 \ldots a_N}^u) \) is a conformal map. Because \( z \in \text{int}(S_{a_0 \ldots a_N}^u) \subset S_{a_0 \ldots a_N}^u \subset \text{int}(S_{a_0 \ldots a_N}^u) \), we conclude \(|(f_\lambda^k)'(z)| > 1 \) by the Schwarz Lemma. Thus, \( z \) is a repelling periodic point. \( \square \)

Proposition 3.2 tells us the combinatorial structure of the cut ray \( \Omega^0_\lambda \). The following proposition shows that the iterated preimages of \( \Omega^0_\lambda \) have the same combinatorial structure as \( \Omega^0_\lambda \) provided that \( \Omega^0_\lambda \) does not meet the critical orbit.

**Proposition 3.5.** For \( \lambda \in \mathcal{H} \) and \( \theta \in \Theta \), suppose the cut ray \( \Omega^0_\lambda \) does not meet the critical orbit. Then, for any \( \alpha \in \bigcup_{k \geq 0} \tau^{-k}(\theta) \), there is a unique ray \( \omega^\alpha_\lambda \) such that:
1. \( \omega^\alpha_\lambda \) is a continuous curve connecting 0 with \( \infty \).
2. \( \omega^\alpha_\lambda^\alpha/2 = -\omega^\alpha_\lambda \).
3. \( f_\lambda(\omega^\alpha_\lambda) = \omega^\tau(\alpha)_\lambda \cup \omega^\tau(\alpha)+1/2 \).
4. \( \omega^\alpha_\lambda \cap B_\lambda = R_\lambda(\alpha) \cup \{ \infty \} \).

For this reason, we still call \( \omega^\alpha_\lambda \) a full ray of angle \( \alpha \) and \( \Omega^\alpha_\lambda = \omega^\alpha_\lambda \cup \omega^\alpha_\lambda+1/2 \) a cut ray of angle \( \alpha \) (or \( \alpha + \frac{1}{2} \)).

**Proof.** The proof is based on an inductive argument. Suppose \( \alpha \in \bigcup_{k \geq 0} \tau^{-k}(\theta) \) is an angle such that the full ray \( \omega^\alpha_\lambda \) and the cut ray \( \Omega^\alpha_\lambda \) satisfy 1, 2, 3, 4. Then, for \( \beta \in \tau^{-1}(\alpha) \), we define \( \omega^\beta_\lambda \) by lifting \( \Omega^\alpha_\lambda \) in the following way:

\[
f_\lambda(\omega^\alpha_\lambda) = \Omega^\alpha_\lambda, \quad \omega^\beta_\lambda \cap B_\lambda = R_\lambda(\beta) \cup \{ \infty \}.
\]

The ray \( \omega^\beta_\lambda \) is unique because we require \( \omega^\beta_\lambda \cap B_\lambda = R_\lambda(\beta) \cup \{ \infty \} \). Also, by uniqueness of lifting maps, we conclude \( \omega^{\beta+1/2}_\lambda = -\omega^\beta_\lambda \) by the fact \( R_\lambda(\beta + 1/2) = -R_\lambda(\beta) \) and \( \Omega^\beta_\lambda = -\Omega^\alpha_\lambda \).

In the following, we show that \( \omega^\beta_\lambda \) connects \( \infty \) and 0. If not, then \( \omega^\beta_\lambda \) must be a curve connecting \( \infty \) with itself, hence a Jordan curve. This implies that \( \omega^\beta_\lambda \) does not meet 0. Because \( \Omega^\alpha_\lambda = -\Omega^\alpha_\lambda \), all curves in the set \( C = \{ e^{k\pi i/n} \omega^\beta_\lambda, \ H_\lambda(e^{k\pi i/n} \omega^\beta_\lambda) : 0 \leq k < 2n \} \) are preimages of \( \Omega^\alpha_\lambda \), where \( H_\lambda(z) = \sqrt[1]{z} \). Because \( \Omega^\alpha_\lambda \) does not meet the critical orbit, we conclude that for any \( \gamma_1, \gamma_2 \in C \) with \( \gamma_1 \neq \gamma_2, \gamma_1 \) and \( \gamma_2 \) are disjoint outside \( [0, \infty) \). This means \#C = 4n. However, this is a contradiction because the degree of \( f_\lambda \) is 2n. \( \square \)

Recall that for any \( \theta \in \Theta \) with itinerary \( s(\theta) = (s_0, s_1, s_2, \ldots) \), the cut ray \( \Omega^\theta_\lambda \) contains at least two points, 0 and \( \infty \), and \( \Omega^\theta_\lambda \setminus [0, \infty) \) is contained in the interior of \( S_{s_0} \cup S_{-s_0} \). Now, given two angles \( \alpha, \beta \in \Theta \) with \( \Omega^\alpha_\lambda \neq \Omega^\beta_\lambda \), suppose \( s(\alpha) = (s_0^\alpha, s_1^\alpha, s_2^\alpha, \ldots), \ s(\beta) = (s_0^\beta, s_1^\beta, s_2^\beta, \ldots) \). Let \( J(\alpha, \beta) \) be the first integer \( k \geq 0 \) such that \( |s_k^\alpha| \neq |s_k^\beta| \). Note that the intersection \( \Omega^\alpha_\lambda \cap \Omega^\beta_\lambda \) consists of at least two points 0 and \( \infty \). Furthermore, if \( J(\alpha, \beta) = 0 \), then \( \Omega^\alpha_\lambda \cap \Omega^\beta_\lambda = [0, \infty) \). The following proposition tells us the number of intersection points in the general case.

**Proposition 3.6.** Let \( \alpha, \beta \in \Theta \) with \( \Omega^\alpha_\lambda \neq \Omega^\beta_\lambda \); then, the intersection \( \Omega^\alpha_\lambda \cap \Omega^\beta_\lambda \) consists of \( 2J(\alpha, \beta)+1 \) points.

**Proof.** We consider the orbit of \( \Omega^\alpha_\lambda \cap \Omega^\beta_\lambda \) under \( f_\lambda \):

\[
\Omega^\alpha_\lambda \cap \Omega^\beta_\lambda \rightarrow \Omega^\tau(\alpha)_\lambda \cap \Omega^\tau(\beta)_\lambda \rightarrow \cdots \rightarrow \Omega^{1(\alpha, \beta)}(\alpha)_\lambda \cap \Omega^{1(\alpha, \beta)}(\beta)_\lambda.
\]

Note that for any \( 0 \leq k \leq J(\alpha, \beta) - 1, f_\lambda : \Omega^\tau k(\alpha)_\lambda \cap \Omega^\tau k(\beta)_\lambda \rightarrow \Omega^\tau k+1(\alpha)_\lambda \cap \Omega^\tau k+1(\beta)_\lambda \) is a two-to-one map; thus, we have

\[
\#(\Omega^\alpha_\lambda \cap \Omega^\beta_\lambda) = 2\#(\Omega^\tau(\alpha)_\lambda \cap \Omega^\tau(\beta)_\lambda) = \cdots = 2^{J(\alpha, \beta)}\#(\Omega^{1(\alpha, \beta)}(\alpha)_\lambda \cap \Omega^{1(\alpha, \beta)}(\beta)_\lambda) = 2^{J(\alpha, \beta)+1}. \quad \square
\]
Remark 3.3. From the proof of Proposition 3.6, we know that any two distinct cut rays $\Omega^\alpha_\lambda$ and $\Omega^\beta_\lambda$ intersect at the preimages of $\infty$. More precisely, $\Omega^\alpha_\lambda \cap \Omega^\beta_\lambda \subseteq \bigcup_{0 \leq k \leq J(\alpha, \beta) + 1} f^{-k}_\lambda(\infty)$, and for $2 \leq k \leq J(\alpha, \beta) + 1$, the intersection $\Omega^\alpha_\lambda \cap \Omega^\beta_\lambda \cap f^{-k}(-1)(0)$ consists of $2^{k-1}$ points.

4. Puzzles, graphs and tableaux

4.1. The Yoccoz puzzle

Let $X_\lambda = \bar{C} \setminus \{z \in B_\lambda; \ G_\lambda(z) \geq 1\} = \mathcal{V}(1)$. Given $N$ periodic angles $\theta_1, \ldots, \theta_N$ that lie in different periodic cycles of $\Theta$, let

$$g_\lambda(\theta_1, \ldots, \theta_N) = \bigcup_{k \geq 0} \left( \bigcup_{i=1}^{\infty} \left( \Omega^\lambda_\theta(\theta_i) \cup \cdots \cup \Omega^\lambda_\theta(\theta_N) \right) \right).$$

Obviously, $g_\lambda(\theta_1, \ldots, \theta_N)$ is $f_\lambda$-invariant. The graph $G_\lambda(\theta_1, \ldots, \theta_N)$ generated by $\theta_1, \ldots, \theta_N$ is defined as follows:

$$G_\lambda(\theta_1, \ldots, \theta_N) = \partial X_\lambda \cup \left( X_\lambda \setminus g_\lambda(\theta_1, \ldots, \theta_N) \right).$$

The Yoccoz puzzle induced by the graph $G_\lambda(\theta_1, \ldots, \theta_N)$ is constructed in the following way. The Yoccoz puzzle of depth zero consists of all connected components of $X_\lambda \setminus G_\lambda(\theta_1, \ldots, \theta_N)$, and each component is called a puzzle piece of depth zero. The Yoccoz puzzle of greater depth can be constructed by induction as follows: if $P^{(1)}_d, \ldots, P^{(m)}_d$ are the puzzle pieces of depth $d$, then the connected components of the set $f^{-1}_\lambda(P^{(j)}_d)$ are the puzzle pieces $P^{(j)}_{d+1}$ of depth $d + 1$. One can verify that the puzzle pieces of depth $d$ consist of all connected components of $f_d^{-d}(X_\lambda \setminus G_\lambda(\theta_1, \ldots, \theta_N))$ and that each puzzle piece is a disk.

In applying the Yoccoz puzzle theory, we should avoid a situation in which the critical orbits touch the set $\Omega_\lambda(\theta_1, \ldots, \theta_N)$. If the critical orbits touch the graph $G_\lambda(\theta_1, \ldots, \theta_N)$, we say the graph $G_\lambda(\theta_1, \ldots, \theta_N)$ is touchable. In this case, we cannot find a sequence of shrinking puzzle pieces such that each piece contains a critical point in its interior (that is to say, we cannot find a non-degenerate critical annulus that plays a crucial role in the Yoccoz puzzle theory). For this reason, because there are infinite periodic angles in $\Theta$, we can change the $N$-tuple $(\theta_1, \ldots, \theta_N)$ to another $N$-tuple $(\theta'_1, \ldots, \theta'_N)$ to make the graph not touchable.

Let $J_0$ be the set of all points on the Julia set $J(f_\lambda)$ with orbits that eventually meet the graph $G_\lambda(\theta_1, \ldots, \theta_N)$. Then $J_0 = \bigcup_{k \geq 0} f^{-k}_\lambda(G_\lambda(\theta_1, \ldots, \theta_N) \cap J(f_\lambda))$. For any $z \in \bar{C} \setminus (A_\lambda \cup J_0)$, there is a unique sequence of puzzle pieces $P_0(z) \supset P_1(z) \supset P_2(z) \supset \cdots$ that contain $z$. By Proposition 3.4, if $f_\lambda$ has a non-repelling cycle in $\mathbb{C}$, say $C = \{z, f_1(z), \ldots, f^P_\lambda(z) = z\}$, then this cycle must avoid the graph $G_\lambda(\theta_1, \ldots, \theta_N)$. This implies that $C \subset \bar{C} \setminus (A_\lambda \cup J_0)$. Thus, for any $d \geq 0$ and any $x \in C$, the puzzle piece $P_d(x)$ is well defined.

Lemma 4.1. Suppose the graph $G_\lambda(\theta_1, \ldots, \theta_N)$ is not touchable, then for any $z \in \bar{C} \setminus (A_\lambda \cup J_0)$, the puzzle pieces satisfy:

$$-P_0(z) = P_0(-z), \quad \omega P_d(z) = P_d(\omega z), \quad \omega^{2n} = 1, \quad d \geq 1.$$
Proof. By the definition of the graph $G_\lambda(\theta_1, \ldots, \theta_N)$ and the symmetry of the Green function $G_\lambda: A_\lambda \to (0, +\infty)$ (see Lemma 2.1), we have $X_\lambda \setminus G_\lambda(\theta_1, \ldots, \theta_N) = -X_\lambda \setminus G_\lambda(\theta_1, \ldots, \theta_N)$. Thus $-P_0(z) = P_0(-z)$. Suppose that for some $d \geq 0$,
\[ f_\lambda^{-d}(X_\lambda \setminus G_\lambda(\theta_1, \ldots, \theta_N)) = f_\lambda^{-d}(X_\lambda \setminus G_\lambda(\theta_1, \ldots, \theta_N)). \]

Because $f_\lambda(\omega z) = \pm f_\lambda(z)$ and $G_\lambda(\omega z) = G_\lambda(z)$, we have $f_\lambda(z) \in f_\lambda^{-d}(X_\lambda \setminus G_\lambda(\theta_1, \ldots, \theta_N))$ if and only if $f_\lambda(z) = f_\lambda^{-d}(X_\lambda \setminus G_\lambda(\theta_1, \ldots, \theta_N))$. Thus
\[ f_\lambda^{-(d+1)}(X_\lambda \setminus G_\lambda(\theta_1, \ldots, \theta_N)) = \omega f_\lambda^{-(d+1)}(X_\lambda \setminus G_\lambda(\theta_1, \ldots, \theta_N)). \]

The conclusion follows by induction. □

**Lemma 4.2.** Suppose the graph $G_\lambda(\theta_1, \ldots, \theta_N)$ is not touchable, then for any $d \geq 0$ and any puzzle piece $P_d$ of depth $d$, the intersection $\bar{P}_d \cap J(f_\lambda)$ is connected.

**Proof.** It is equivalent to prove that every connected component of $\bar{C} \setminus (\bar{P}_d \cap J(f_\lambda))$ is simply connected. Because the Julia set $J(f_\lambda)$ is connected, every component of $\bar{C} \setminus (\bar{P}_d \cap J(f_\lambda))$ that lies inside $P_d$ is simply connected. Therefore, we only need to consider the components of $\bar{C} \setminus (\bar{P}_d \cap J(f_\lambda))$ that intersect with $\partial P_d$. Note that the puzzle piece $P_d$ is bounded by finitely many cut rays, say $\Omega_\lambda^{\beta_1}, \ldots, \Omega_\lambda^{\beta_s}$, together with finitely many equipotential curves $e(U_1, v), \ldots, e(U_t, v)$. By the structure of cut rays (Proposition 3.2), there is exactly one component of $\bar{C} \setminus (\bar{P}_d \cap J(f_\lambda))$ that intersects with the boundary $\partial P_d$. This component is the union of $\bar{P}_d$ and countably many Fatou components that intersect with the cut rays $\Omega_\lambda^{\beta_1}, \ldots, \Omega_\lambda^{\beta_s}$. Thus, it is also simply connected. □

### 4.2. Admissible graphs

Given the point $z \in \bar{C} \setminus (A_\lambda \cup J_0)$, the difference set $A_d(z) = P_d(z) \setminus \overline{P}_{d+1}(z)$ is an annulus, either degenerate or of positive modulus. Here, $d$ is called the depth of $A_d(z)$. For $d \geq 1$ and $c \in C_\lambda$, the annulus $A_d(z)$ is called off-critical, c-critical or c-semi-critical if $P_d(z)$ contains no critical points, $P_{d+1}(z)$ contains the critical point $c$ or $A_d(z)$ contains the critical point $c$, respectively.

Because the critical annuli play a crucial role in our discussion, we will devote ourselves to finding a graph such that with respect to the Yoccoz puzzle induced by such a graph, the critical annulus $A_d(c)$ is non-degenerate for some $d \geq 1$. By Lemma 4.1, if some critical annulus $A_d(c)$ of depth $d \geq 1$ is non-degenerate, then all critical annuli of the same depth are non-degenerate. The graph that satisfies this property is of special interest.

**Definition 4.1.** We say the graph $G_\lambda(\theta_1, \ldots, \theta_N)$ is admissible if it is not touchable and if with respect to the Yoccoz puzzle induced by $G_\lambda(\theta_1, \ldots, \theta_N)$ there exists a non-degenerate critical annulus $A_d(c)$ for some critical point $c \in C_\lambda$ and some depth $d \geq 1$. Otherwise, we say the graph $G_\lambda(\theta_1, \ldots, \theta_N)$ is non-admissible.

By definition, a non-admissible graph either is touchable or contains no non-degenerate critical annulus of depth greater than one with respect to its induced Yoccoz puzzle. In the definition
of an admissible graph, we require that the critical annulus \(A_d(c)\) is non-degenerate for some depth \(d \geq 1\) rather than \(d = 0\) because the puzzle pieces of depth zero have only two-fold symmetry and the puzzle pieces of depth greater than zero have \(2n\)-fold symmetry (see Lemma 4.1).

The following remark tells us that a graph may be non-admissible in some cases.

**Remark 4.1.** There exist non-admissible graphs. For example, for any \(n \geq 3\), suppose \(f_\lambda\) is 1-renormalizable at \(c_0\) (see Section 5 for definition). Then, the graph \(G_\lambda(1)\) is non-admissible because \(A_d(c_0)\) is degenerate for all depths \(d \geq 1\) (see Fig. 6). One should note that \(A_0(c_1)\) is non-degenerate and \(A_d(c_1) = e^{\pi i/3} A_d(c_0)\) is degenerate for all \(d \geq 1\).

However, even if non-admissible graphs exist, we can always find an admissible graph based on an elaborate choice. The aim of this section is to prove the existence of admissible graphs for \(n \geq 3\).

**Proposition 4.1.** For any \(n \geq 3\) and any \(\lambda \in \mathcal{H}\), if \(f_\lambda\) is not critically finite, then there always exists an admissible graph.

The proof is divided into three lemmas: Lemma 4.3, Lemma 4.4 and Lemma 4.5. In fact, these lemmas enable us to prove much more: when \(n \geq 5\), there always exist infinitely many admissible graphs \(f^{k+1}_\lambda(\sqrt[n]{\lambda}) = f^k_\lambda(\sqrt[n]{\lambda})\) or \(f^{k+2}_\lambda(\sqrt[n]{\lambda}) = f^k_\lambda(\sqrt[n]{\lambda})\) for some \(k \geq 1\).

**Lemma 4.3.** When \(n = 3\), there exists an admissible graph except when the critical orbit of \(f_\lambda\) eventually lands at a repelling cycle of period one or two. More precisely,

1. If neither \(G_\lambda(1/4)\) nor \(G_\lambda(1/2)\) is touchable, then at least one of the graphs \(G_\lambda(1/4), G_\lambda(1/2), G_\lambda(1/4, 1/2)\) is admissible.
2. If \(G_\lambda(1/2)\) is touchable, then either \(G_\lambda(1/4)\) is admissible or the critical orbit of \(f_\lambda\) eventually lands at a repelling cycle of period two.
3. If $G_\lambda(1/4)$ is touchable, then either $G_\lambda(1/2)$ is admissible or the critical orbit of $f_\lambda$ eventually lands at a repelling fixed point.

**Proof.** First, note that

$$f_\lambda^{-1}(\Omega_\lambda^{1/4}) = \Omega_\lambda^{1/12} \cup \Omega_\lambda^{1/4} \cup \Omega_\lambda^{5/12}, \quad f_\lambda^{-1}(\Omega_\lambda^{1/2}) = \Omega_\lambda^{1/6} \cup \Omega_\lambda^{1/3} \cup \Omega_\lambda^{1/2}.$$  

1. In this case, the full rays $\omega_\lambda^{1/12}$ and $\omega_\lambda^{1/6}$ decompose $S_0$ into four domains: $D_1, D_2, D_3$ and $D_4$ (see Fig. 7). If neither $G_\lambda(1/4)$ nor $G_\lambda(1/2)$ is touchable, then the critical orbit has no intersection with $\Omega_\lambda^{1/4} \cup \Omega_\lambda^{1/2}$.

We consider the location of the critical value $v_\lambda^+$; there are four possibilities:

**Case 1.** $v_\lambda^+ \in D_1$. In this case, the annulus $A_0(v_\lambda^+) = P_0(v_\lambda^+) \setminus P_1(v_\lambda^+)$ is non-degenerate with respect to the Yoccoz puzzle as induced by either of the graphs $G_\lambda(1/4), G_\lambda(1/2)$ and $G_\lambda(1/4, 1/2)$. It turns out that the critical annulus $A_1(c)$ is non-degenerate for all $c \in C_\lambda$. Thus, in this case, all the graphs $G_\lambda(1/4), G_\lambda(1/2), G_\lambda(1/4, 1/2)$ are admissible.

**Case 2.** $v_\lambda^+ \in D_2$. The annulus $A_0(v_\lambda^+) = P_0(v_\lambda^+) \setminus P_1(v_\lambda^+)$ is non-degenerate with respect to the Yoccoz puzzle induced by the graph $G_\lambda(1/4)$. Therefore, all critical annuli $A_1(c)$ are non-degenerate. Thus, the graph $G_\lambda(1/4)$ is admissible.

**Case 3.** $v_\lambda^+ \in D_3$. The annulus $A_0(v_\lambda^+) = P_0(v_\lambda^+) \setminus P_1(v_\lambda^+)$ is non-degenerate with respect to the Yoccoz puzzle induced by the graph $G_\lambda(1/4, 1/2)$. Therefore, all critical annuli $A_1(c)$ are non-degenerate, and the graph $G_\lambda(1/4, 1/2)$ is admissible.
Case 4. $v^\perp_\lambda \in D_4$. Based on an argument similar to that used above, we conclude that the graph $G_2(1/2)$ is admissible.

Moreover, for any $z \in (\Omega^{1/2}_\lambda(2, 2) \cup \Omega^{1/2}_\lambda(2, -2) \cap J(f_{\lambda}))$, the annulus $A_0(z)$ is non-degenerate with respect to the Yoccoz puzzle induced by the graph $G_2(1/2)$.

Because $G_2(1/2)$ is touchable, there exist an integer $p \geq 1$ and a critical point $c \in C_\lambda$ such that $f_{\lambda}^p(c) \in \Omega^{1/2}_\lambda$. Consider the itinerary of $f_{\lambda}^p(c)$, say $s_{\lambda}(f_{\lambda}^p(c)) = (s_0, s_1, s_2, \ldots)$. There are two possibilities:

Case 1. There is an integer $n \geq 0$ such that $(s_n, s_{n+1}) = (2, 2)$ or $(-2, -2)$. In this case, $f_{\lambda}^{n+p}(c) \in (\Omega^{1/2}_\lambda(2, 2) \cup \Omega^{1/2}_\lambda(2, -2)) \cap J(f_{\lambda})$; thus, the annulus $A_0(f_{\lambda}^{n+p}(c))$ is non-degenerate. It turns out that the critical annulus $A_{n+p}(c)$ is non-degenerate. Therefore, the graph $G_2(1/4)$ is admissible.

Case 2. For any integer $n \geq 0$, $(s_n, s_{n+1}) = (2, 2)$ or $(-2, -2)$. In this case, either $s_{\lambda}(f_{\lambda}^p(c)) = (2, -2, 2, -2, \ldots) = (2, -2)$ or $s_{\lambda}(f_{\lambda}^p(c)) = (-2, 2, -2, 2, \ldots) = (-2, 2)$. By Proposition 3.4, $f_{\lambda}^p(c)$ lies in a repelling cycle of period two.

3. The proof is similar to the proof of 2. In this case, the graph $G_2(1/2)$ is necessarily untouchable. First, note that the cut ray $\Omega^{5/12}_\lambda$ decomposes $\Omega^{1/2}_\lambda$ into four parts: $\Omega^{1/2}_\lambda(2, 2)$, $\Omega^{1/2}_\lambda(2, -2)$, $\Omega^{1/2}_\lambda(-2, 2)$ and $\Omega^{1/2}_\lambda(-2, -2)$, where

$$
\Omega^{1/2}_\lambda(\epsilon_0, \epsilon_1) = \{z \in \Omega^{1/2}_\lambda \setminus O_\lambda; \ s_{\lambda}(z) = (\epsilon_0, \epsilon_1, \pm 2, \pm 2, \ldots)\}, \ \epsilon_0, \epsilon_1 = \pm 2.
$$

Moreover, for any $z \in (\Omega^{1/2}_\lambda(2, 2) \cup \Omega^{1/2}_\lambda(-2, -2) \cap J(f_{\lambda}))$, the annulus $A_0(z)$ is non-degenerate with respect to the Yoccoz puzzle induced by the graph $G_2(1/2)$.

Because $G_2(1/4)$ is touchable, there exist an integer $p \geq 1$ and a critical point $c \in C_\lambda$ such that $f_{\lambda}^p(c) \in \Omega^{1/4}_\lambda$. Consider the itinerary of $f_{\lambda}^p(c)$, say $s_{\lambda}(f_{\lambda}^p(c)) = (s_0, s_1, s_2, \ldots)$. There are two possibilities:

Case 1. There is an integer $n \geq 0$ such that $(s_n, s_{n+1}) = (-1, 1)$ or $(1, -1)$. In this case, $f_{\lambda}^{n+p}(c) \in (\Omega^{1/4}_\lambda(1, -1) \cup \Omega^{1/4}_\lambda(-1, 1)) \cap J(f_{\lambda})$; thus, the annulus $A_0(f_{\lambda}^{n+p}(c))$ is non-degenerate. It turns out that the critical annulus $A_{n+p}(c)$ is non-degenerate. Therefore, the graph $G_2(1/2)$ is admissible.

Case 2. For any integer $n \geq 0$, $(s_n, s_{n+1}) = (1, 1)$ or $(-1, -1)$. In this case, either $s_{\lambda}(f_{\lambda}^p(c)) = (1, 1, \ldots) = (1)$ or $s_{\lambda}(f_{\lambda}^p(c)) = (-1, -1, \ldots) = (-1)$. By Proposition 3.4, $f_{\lambda}^p(c)$ is a repelling fixed point.
Fig. 8. Candidates for admissible graph when \( n = 4 \).

**Lemma 4.4.** When \( n = 4 \), if \( G_\lambda(1/3) \) is not touchable, then \( G_\lambda(1/3) \) is admissible; if \( G_\lambda(1/3) \) is touchable, then \( G_\lambda(2/3, 1) \) is admissible.

**Proof.** First, note that \( s(1/3) = (2, 2, \ldots) = (\overline{2}) \), \( s(2/3) = (-1, -1, \ldots) = (\overline{-1}) \) and \( s(1) = (-3, -3, \ldots) = (\overline{-3}) \). Thus, \( \Omega^{1/3}_\lambda \subset S_2 \cup S_{-2}, \Omega^{2/3}_\lambda \subset S_1 \cup S_{-1} \) and \( \Omega^{1}_\lambda \subset S_3 \cup S_{-3} \) (see Fig. 8). It is easy to verify

\[
\Omega^{1/3}_\lambda = \Omega^{1/12}_\lambda \cup \Omega^{5/24}_\lambda \cup \Omega^{1/3}_\lambda \cup \Omega^{11/24}_\lambda.
\]

If the graph \( G_\lambda(1/3) \) is not touchable, then the critical orbit has no intersection with \( \Omega^{1/3}_\lambda \). With respect to the Yoccoz puzzle induced by \( G_\lambda(1/3) \), the puzzle piece \( P_1(v_\lambda^{-1}) \) is a subset of the domain bounded by \( \omega^{5/24}_\lambda \) and \( \omega^{23/24}_\lambda \) together with the equipotential curves \( e(B_\lambda, 1/n) \) and \( e(T_\lambda, 1/n) \). Thus, the annulus \( A_0(v_\lambda^{-1}) \) is non-degenerate. It turns out that all critical annuli \( A_1(c) \) are non-degenerate. Therefore, the graph \( G_\lambda(1/3) \) is admissible.

If the graph \( G_\lambda(1/3) \) is touchable, then there exist an integer \( p \geqslant 1 \) and a critical point \( c \in C_\lambda \) such that \( f^p_\lambda(c) \in \Omega^{1/3}_\lambda \). Note that the preimage of \( \Omega^{2/3}_\lambda \) that lies in \( S_2 \cup S_{-2} \) is \( \Omega^{7/24}_\lambda \) and the preimage of \( \Omega^1_\lambda \) that lies in \( S_3 \cup S_{-3} \) is \( \Omega^{3/8}_\lambda \). In this case, with respect to the Yoccoz puzzle induced by the graph \( G_\lambda(2/3, 1) \), the puzzle piece \( P_1(f^p_\lambda(c)) \) is bounded by \( \Omega^{7/24}_\lambda \) and \( \Omega^{3/8}_\lambda \); thus, the annulus \( A_0(f^p_\lambda(c)) \) is non-degenerate. It follows that all critical annuli \( A_p(c) \) are non-degenerate, and the graph \( G_\lambda(2/3, 1) \) is admissible. \( \Box \)

In the following, we will consider the case when \( n \geqslant 5 \). Let

\[
\hat{\Theta} = \bigcap_{j \geqslant 0} \tau^{-j} \left( \bigcup_{2 \leq k \leq n-2} (\Theta_k \cup \Theta_{-k}) \right).
\]
be the set of all angles in \( \Theta \) whose orbits remain in \( \bigcup_{2 \leq k \leq n-2} (\Theta_k \cup \Theta_{-k}) \) under all iterations of \( \tau \), and let \( \hat{\Theta}_{per} \) be the set of all periodic angles in \( \hat{\Theta} \). Based on a similar argument as for Lemma 3.2, we can show that \( \hat{\Theta}_{per} \) is a dense subset of \( \hat{\Theta} \). By Lemma 3.1, one can check that the set \( \hat{\Theta}_{per} \) can be written as

\[
\hat{\Theta}_{per} = \bigcup_{p \geq 1} \{ \kappa(s) : s = (s_0, \ldots, s_{p-1}) \in \Sigma_0 \text{ and } s_0, \ldots, s_{p-1} \in \{ \pm 2, \ldots, \pm (n-2) \} \}
\]

and that any angle \( \theta \in \hat{\Theta}_{per} \) is of the form

\[
\theta = \frac{1}{2} \left( \frac{\chi(s_0)}{n} + \frac{|s_0|}{n(n^p-1)} + \frac{n^p}{n^p-1} \sum_{1 \leq k < p} \frac{|s_k|}{n^{k+1}} \right).
\]

Lemma 4.5. When \( n \geq 5 \), there are infinitely many periodic angles \( \theta \in \Theta \) such that the graph \( G_\lambda(\theta) \) is admissible.

Proof. We can choose an angle \( \theta \in \hat{\Theta}_{per} \) such that the critical orbit avoids the graph \( G_\lambda(\theta) \). (Note that there are infinitely many such choices of angle \( \theta \).) When \( n \geq 5 \), the set \( \bigcup_{j \geq 0} \Omega_{\lambda}^{\tau^j(\theta)} - \{0, \infty\} \) lies outside \( S_1 \cup S_0 \cup S_{-(n-1)} \) (see Fig. 9). Then, with respect to the Yoccoz puzzle induced by the graph \( G_\lambda(\theta) \), \( P_1(v_\lambda^+) \) is contained in the interior of \( S_1 \cup S_0 \cup S_{-(n-1)} \) and is a proper subset of \( P_0(v_\lambda^+) \). Because \( f_\lambda(P_2(c_0)) = P_1(v_\lambda^+) \) and \( f_\lambda(P_1(c_0)) = P_0(v_\lambda^+) \), we know that \( A_1(c_0) = P_1(c_0) \setminus P_2(c_0) \) is non-degenerate. Thus, the graph \( G_\lambda(\theta) \) is admissible. \( \square \)
In the remainder of this section, we prove an important property of the cut rays that are used to generate admissible graphs. Let

\[ \Theta_{ad} = \begin{cases} \{ \frac{1}{4}, \frac{1}{2} \}, & n = 3, \\ \{ \frac{1}{3}, \frac{2}{3}, 1 \}, & n = 4, \\ \Theta_{per}, & n \geq 5. \end{cases} \]

Note that for any admissible graph \( G_\lambda(\theta_1, \ldots, \theta_N) \) constructed by Lemma 4.3, Lemma 4.4 and Lemma 4.5, \( \{\theta_1, \ldots, \theta_N\} \subset \Theta_{ad} \). In the following, we will prove

**Proposition 4.2.** For any \( \theta \in \Theta_{ad} \), the intersection \( \Omega^\theta_\lambda \cap \partial B_\lambda \) consists of two points.

The proof is based on the following:

**Lemma 4.6.** Suppose \( \theta \in \Theta \), and \( \theta \) satisfies one of the following conditions:

1. There are two sequences, \( \{\theta^+_k\}_{k \geq 1}, \{\theta^-_k\}_{k \geq 1} \subset \Theta \) such that for all \( k \geq 1 \), \( \theta^-_k < \theta < \theta^+_k \) and \( J(\theta^+_k, \theta) = J(\theta^-_k, \theta) \to \infty \) as \( k \to \infty \).
2. There is a sequence \( \{\theta_k\}_{k \geq 1} \subset \Theta \) such that \( \theta_1 < \theta_2 < \theta_3 < \cdots \) or \( \theta_1 > \theta_2 > \theta_3 > \cdots \) and \( J(\theta_k, \theta) = k \) for any \( k \geq 1 \).

Then the intersection \( \Omega^\theta_\lambda \cap \partial B_\lambda \) consists of two points.

**Proof.** 1. Suppose \( \theta \) satisfies C1 and \( s(\theta) = (s_0, s_1, s_2, \ldots) \). By Proposition 3.6, the cut rays \( \Omega^\theta_\lambda\)

and \( \Omega^-_\lambda \) both intersect with \( \Omega^\theta_\lambda \) at \( 2J(\theta^+_-k, \theta) + 1 \) points; they hence decompose \( \Omega^\theta_\lambda \) into \( 2J(\theta^+_-k, \theta) + 1 \) parts:

\[ \Omega^\theta_\lambda(\epsilon_0, \epsilon_1, \ldots, \epsilon_{J(\theta^+_-k, \theta)}), \quad \epsilon_j = \pm s_j, \quad 0 \leq j \leq J(\theta^+_-k, \theta). \]

Here \( \Omega^\theta_\lambda(\epsilon_0, \epsilon_1, \ldots, \epsilon_p) := \{ z \in \Omega^\theta_\lambda \setminus \Omega_\lambda; \ s_\lambda(z) = (\epsilon_0, \epsilon_1, \ldots, \epsilon_p, \pm s_{p+1}, \pm s_{p+2}, \ldots) \} \).

Based on the structure of the cut rays (Proposition 3.2) and because the angle \( \theta \) satisfies condition C1, we conclude that of these \( 2J(\theta^+_-k, \theta) + 1 \) parts, only two intersect with \( B_\lambda \): \( \Omega^\theta_\lambda(s_0, s_1, \ldots, s_{J(\theta^+_-k, \theta)}) \) and \( \Omega^-_\lambda(-s_0, (-1)^n s_1, \ldots, (-1)^n s_{J(\theta^+_-k, \theta)}) \). We should remark that here we use two cut rays \( \Omega^\theta_\lambda, \Omega^-_\lambda \) with \( J(\theta^+_-k, \theta) = J(\theta^-_k, \theta) \) to separate the other segments of \( \Omega^\theta_\lambda \) from \( B_\lambda \) (see Fig. 10). Moreover, \( \Omega^\theta_\lambda \cap B_\lambda \subset \Omega^\theta_\lambda(s_0, s_1, \ldots, s_{J(\theta^+_-k, \theta)}) \cup \Omega^-_\lambda(-s_0, (-1)^n s_1, \ldots, (-1)^n s_{J(\theta^+_-k, \theta)}) \) for any \( k \geq 1 \). It turns out that

\[ \Omega^\theta_\lambda \cap B_\lambda \subset \bigcap_{k \geq 1} \left( \Omega^\theta_\lambda(s_0, s_1, \ldots, s_{J(\theta^+_-k, \theta)}) \cup \Omega^-_\lambda(-s_0, (-1)^n s_1, \ldots, (-1)^n s_{J(\theta^+_-k, \theta)}) \right) \]

\[ = \{ z \in \Omega^\theta_\lambda; \ s_\lambda(z) = (s_0, s_1, s_2, \ldots) \text{ or } (-s_0, (-1)^n s_1, (-1)^n s_2, \ldots) \} \cup R_\lambda(\theta + 1/2). \]
Fig. 10. Three cuts rays with angles $\theta^+_k > \theta > \theta^-_k$. In this figure, $J(\theta^+_k, \theta) = J(\theta^-_k, \theta) = 1$. Exactly two segments of $\Omega^\theta_\lambda$ intersect with $B_\lambda$: $\Omega^\theta_\lambda(s_0, s_1)$ and $\Omega^\theta_\lambda(-s_0, (-1)^n s_1)$.

Fig. 11. Cut rays with angles $\theta_1 < \theta_2 < \cdots < \theta$, $J(\theta, \theta_1) = 1$, $J(\theta, \theta_2) = 2$, ... Moreover, $B_\lambda$ has no intersection with the bounded components $W_1$ and $W_2$ of $\bar{C} \setminus (\Omega^\theta_1 \cup \Omega^\theta_2)$.

By Proposition 3.2, the intersection $\Omega^\theta_\lambda \cap \partial B_\lambda$ consists of two points. These two points are the landing points of the external rays $R_\lambda(\theta)$ and $R_\lambda(\theta + 1/2)$.

2. Now we suppose that $\theta$ satisfies C2 and $s(\theta) = (s_0, s_1, s_2, \ldots)$. We only prove the case when $n$ is odd. The argument applies equally well to the case when $n$ is even. Let $\{\theta_k\}_{k \geq 1} \subset \Theta$ be a sequence such that $\theta_1 < \theta_2 < \theta_3 < \cdots$ and $J(\theta_k, \theta) = k$ for any $k \geq 1$. The following facts are straightforward:

**Fact 1.** Let $z \in \Omega^\theta_\lambda$. If the itinerary $s_k(z)$ is of the form $(\epsilon_0, \ldots, \epsilon_k, s_{k+1}, s_{k+2}, \ldots)$ or $(\epsilon_0, \ldots, \epsilon_k, -s_{k+1}, -s_{k+2}, \ldots)$ for some $k \geq 0$, then $s_k(f_{\lambda}^{k+1}(z)) = \pm(s_{k+1}, s_{k+2}, \ldots) = s(\tau^{k+1}(\theta))$ or $s(\tau^{k+1}(\theta) + \frac{1}{2})$. By Proposition 3.1, $f_{\lambda}^{k+1}(z) \in R_\lambda(\tau^{k+1}(\theta)) \cup R_\lambda(\tau^{k+1}(\theta) + \frac{1}{2})$. Thus, $z$ lies in the closure of some external ray or radial ray $R_U(\theta_U)$ for $U \in \mathcal{P}$.

**Fact 2.** For any $k > 1$, $B_\lambda$ has no intersection with any bounded component of $\bar{C} \setminus \bigcup_{1 \leq j \leq k} \Omega^\theta_j$; see Fig. 11. (The proof is almost immediate from Proposition 3.1.)

**Fact 3.** The sections of $\Omega^\theta_\lambda$ that intersect with the unbounded component of $\bar{C} \setminus \bigcup_{1 \leq j \leq k} \Omega^\theta_j$ are as follows:

$$
\Omega^\theta_\lambda(s_0, \ldots, s_k), \quad \Omega^\theta_\lambda(-s_0, \ldots, -s_k),
$$

$$
\Omega^\theta_\lambda(s_0, \ldots, s_j, -s_{j+1}, \ldots, -s_k), \quad \Omega^\theta_\lambda(-s_0, \ldots, -s_j, s_{j+1}, \ldots, s_k), \quad 0 \leq j < k.
$$

Let $E_k$ be the collection of these sections.
Based on Facts 2 and 3, we have $B_λ ∩ Ω^λ_λ ⊂ \bigcup_{E ∈ E^λ_λ} E$ for any $k > 1$. It follows that $B_λ ∩ Ω^λ_λ ⊂ \bigcap_{k > 1, E ∈ E^λ_λ} E = \{ z ∈ Ω^λ_λ, s_k(z) \text{ is of the form } ±s(θ) \text{ or } ±(s_0, s_1, \ldots, s_k, -s_{k+1}, -s_{k+2}, \ldots) \text{ for some } k ≥ 0 \}$. 

By Fact 1, for any $z ∈ B_λ ∩ Ω^λ_λ$, either $z ∈ R_λ(θ) ∪ R_λ(θ + 1/2)$ or there exist $U ∈ P \setminus \{ B_λ \}$ and an angle $θ_U$ such that $z ∈ R_λ(θ_U)$. In the following, we show that the latter is impossible. In fact, if $z ∈ B_λ ∩ Ω^λ_λ ∩ R_λ(θ_U)$, then $z ∈ \partial B_λ ∩ \partial U$. Let $p ≥ 0$ be the first integer such that $f^p_λ(U) = T_λ$.

After $p$ iterations, we see that $f^p_λ(z) ∈ \partial B_λ ∩ \partial T_λ$ and $f^p_λ(z)$ is the landing point of the radial ray $R_λ(θ_Tλ) = f^p_λ(R_λ(θ_U))$. On the other hand, $f^{p+1}_λ(z)$ is the landing point of the external ray $R_λ(θ_λ) = f^{p+1}_λ(R_λ(θ_U))$. Therefore, $f^p_λ(z)$ is also a landing point of some external ray $R_λ(β)$, $β ∈ τ^{-1}(θ_λ)$. Because both $R_λ(θ_Tλ)$ and $R_λ(β)$ land at $f^p_λ(z)$, and $f_λ(R_λ(θ_Tλ)) = f_λ(R_λ(β)) = R_λ(θ_λ)$, $f^p_λ(z)$ is necessarily a critical point in $C_λ$.

However, the result that $f^p_λ(z) ∈ f^p_λ(Ω^λ_λ) ∩ C_λ$ leads to a contradiction because for any $α ∈ Θ$, the cut ray $Ω^α_λ$ avoids the critical set $C_λ$.

Now, we are in the situation $B_λ ∩ Ω^λ_λ ⊂ R_λ(θ) ∪ R_λ(θ + 1/2)$, and the conclusion follows.

**Proof of Proposition 4.2.** It suffices to verify that for any $θ ∈ Θ_{ad}$, $θ$ satisfies either C1 or C2 by Lemma 4.6.

When $n = 3$, $s(1/4) = (\overline{1}, \underline{1})$, $s(1/2) = (\overline{2})$. Define two sequences of angles $\{α_k\}_{k ≥ 1}, \{β_k\}_{k ≥ 1} ⊂ Θ$ such that

$$s(α_1) = (1, −2, −1, 1, −1, 1, \ldots), \quad s(β_1) = (2, 1, −1, 2, 2, 2, \ldots),$$

$$s(α_2) = (1, −1, 2, −1, 1, −1, 1, \ldots), \quad s(β_2) = (2, 2, 1, −1, 2, 2, \ldots),$$

$$s(α_3) = (1, −1, 1, −2, −1, 1, \ldots), \quad s(β_3) = (2, 2, 2, 1, −1, 2, \ldots),$$

$$\ldots$$

Then, $α_1 > α_2 > α_3 > \cdots$ and $J(α_k, 1/4) = k$ for any $k ≥ 1$; $β_1 < β_2 < β_3 < \cdots$ and $J(β_k, 1/2) = k$. Thus, both $1/4$ and $1/2$ satisfy condition C2.

When $n = 4$, $s(1/3) = (\overline{2}), s(2/3) = (\overline{1}), s(1) = (\overline{3})$. Define three sequences of angles $\{α_k\}_{k ≥ 1}, \{β_k\}_{k ≥ 1}, \{γ_k\}_{k ≥ 1} ⊂ Θ$ such that

$$s(α_1) = (2, 1, −2, 2, 2, \ldots), \quad s(β_1) = (−1, −3, −1, −1, 1, \ldots),$$

$$s(γ_1) = (−3, −1, −3, −3, \ldots), \quad s(α_2) = (2, 2, 1, −2, 2, \ldots),$$

$$s(β_2) = (−1, −1, −3, 1, −1, 3, \ldots), \quad s(γ_2) = (−3, −3, −1, −3, \ldots),$$

$$s(α_3) = (2, 2, 2, 1, −2, \ldots), \quad s(β_3) = (−1, −1, −1, −3, \ldots),$$

$$s(γ_3) = (−3, −3, −3, −1, \ldots), \quad \ldots$$

Then $α_1 < α_2 < α_3 < \cdots$ and $J(α_k, 1/3) = k; β_1 > β_2 > β_3 > \cdots$ and $J(β_k, 2/3) = k; γ_1 < γ_2 < γ_3 < \cdots$ and $J(γ_k, 1) = k$. Thus, $1/3, 2/3, 1$ all satisfy condition C2.

When $n ≥ 5$, we can prove that for any $θ ∈ Θ_{per}$, $θ$ satisfies condition C1. (In fact, this is true for all $θ ∈ Θ_0$.) The proof is as follows. Suppose $s(θ) = (s_0, s_1, s_2, \ldots)$. For any $k ≥ 1$, we choose $s^+_k, s^-_k ∈ \{±1, ±(n − 1)\}$ and $s^+_k, s^-_k \in I \setminus \{0, n\}$ such that
is a Jordan curve. Denote these two boundary curves by $\gamma_{\lambda,m}(\theta)$.

(2) $(s_0, \ldots, s_{k-1}, s_k^+, s_{k+1}^-, s_{k+2}, s_{k+3}, \ldots), (s_0, \ldots, s_{k-1}, s_k^-, s_{k+1}^+, s_{k+2}, s_{k+3}, \ldots) \in \Sigma_0$. Let

\[
\begin{align*}
\theta_k &= \kappa ((s_0, \ldots, s_{k-1}, s_k^+, s_{k+1}^+, s_{k+2}, s_{k+3}, \ldots)), \\
\theta_k^- &= \kappa ((s_0, \ldots, s_{k-1}, s_k^-, s_{k+1}^-, s_{k+2}, s_{k+3}, \ldots)).
\end{align*}
\]

It is easy to check that $\theta_k^- < \theta < \theta_k^+$ and $J(\theta_k^+, \theta) = J(\theta_k^-, \theta) = k \rightarrow \infty$ as $k \rightarrow \infty$. □

4.3. Modified puzzle piece

Consistent with the idea of the ‘thickened puzzle piece’ used in [21] to study the quadratic Julia set, we construct the ‘modified puzzle piece’ for McMullen maps. The ‘modified puzzle piece’ can be used to study the local connectivity of $J(f_{\lambda})$ in the non-renormalizable case (see Lemma 7.1). It is also used to define renormalizations (see Remark 5.1).

Given an angle $\theta \in \Theta$ with itinerary $s(\theta) = (s_0, s_1, s_2, \ldots)$, the cut ray $\Omega_\lambda^\theta$ is identified as

\[
\Omega_\lambda^\theta = \bigcap_{k \geq 0} f_{\lambda,k}^{-1}(S_{s_k} \cup S_{-s_k});
\]

it can be approximated by the sequence of compact sets $\{\Omega_{\lambda,m}^\theta = \bigcap_{0 \leq k \leq m} f_{\lambda,k}^{-1}(S_{s_k} \cup S_{-s_k}) \}_{m \geq 0}$ in Hausdorff topology. Now, we consider the set $\hat{\mathbb{C}} \setminus \Omega_\lambda^\theta$. The open set $\hat{\mathbb{C}} \setminus \Omega_{\lambda,m}^\theta$ consists of two connected components, and the boundary of each component is a Jordan curve. Denote these two boundary curves by $\gamma_{1,m}^\lambda(\theta)$ and $\gamma_{2,m}^\lambda(\theta)$. Let $V_m(\theta) = \gamma_{1,m}^\lambda(\theta) \cap \gamma_{2,m}^\lambda(\theta)$ be the intersection of these two curves. It is obvious that $V_m(\theta)$ consists of finitely many points and that $V_m(\theta) = \Omega_\lambda^\theta \cap (\bigcup_{0 \leq k \leq m} f_{\lambda,k}^{-1}(\infty))$. For any $v \in V_m(\theta)$, let $D(v)$ be the connected component of $\{z \in A_\lambda; G_\lambda(z) > 1\}$ that contains $v$. Obviously, $D(v)$ is a disk.

In the following, we construct the ‘modified puzzle piece’. For the Yoccoz puzzle induced by the graph $G_\lambda(\theta_1, \ldots, \theta_N)$, recall that each puzzle piece $P_0$ of depth zero is contained in a unique component of $\hat{\mathbb{C}} \setminus g_\lambda(\theta_1, \ldots, \theta_N)$. This component is simply connected and is denoted by $Q_0$. We may choose a $m$ large enough so that for any $\alpha, \beta \in \{\tau^k(\theta); 1 \leq j \leq N, k \geq 0\}$ with $\Omega_\alpha^\alpha \neq \Omega_\lambda^\beta$,

\[
\Omega_{\lambda,m}^\alpha \cap \Omega_{\lambda,m}^\beta = \Omega_{\lambda}^\alpha \cap \Omega_{\lambda}^\beta.
\]

The disk $Q_0$ is bounded by some collection of cut rays, say $\{\Omega_{\lambda}^\alpha; \alpha \in \Lambda(Q_0)\}$, where $\Lambda(Q_0)$ is an index set induced by $Q_0$. For any $\alpha \in \Lambda(Q_0)$, choose a curve $\gamma(\alpha) \in \{\gamma_{1,m}^\lambda(\alpha), \gamma_{2,m}^\lambda(\alpha)\}$ such that $\gamma(\alpha) \cap Q_0 = \emptyset$. Let $Q_0$ be the connected component of $\hat{\mathbb{C}} \setminus \bigcup_{\alpha \in \Lambda(Q_0)} \gamma(\alpha)$ that contains $Q_0$, and let $V(Q_0) = \bigcup_{\alpha \in \Lambda(Q_0)} (V_m(\alpha) \cap \partial Q_0)$. The modified puzzle piece $\hat{P}_0$ of $P_0$ is defined as follows:

\[
\hat{P}_0 = \hat{Q}_0 - \bigcup_{v \in V(Q_0)} D(v).
\]

Roughly speaking, we can obtain $\hat{P}_0$ from $Q_0$ by thickening $Q_0$ near $\partial Q_0 \setminus V(Q_0)$ and truncating $Q_0$ near the points in $V(Q_0)$. The puzzle piece $P_0$ is not contained in $\hat{P}_0$; for this reason, we call $\hat{P}_0$ the ‘modified puzzle piece’ of $P_0$ rather than the ‘thickened puzzle piece’ of $P_0$.

Modified puzzle pieces of greater depth can be constructed by the usual inductive procedure; if $\hat{P}_d^{(j)}$ is the modified puzzle piece of depth $d$, then each component of $f_{\lambda}^{-1}(\hat{P}_d^{(j)})$ is the modified puzzle piece of depth $d + 1$ (see Fig. 12).
The advantage of these modified puzzle pieces is as follows: if a puzzle piece $P_d^{(j)}$ contains $P_d^{(k)}$, then the modified puzzle piece $\hat{P}_d^{(j)}$ contains $\hat{P}_d^{(k)}$, which can be easily proved by induction. In other words, this construction replaces all of our annuli with non-degenerate annuli.

For $z \in \mathbb{C} \setminus (A_\lambda \cup J_0)$, let $\hat{P}_d(z)$ be the modified puzzle piece of $P_d(z)$. We will only make use of modified puzzle pieces that are small enough to satisfy the following additional restriction: if $\hat{P}_d(z)$ contains a critical point, then $P_d(z)$ must already contain this critical point. Note that if the graph $G_\lambda(\theta_1, \ldots, \theta_N)$ is not touchable, then this requirement is easily satisfied for any bounded value of depth $d$ by choosing $m$ large enough, which will suffice for the applications.

Based on construction, the puzzle piece $P_d(z)$ and the modified puzzle piece $\hat{P}_d(z)$ satisfy the following relation:

$$P_d(z) \subset \hat{P}_d(z) \cup A_\lambda, \quad \bigcap_{d \geq 0} P_d(z) \subset \bigcap_{d \geq 0} \hat{P}_d(z).$$

The modified puzzle pieces also satisfy the following symmetry properties: For any $z \in \mathbb{C} \setminus (A_\lambda \cup J_0)$,

$$-\hat{P}_0(z) = \hat{P}_0(-z), \quad \omega \hat{P}_d(z) = \hat{P}_d(\omega z), \quad \omega^{2n} = 1, \quad d \geq 1.$$

### 4.4. Tableaux

In this section, we present some basic information on tableaux, based on Milnor’s Lecture [21]. Applications of tableau analysis combined with puzzle techniques can be found in [2, 14, 21, 22, 25–27, 32] and many other papers.

Recall that $J_0$ is the set of all points on $J(f_\lambda)$ with orbits that eventually touch the graph $G_\lambda(\theta_1, \ldots, \theta_N)$. For $x \in \mathbb{C} \setminus (A_\lambda \cup J_0)$, the tableau $T(x)$ is defined as the two-dimensional array $(P_{d,l}(x))_{d,l \geq 0}$, where $P_{d,l}(x) = f_\lambda^l(P_{d+l}(x)) = P_d(f_\lambda^l(x))$. The position $(d, l)$ is called critical if $P_{d,l}(x)$ contains a critical point in $C_\lambda$. If $P_{d,l}(x)$ contains a critical point $c \in C_\lambda$, the position $(d, l)$ is called a $c$-position.

For any $x \in \mathbb{C} \setminus (A_\lambda \cup J_0)$, the tableau $T(x)$ satisfies the following three rules:
(T1) For each column \( l \geq 0 \), either the position \((d, l)\) is critical for all \( d \geq 0 \) or there is a unique integer \( d_0 \geq 0 \) such that the position \((d, l)\) is critical for all \( d < d_0 \) and not critical for \( d \geq d_0 \).

(T2) If \( P_{d,l}(x) = P_d(y) \) for some \( y \in \overline{C} \setminus (A_\lambda \cup J_0) \), then \( P_{l,l+j}(x) = P_{l,j}(y) \) for \( 0 \leq i + j < d \).

(T3) Let \( T(c) \) be a tableau with \( c \in C_\lambda \). Assume

(a) \( P_{d+1-l,l}(c) = P_{d+1-l}(c') \) for some critical point \( c' \in C_\lambda \), \( 0 \leq l < d \), and \( P_{d-1,i}(c) \) contains no critical points for \( 0 < i < l \).

(b) \( P_{d,m}(x) = P_d(c) \) and \( P_{d+1,m}(x) \neq P_{d+1}(c) \) for some \( m > 0 \).

Then, \( P_{d+1-l,m+1}(x) \neq P_{d+1-l}(c') \).

**Remark 4.2.** The tableau rule (T3) is based on the fact that every puzzle piece of depth \( d \geq 1 \) contains at most one critical point in \( C_\lambda \).

**Definition 4.2.** 1. The tableau \( T(x) \) is non-critical if there is an integer \( d_0 \geq 0 \) such that \((d_0, j)\) is not critical for all \( j > 0 \). Otherwise, \( T(x) \) is called critical. (One should be careful to note that \( T(x) \) is critical does not mean \( x \in C_\lambda \).

2. The tableau \( T(x) \) is called pre-periodic if there exist two integers \( l \geq 0 \) and \( p \geq 1 \) such that \( P_{d,l+p}(x) = P_{d,l}(x) \) for all \( d \geq 0 \). In this case, if \( l = 0 \), \( T(x) \) is called periodic, and the smallest integer \( p \geq 1 \) is called the period of \( T(x) \).

3. Let \( Row_c(d) \) be the \( d \)-th row of the tableau \( T(c) \) with \( c \in C_\lambda \). We say \( Row_c(d+l) \) with \( l > 0 \) is a child of \( Row_c(d) \) if there is a critical point \( c' \in C_\lambda \) such that \( A_d(f^1_{\lambda}(c)) = A_d(c') \) and \( f^1_{\lambda} : A_{d+l}(c) \rightarrow A_d(c') \) is a degree two covering map.

4. Let \( c \in C_\lambda \). For \( d \geq 1 \), we say \( Row_c(d) \) is excellent if \( A_d(f^1_{\lambda}(c)) \) is not semi-critical for all \( l \geq 0 \).

**Remark 4.3.** By Lemma 4.1 and the fact that \( f^k_{\lambda}(\omega z) = \pm f^k_{\lambda}(z) \) for \( k \geq 1 \), \( \omega^{2n} = 1 \), we have

1. If \( (d, l) \) is a critical position for some tableau \( T(c) \) with \( c \in C_\lambda \), then \( (d, l) \) is a critical position of \( T(c') \) for every \( c' \in C_\lambda \).

2. If there is \( c \in C_\lambda \) such that the tableau \( T(c) \) is critical, non-critical or pre-periodic, then for every \( c' \in C_\lambda \), the tableau \( T(c') \) is critical, non-critical or pre-periodic, respectively.

3. If \( Row_c(d) \) is excellent or has a child \( Row_c(d+l) \) for some critical point \( c \in C_\lambda \), then for every \( c' \in C_\lambda \), \( Row_{c'}(d+l) \) is excellent or has a child \( Row_{c'}(d+l) \), respectively.

**Lemma 4.7.** Suppose some tableau \( T(c) \) with \( c \in C_\lambda \) is critical but not pre-periodic, then

1. For every \( d \geq 1 \), \( Row_c(d) \) has at least one child.
2. If \( Row_c(d) \) is excellent, then \( Row_c(d) \) has at least two children.
3. If \( Row_c(d) \) is excellent and \( Row_c(d+l) \) is its child, then \( Row_c(d+l) \) is also excellent.
4. If \( Row_c(d) \) has only one child, say \( Row_c(d+l) \), then \( Row_c(d+l) \) is excellent.

**Proof.** 1. By hypothesis, for every \( d \geq 1 \), we can find a smallest integer \( l > 0 \) such that the annulus \( A_d(f^1_{\lambda}(c)) \) is \( c' \)-critical for some \( c' \in C_\lambda \). The map \( f^1_{\lambda} : A_{d+l}(c) \rightarrow A_d(c') \) is a degree two covering map, which implies that \( Row_c(d+l) \) is a child of \( Row_c(d) \).

2. Following 1, there exists \( d' > d \) such that the annulus \( A_{d'}(f^1_{\lambda}(c)) \) is \( c' \)-semi-critical. Because \( Row_c(d) \) is excellent, by tableau rule (T3), \( A_{d'-1}(f_{\lambda}^{1+t}(c)) \) is either off-critical or semi-critical for \( 0 < t \leq d' - d \). In particular, \( A_d(f^{1+d'-d}_{\lambda}(c)) \) is off-critical. Hence, we can find a
smallest integer \( l' > l + d' - d \) such that the annulus \( A_d(f^l_\lambda(c)) \) is critical; therefore, \( \text{Row}_c(d + l') \) is another child of \( \text{Row}_c(d) \).

3. If \( \text{Row}_c(d + l) \) is not excellent, then there is a column \( l' \geq l \) such that \( A_{d+l}(f^l_\lambda(c)) \) is semi-critical. By tableau rule (T3), \( A_d(f^{l+l'}_\lambda(c)) \) is also semi-critical, which contradicts the fact that \( \text{Row}_c(d) \) is excellent.

4. If \( \text{Row}_c(d + l) \) is not excellent, then as in (3), \( A_d(f^{l+l'}_\lambda(c)) \) is semi-critical for some \( l' \geq l \).

\( \tilde{c} \) is another child of Row \( c(d) \) such that the annulus \( \text{Ad}(f^l_\lambda(c)) \) is critical; therefore, Row \( c(d) \) is excellent.

Suppose \( l' > l \) is the smallest integer. We can find a smallest integer \( t > l' + l \) such that \( A_d(f^l_\lambda(c)) \) is \( c' \)-critical for some \( c' \in C_\lambda \). Then \( \text{Row}_c(d + t) \) is also a child of \( \text{Row}_c(d) \), which is a contradiction. \( \square \)

**Lemma 4.8.** Suppose some tableau \( T(c) \) with \( c \in C_\lambda \) is critical and pre-periodic.

1. If \( n \) is odd, then there exist exactly two critical points \( \pm c' \in C_\lambda \) such that \( T(c') \) and \( T(-c') \) are periodic.

2. If \( n \) is even, then there is a unique critical point \( \tilde{c} \in C_\lambda \) such that \( T(\tilde{c}) \) is periodic.

**Proof.** Because \( T(c) \) is critical and pre-periodic, there exist a smallest integer \( p \geq 1 \) and a unique critical point \( c' \in C_\lambda \) such that \( (d, p) \) is a \( c' \)-position for all \( d \geq 0 \).

1. If \( n \) is odd, there are two possibilities: either \( f_\lambda(c) = f_\lambda(c') \) or \( f_\lambda(c) + f_\lambda(c') = 0 \).

If \( f_\lambda(c) = f_\lambda(c') \), then both \( T(c') \) and \( T(-c') \) are periodic with period \( p \). In this case, there is an integer \( d_0 \geq 0 \) such that for any \( d \geq d_0 \), \( 0 < l < p \), the position \( (d, l) \) is not critical. It is easy to check that for any \( \tilde{c} \in C_\lambda \setminus \{\pm c'\} \), the tableau \( T(\tilde{c}) \) is strictly pre-periodic. In particular, if \( p = 1 \), then \( P_d(c') = P_d(f_\lambda(c')) \) for all \( d \geq 0 \). This means that for any \( d \geq 0 \), \( c' \) and \( f_\lambda(c') \) lie in the same puzzle piece of depth \( d \). Thus, we conclude \( \{\pm c'\} = \{c_0, c_n\} \).

If \( f_\lambda(c) + f_\lambda(c') = 0 \), then both \( T(c') \) and \( T(-c') \) are periodic with period \( 2p \). Consider the tableau \( T(c') \); there is an integer \( d_0 \geq 0 \) such that for any \( d \geq d_0 \), \( 0 < l < p \), the position \( (d, l) \) is not critical and for any \( d \geq 0 \) the position \( (d, p) \) is \( (-c') \)-critical. It is easy to confirm that for any \( \tilde{c} \in C_\lambda \setminus \{\pm c'\} \), the tableau \( T(\tilde{c}) \) is strictly pre-periodic. In particular, if \( p = 1 \), then \( P_d(-c') = P_d(f_\lambda(c')) \) for all \( d \geq 0 \). Therefore, for any \( d \geq 0 \), \( -c' \) and \( f_\lambda(c') \) lie in the same puzzle piece of depth \( d \). Thus, we conclude \( \{\pm c'\} = \{c_1, c_{n+1}\} \).

2. \( n \) is even. In this case, based on the fact that \( f^k_\lambda(v^-_\lambda) = f^k_\lambda(v^+_\lambda) \) for all \( k \geq 1 \), we conclude the tableau \( T(f_\lambda(c')) \) is periodic. When a period \( p \) and the tableau \( T(-f_\lambda(c')) \) is strictly pre-periodic. There is thus a unique critical point \( \tilde{c} \in f^{-1}_\lambda(f_\lambda(c')) \) such that \( T(\tilde{c}) \) is periodic. For this tableau, there is an integer \( d_0 \geq 0 \) such that for any \( d \geq d_0 \), \( 0 < l < p \), the position \( (d, l) \) is not critical. It is easy to check that for any \( c'' \in C_\lambda \setminus \{\tilde{c}\} \), the tableau \( T(c'') \) is strictly pre-periodic. In particular, if \( p = 1 \) and \( T(v^+_\lambda) \) is periodic, then \( \tilde{c} = c_0 \); if \( p = 1 \) and \( T(v^-_\lambda) \) is periodic, then \( \tilde{c} = c_{n+1} \). \( \square \)

## 5. Renormalizations

In this section, we discuss the renormalization of McMullen maps with respect to the puzzle piece.

**Definition 5.1.** If there exist a critical point \( c \) of \( f_\lambda \), an integer \( p \geq 1 \) and two disks \( U \) and \( V \) containing \( c \) such that

\[
e_{f^p_\lambda} : U \to V
\]
is a quadratic-like map whose Julia set is connected (here $\epsilon \in \{\pm 1\}$ is a symbol), then we say $f_\lambda$ is $p$-renormalizable at $c$ if $\epsilon = 1$ and $f_\lambda$ is $p$-\(\ast\)-renormalizable at $c$ if $\epsilon = -1$. In the former case, the triple $(f_\lambda^p, U, V)$ is called a $p$-renormalization of $f_\lambda$ at $c$. In the latter case, the triple $(-f_\lambda^p, U, V)$ is called a $p$-\(\ast\)-renormalization of $f_\lambda$ at $c$.

In the following, we use $K_c = \{z \in U; (\epsilon f_\lambda^p)^k(z) \in U, \forall k \geq 0\} = \bigcap_{k \geq 0}(\epsilon f_\lambda^p)^{-k}(U)$ to denote the small filled Julia set of the ($\ast$-)renormalization $(\epsilon f_\lambda^p, U, V)$. By the straightening theorem of Douady and Hubbard [11], if $(\epsilon f_\lambda^p, U, V)$ is a $p$-\(\ast\)-renormalization of $f_\lambda$, then $\epsilon f_\lambda^p$ is conjugated by a quasi-conformal map $\sigma$ to a unique quadratic polynomial $p_\mu(z) = z^2 + \mu$ in a neighborhood of the filled Julia set $K_c$. Let $\beta$ be the $\beta$-fixed point (i.e., the landing point of the zero external ray) of $p_\mu$ and $\beta'$ be the other preimage of $\beta$. We call $\beta_c = \sigma^{-1}(\beta)$ the $\beta$-fixed point of the renormalization $(\epsilon f_\lambda^p, U, V)$. The other preimage of $\beta_c$ under the renormalization is $\beta'_c = \sigma^{-1}(\beta')$.

In this section, we always assume that the graph $G_\lambda(\theta_1, \ldots, \theta_N)$ is admissible.

### 5.1. From tableau to renormalizations

**Lemma 5.1.** Suppose some tableau $T(c)$ with $c \in C_\lambda$ is pre-periodic.

1. If $T(c)$ is non-critical, then $f_\lambda$ is critically finite.
2. If $T(c)$ is critical, then $f_\lambda$ is either renormalizable or \(\ast\)-renormalizable.

**Proof.** Because $T(c)$ is pre-periodic, there exist two integers $l \geq 0$ and $p \geq 1$ such that $P_d(f_\lambda^{l+p}(c)) = P_d,f_\lambda^p(c) = P_d,f_\lambda^l(c) = P_d(f_\lambda^l(c))$ for all $d \geq 0$.

1. $T(c)$ is non-critical. In this case, the tableaux $T(f_\lambda^l(c))$ and $T(f_\lambda^{l+p}(c))$ are also non-critical. Based on Lemma 7.1, $\{f_\lambda^{l+p}(c)\} = \bigcap_{d \geq 0}P_d(f_\lambda^{l+p}(c)) = \bigcap_{d \geq 0}P_d(f_\lambda^l(c)) = \{f_\lambda^l(c)\}$. Therefore, $f_\lambda^{l+p}(c) = f_\lambda^l(c)$, and $f_\lambda$ is critically finite.

2. $T(c)$ is critical. If $n$ is odd, then based on Lemma 4.8, there are exactly two critical points $\pm c' \in C_\lambda$ such that $T(c')$ and $T(-c')$ are periodic. Suppose the period is $p$, and consider the tableau $T(c')$. There are two possibilities:

**Case 1.** There is an integer $d_0 \geq 0$ such that for any $d \geq d_0$, $0 < l < p$, the position $(d, l)$ is not critical. Then, $f_\lambda^p : P_{d_0} -\text{p}(c') \rightarrow P_{d_0}(c')$ is a quadratic-like map and $\{f_\lambda^{k}(c'); k \geq 0\} \subset P_{d_0+p}(c')$. Thus, $(f_\lambda^p, P_{d_0+p}(c'), P_{d_0}(c'))$ is a $p$-renormalization of $f_\lambda$ at $c'$. Because $f_\lambda$ is an odd function, $(f_\lambda^p, P_{d_0+p}(c'), P_{d_0}(c'))$ is a $p$-renormalization of $f_\lambda$ at $-c'$.

**Case 2.** $p$ is even and there is an integer $d_0 \geq 0$ such that for any $d \geq d_0$, $0 < l < p/2$, the position $(d, l)$ is not critical, and for any $d \geq 0$, the position $(d, p/2)$ is $(-c')$-critical. Then, $f_\lambda^{p/2} : P_{d_0+p/2}(c') \rightarrow P_{d_0}(c')$ is a quadratic-like map with $\{-1\}^{k}f_\lambda^{k}(c'); k \geq 0\} \subset P_{d_0+p/2}(c')$. Thus, $(-f_\lambda^{p/2}, P_{d_0+p/2}(c'), P_{d_0}(c'))$ is a $p/2$-\(\ast\)-renormalization of $f_\lambda$ at $c'$. It turns out that $(-f_\lambda^{p/2}, P_{d_0+p/2}(c'), P_{d_0}(c'))$ is a $p/2$-\(\ast\)-renormalization of $f_\lambda$ at $-c'$.

If $n$ is even, then based on Lemma 4.8, there is a unique critical point $\tilde{c} \in C_\lambda$ such that $T(\tilde{c})$ is periodic. Suppose the period is $p$; there is then an integer $d_0 \geq 0$ such that for any $d \geq d_0$, $0 < l < p$, the position $(d, l)$ is not critical. Then, $f_\lambda^p : P_{d_0+p}(\tilde{c}) \rightarrow P_{d_0}(\tilde{c})$ is a quadratic-like
Lemma 5.1. The map \( \{ f^k_\lambda(\tilde{c}); \ k \geq 0 \} \subset P_{d_0+p}(\tilde{c}) \). Thus, \( (f^P_\lambda, P_{d_0+p}(\tilde{c}), P_{d_0}(\tilde{c})) \) is a \( p \)-renormalization of \( f_\lambda \) at \( \tilde{c} \). Because \( f_\lambda \) is an even function, \( (-f^P_\lambda, P_{d_0+p}(-\tilde{c}), P_{d_0}(-\tilde{c})) \) is a \( p \)-*-renormalization of \( f_\lambda \) at \( -\tilde{c} \). \( \Box \)

Remark 5.1. Lemma 5.1 also holds when the graph \( G_\lambda(\theta_1, \ldots, \theta_N) \) is not touchable. Indeed, in this case, we can use modified puzzle pieces to define renormalizations.

Proposition 5.1. Suppose \( f_\lambda \) has a non-repelling cycle in \( \mathbb{C} \); then \( f_\lambda \) is either renormalizable or \( *\)-renormalizable. In this situation, there are three possibilities:

1. If \( f_\lambda \) is renormalizable and \( n \) is odd, then \( f_\lambda \) has exactly two non-repelling cycles in \( \mathbb{C} \).
2. If \( f_\lambda \) is \( *\)-renormalizable and \( n \) is odd, then \( f_\lambda \) has exactly one non-repelling cycle in \( \mathbb{C} \).
3. If \( f_\lambda \) is renormalizable and \( n \) is even, then \( f_\lambda \) has exactly one non-repelling cycle in \( \mathbb{C} \).

Proof. Let \( C = \{ z_0, f_\lambda(z_0), \ldots, f^q_\lambda(z_0) = z_0 \} \) be the non-repelling cycle of \( f_\lambda \) in \( \mathbb{C} \). By Proposition 4.1, we can find an admissible graph \( G_\lambda(\theta_1, \ldots, \theta_N) \). By Proposition 3.4, the cycle \( C \) avoids the graph \( G_\lambda(\theta_1, \ldots, \theta_N) \). Thus, for any \( z \in \mathbb{C} \) and any integer \( d \geq 0 \), the puzzle piece \( P_d(z) \) is well defined.

We claim that there exist \( z \in \mathbb{C} \) and a critical point \( c \in C_\lambda \) such that \( P_d(z) = P_d(c) \) for all \( d \geq 0 \). Otherwise, the tableau \( T(z) \) is non-critical for any \( z \in \mathbb{C} \). It follows that there is an integer \( d_0 \geq 0 \) such that the map \( f^q_\lambda: P_{d_0+q}(z_0) \to P_{d_0}(z_0) \) is conformal. Based on the Schwarz Lemma, \( |f^q_\lambda(z_0)| > 1 \), which is a contradiction.

In this way, we can find a critical point \( c \in C_\lambda \) with tableau \( T(c) \) that is periodic. Based on Lemma 5.1, \( f_\lambda \) is either renormalizable or \( *\)-renormalizable.

To continue, suppose the period of \( T(c) \) is \( p \), which is necessarily a divisor of \( q \). Based on Lemma 5.1, there are three possibilities:

P1. \( n \) is odd and \( (f^P_\lambda, P_{d_0+p}(c), P_{d_0}(c)) \) is a \( p \)-renormalization of \( f_\lambda \) at \( c \). In this case, \( (f^P_\lambda, P_{d_0+p}(c), P_{d_0}(c)) \) is quasi-conformally conjugate to a polynomial \( z \mapsto z^2 + \mu \). Because a quadratic polynomial has at most one non-repelling cycle (see [3] or [28]), it turns out that \( C \) is the only non-repelling cycle contained in \( \bigcup_{0 \leq j < p} f^j_\lambda(K_c) \). On the other hand, \( -C \) is the only non-repelling cycle contained in \( \bigcup_{0 \leq j < p} f^j_\lambda(-K_c) \). Because there are exactly two critical points whose tableaux are periodic in this case and \( (\bigcup_{0 \leq j < p} f^j_\lambda(K_c)) \cap (\bigcup_{0 \leq j < p} f^j_\lambda(-K_c)) = \emptyset \), we conclude that \( f_\lambda \) has exactly two non-repelling cycles in \( \mathbb{C} \).

P2. \( n \) is odd and \( (-f^P_\lambda, P_{d_0+p/2}(c), P_{d_0}(c)) \) is a \( p/2 \)-*-reorganization of \( f_\lambda \) at \( c \). In this case, the cycle \( C \) meets both \( K_c \) and \( -K_c \). By a similar argument as above, one sees that \( C \) is the only non-repelling cycle contained in \( \bigcup_{0 \leq j < p} f^j_\lambda(K_c) \). Because the cycle \( -C \) is also contained in \( \bigcup_{0 \leq j < p} f^j_\lambda(K_c) \), it turns out that \( C = -C \).

P3. \( n \) is even and \( (f^P_\lambda, P_{d_0+p}(c), P_{d_0}(c)) \) is a \( p \)-renormalization of \( f_\lambda \) at \( c \). In this case, \( c \) is the only critical point whose tableau \( T(c) \) is periodic. Based on a similar argument as made above, we show that \( C \) is the only non-repelling cycle in \( \mathbb{C} \). \( \Box \)

In the following, we discuss the case when \( f_\lambda \) has an indifferent cycle of multiplier \( e^{2\pi i \theta} \). Douady [9] conjectured that for any rational map, whenever it is linearizable (i.e., the map is conformally conjugate to an irrational rotation) near an indifferent fixed point of multiplier \( e^{2\pi i \theta} \), then \( \theta \) must be a Brjuno number. Here, an irrational number \( \theta \) of convergents \( p_k/q_k \) (rational
approximations obtained by the continued fraction expansion) is a Brjuno number (denoted by $\mathcal{B}$) if
\[
\sum_{k \geq 1} \frac{\log q_{k+1}}{q_k} < +\infty.
\]
According to Cremer, Siegel and Brjuno, if $\theta \in \mathcal{B}$, then every germ $f(z) = e^{2\pi i \theta} z + \mathcal{O}(z^2)$ is linearizable. Yoccoz [33] shows that if the quadratic polynomial $z \mapsto e^{2\pi i \theta} z + z^2$ is linearizable, then $\theta \in \mathcal{B}$. For a general case, Geyer [12] shows that for any $d \geq 2$, if $z \mapsto z^d + c$ has an indifferent cycle of multiplier $e^{2\pi i \theta}$ near which the map is linearizable, then $\theta \in \mathcal{B}$. Based on these results and Proposition 5.1, we immediately establish:

**Proposition 5.2.** Suppose $f_\lambda$ has an indifferent cycle of multiplier $e^{2\pi i \theta}$; then $f_\lambda$ is linearizable near the indifferent cycle if and only if $\theta \in \mathcal{B}$.

### 5.2. Properties of renormalizations

In this section, we assume that some tableau $T(c)$ with $c \in C_\lambda$ is periodic with period $k$. By Lemma 5.1, $f_\lambda$ is either $k$-renormalizable at $c$ or $k/2$-*renormalizable at $c$. Let $(\epsilon f_\lambda^p, P_{d_0+p}(c), P_{d_0}(c))$ be the corresponding renormalization, where
\[
(\epsilon, p) = \begin{cases} (1, k), & \text{if } f_\lambda \text{ is } k\text{-renormalizable at } c, \\ (-1, k/2), & \text{if } f_\lambda \text{ is } k/2\text{-*renormalizable at } c. \end{cases}
\]

The small filled Julia set $K_c = \bigcap_{d \geq 0} P_d(c) = \bigcap_{d \geq 0} P_{d_0}(c)$.

If $K_c \cap \partial B_\lambda \neq \emptyset$, we will show that there is a unique external ray in $B_\lambda$ converging on $K_c$. Before the proof, we need a classic result for quadratic polynomials:

**Lemma 5.2.** Let $p_\mu(z) = z^2 + \mu$ be a quadratic polynomial with a connected filled Julia set $K$. If there is a curve $\delta \subset \mathbb{C} \setminus K$ converging to $x \in K$ and $p_\mu(\delta) \supset \delta$, then $x$ is the $\beta$-fixed point of $p_\mu$.

Here, a curve $\delta \subset \mathbb{C} \setminus K$ converges to $x \in K$ means that $\delta$ can be parameterized as $\delta : [0, 1) \to \mathbb{C} \setminus K$ such that $\lim_{t \to 1} \delta(t)$ exists and $\lim_{t \to 1} \delta(t) = x \in K$. See [19] for a proof of Lemma 5.2. The conclusion also holds for quadratic-like maps.

**Lemma 5.3.** Suppose some tableau $T(c)$ with $c \in C_\lambda$ is $k$-periodic and $K_c \cap \partial B_\lambda \neq \emptyset$, then

1. The small filled Julia sets $K_c, f_\lambda(K_c), \ldots, f^{k-1}_\lambda(K_c)$ are pairwise disjoint.
2. There is a unique external ray $R_\lambda(t)$ in $B_\lambda$ accumulating on $K_c$. This external ray lands at $\beta_c \in K_c$ and the angle $t$ is $k$-periodic.

**Proof.** 1. If $f^i_\lambda(K_c) \cap f^j_\lambda(K_c) \neq \emptyset$ for some $0 \leq i < j < k$, then $K_c \cap f^{k+i-j}_\lambda(K_c) \neq \emptyset$. Thus, $P_{d,k+i-j}(c) = f^{k+i-j}_\lambda(P_{d+k+i-j}(c)) = P_d(c)$ for all $d \geq 0$. This implies that the tableau $T(c)$ is $(k+i-j)$-periodic, which is a contradiction.
2. First, note that $f_k^d(P_{d+k}(c)) = P_d(c)$ for $d \geq 0$. Because $K_c \cap \partial B_\lambda \neq \emptyset$, $P_{mk}(c) \cap B_\lambda$ is nonempty and bounded by two external rays, say $R_\lambda(\theta_m^-)$ and $R_\lambda(\theta_m^+)$ with $\theta_m^- < \theta_m^+$. Let $Q(\theta_m^-, \theta_m^+) = P_{mk}(c) \cap B_\lambda$, $m \geq 1$. Because $f_k^d(Q(\theta_{m+1}^-, \theta_{m+1}^+)) = Q(\theta_m^-, \theta_m^+)$, we have
\[
\theta_m^- \leq \theta_{m+1}^- \leq \cdots \leq \theta_{m+1}^+ \leq \theta_m^+,
\]
\[
\theta_m^+ - \theta_m^- = n_k(\theta_{m+1}^+ - \theta_{m+1}^-).
\]
Thus, there is a common limit $t = \lim \theta_m^+ = \lim \theta_m^-$. Because $\theta_m^- \leq t \leq \theta_m^+$ for any $m$, we have $n_k t \equiv t \pmod{Z}$. Thus, $t$ is a periodic angle and the external ray $R_\lambda(t)$ lands at a point $z \in K_c \cap \partial B_\lambda$ (because rational external rays always land). Because $R_\lambda(n_j t)$ lands at $f_j^d(z) \in f_j^d(K_c) \cap \partial B_\lambda$, $0 \leq j < k$ and the small filled Julia sets $K_c$, $f_k(K_c)$, $\ldots$, $f_{k-1}(K_c)$ are pairwise disjoint, we conclude that the angles $t, nt, \ldots, n^{k-1}t$ are distinct. Thus, $t$ is $k$-periodic.

Suppose $\theta$ is another angle such that the external ray $R_\lambda(\theta)$ accumulates on $K_c$. Then, $\theta_m^- \leq \theta \leq \theta_m^+$ for any $m$. Thus, $\theta = \lim \theta_m^+ = \lim \theta_m^- = t$.

To finish, we show $z = \beta_c$. Because $T(c)$ is $k$-periodic, $f_k$ is either $k$-renormalizable or $k/2$-renormalizable. In the former case, $f_k^d(R_\lambda(t)) = R_\lambda(t)$, and, based on Lemma 5.2, $z = \beta_c$. In the latter case, because $R_\lambda(t)$ is the unique external ray accumulating on $K_c$, we conclude that $R_\lambda(t + 1/2) = -R_\lambda(t)$ is the unique external ray accumulating on $-K_c$. On the other hand, $f_k^{k/2}(R_\lambda(t))$ is also an external ray accumulating on $-K_c$, and we have $f_k^{k/2}(R_\lambda(t)) = R_\lambda(t + 1/2) = -R_\lambda(t)$. In this case, $-f_k^{k/2}(R_\lambda(t)) = R_\lambda(t)$. Again, based on Lemma 5.2, $z = \beta_c$.  

6. A criterion of local connectivity

In this section, we present a criterion for the characterization of the local connectivity of the immediate basin of attraction. This criterion can be applied together with Yoccoz puzzle techniques to study the local connectivity and higher regularity of the boundary $\partial B_\lambda$.

In the following discussion, let $f$ be a rational map of degree at least two, $C(f)$ be the critical set of $f$ and $P(f) = \bigcup_{k \geq 1} f^k(C(f))$ be the post-critical set. Suppose that $f$ has an attracting periodic point $z_0$ and the immediate basin $B(z_0)$ is simply connected. Let $B(z, \delta) = \{x \in \mathbb{C}; |x - z| < \delta\}$.

**Definition 6.1.** We say $f$ satisfies the **BD** (bounded degree) condition on $\partial B$ if for any $u \in \partial B$ there is a number $\varepsilon_u > 0$ such that for any integer $m \geq 0$ and any component $U_m(u)$ of $f^{-m}(B(u, \varepsilon_u))$ intersecting with $\partial B$, $U_m(u)$ is simply connected and the degree $\deg(f^m : U_m(u) \to B(u, \varepsilon_u))$ is bounded by some constant $D$ that is independent of $u$, $m$ and $U_m(u)$.

The following is a remark on the definition: because $f^m : U_m(u) \to B(u, \varepsilon_u)$ is a proper map between two disks, we conclude by the Maximum Principle that for any disk $W \subset B(u, \varepsilon_u)$ and any component $V$ of $f^{-m}(W)$ that lies inside $U_m(u)$, $V$ is also a disk.

The aim of this section is to prove the following:

**Proposition 6.1.** If $f$ satisfies the **BD** condition on $\partial B$, then

1. $\partial B$ is locally connected.
2. If, furthermore, $\partial B$ is a Jordan curve, then $\partial B$ is a quasi-circle.
Before presenting the proof, we introduce a distortion lemma. Let $U$ be a hyperbolic disk in $\mathbb{C}$ and $z \in U$. The shape of $U$ about $z$ is defined by:

$$\text{Shape}(U, z) = \sup_{x \in \partial U} \frac{|x - z|}{\inf_{x \in \partial U} |x - z|}.$$  

It is obvious that $\text{Shape}(U, z) = \infty$ if and only if $U$ is unbounded and $\text{Shape}(U, z) = 1$ if and only if $U$ is a round disk centered at $z$. In all other cases, $1 < \text{Shape}(U, z) < \infty$.

Let $K$ be a connected and compact subset of $U$ containing at least two points. For any $z_1, z_2 \in K$, define the turning of $K$ about $z_1$ and $z_2$ by:

$$\Delta(K; z_1, z_2) = \text{diam}(K)/|z_1 - z_2|,$$

where diam(·) is the Euclidean diameter. It is obvious that $1 \leq \Delta(K; z_1, z_2) \leq \infty$ and $\Delta(K; z_1, z_2) = \infty$ if and only if $z_1 = z_2$.

**Lemma 6.1.** For $i \in \{1, 2\}$, let $(V_i, U_i)$ be a pair of hyperbolic disks in $\mathbb{C}$ with $\overline{U_i} \subset V_i$, $g : V_1 \to V_2$ is a proper holomorphic map of degree $d$, and $U_1$ is a component of $g^{-1}(U_2)$. Suppose $\text{mod}(V_2 \setminus \overline{U_2}) \geq m > 0$. Then,

1. (Shape distortion) There is a constant $C(d, m) > 0$ such that for all $z \in U_1$,

$$\text{Shape}(U_1, z) \leq C(d, m)\text{Shape}(U_2, g(z)).$$

2. (Turning distortion) There is a constant $D(d, m) > 0$ such that for any connected and compact subset $K$ of $U_1$ with $\#K \geq 2$ and any $z_1, z_2 \in K$,

$$\Delta(K; z_1, z_2) \leq D(d, m)\Delta(g(K); g(z_1), g(z_2)).$$

**Proof.** A complete proof of 1 can be found in [30], Theorem 2.3.2. In the following, we prove 2. We assume that $g(z_1) \neq g(z_2)$. Otherwise, $\Delta(g(K); g(z_1), g(z_2)) = \infty$, and the conclusion follows. Let $\rho(x, y)$ be the hyperbolic distance in $V_2$, and let $B_1, B_2$ be two hyperbolic disks both centered at $g(z_1)$, with radii $\max_{\xi \in g(K)} \rho(g(z_1), \xi)$ and $\rho(g(z_1), g(z_2))$, respectively. Let $\varphi : V_2 \to D$ be the Riemann mapping with $\varphi(g(z_1)) = 0$, and let $W = \varphi(U_2)$. Because $\text{mod}(\overline{D} \setminus \overline{W}) = \text{mod}(V_2 \setminus \overline{U_2}) \geq m$, we conclude by the Grötzsch Theorem that there is a constant $r(m) \in (0, 1)$ such that $W \subset D_r(m)$; here, we use $D_r$ to denote the disk $\{z; |z| < r\}$.

Note that $\varphi(B_1), \varphi(B_2)$ are two round disks, say $D_R$ and $D_r$, centered at 0. Based on Koebe distortion, there exist three constants $C_1(m), C_2(m), C_3(m) > 0$ such that

$$\text{Shape}(B_1, g(z_1)) \leq C_1(m), \quad \text{Shape}(B_2, g(z_1)) \leq C_2(m),$$

$$R/r \leq C_3(m) \max_{\xi \in g(K) \cap \partial B_1} \left|g(z_1) - \xi\right|/\left|g(z_1) - g(z_2)\right| \leq C_3(m)\Delta(g(K); g(z_1), g(z_2)).$$

For $i \in \{1, 2\}$, let $W_i$ be the component of $g^{-1}(B_i)$ that contains $z_1$. Based on the Maximum Principle, $W_1$ and $W_2$ are simply connected. We may assume that $K \subset \overline{W_1}$ (otherwise, we can replace $B_1$ by $\hat{B}_1$, a hyperbolic disk centered at $g(z_1)$ with radius $\epsilon + \max_{\xi \in g(K)} \rho(g(z_1), \xi)$, where $\epsilon$ is a small positive constant and then let $\epsilon \to 0^+$. Thus, $\text{diam}(K) \leq \text{diam}(W_1) \leq$
2 \sup_{\zeta \in \partial W_1} |\zeta - z_1|. Consider the location of \( z_2 \), which by the Maximum Principle is either \( z_2 \in \partial W_2 \) or \( z_2 \in U_1 \setminus \overline{W_2} \). In either case, \( |z_1 - z_2| \geq \inf_{\zeta \in \partial W_2} |\zeta - z_1| \). Thus, by Shape distortion,

\[
\Delta(K; z_1, z_2) \leq 2 \sup_{\zeta \in \partial W_1} |\zeta - z_1|/ \inf_{\zeta \in \partial W_2} |\zeta - z_1|
\]

\[
= 2 \text{Shape}(W_1, z_1) \text{Shape}(W_2, z_2) Q(W_1, W_2, z_1)
\]

\[
\leq C_1 (d, m) \text{Shape}(B_1, g(z_1)) \text{Shape}(B_2, g(z_2)) Q(W_1, W_2, z_1)
\]

\[
\leq C_2 (d, m) Q(W_1, W_2, z_1)
\]

where \( Q(W_1, W_2, z_1) = \inf_{\zeta \in \partial W_1} |\zeta - z_1|/ \sup_{\zeta \in \partial W_2} |\zeta - z_1| \). To finish, in the following we show that there is a constant \( c(m) > 0 \) such that

\[
Q(W_1, W_2, z_1) \leq c(m) \Delta (g(K); (g(z_1), g(z_2))).
\]

In fact, we only need to consider the case \( Q(W_1, W_2, z_1) > 1 \). In this case, the annulus \( W_1 \setminus \overline{W_2} \) contains the round annulus \( \{ w \in \mathbb{C}; \sup_{z \in \partial W_2} |\zeta - z_1| < |w - z_1| < \inf_{\zeta \in \partial W_1} |\zeta - z_1| \} \). It turns out that

\[
\frac{1}{2\pi} \log Q(W_1, W_2, z_1) \leq \text{mod}(W_1 \setminus \overline{W_2}) \leq \text{mod}(B_1 \setminus \overline{B_2}) = \frac{1}{2\pi} \log \frac{R}{r}
\]

\[
\leq \frac{1}{2\pi} \log \left( C_3 (m) \Delta (g(K); (g(z_1), g(z_2))) \right).
\]

The conclusion follows. \( \square \)

**Proof of Proposition 6.1.** By replacing \( f \) with \( f^k \), we assume \( z_0 \) is a fixed point of \( f \). Based on quasi-conformal surgery, we assume \( z_0 \) is a superattracting fixed point with local degree \( d = \deg(f : B \to B) \geq 2 \). Thus, \( B \) contains no critical points other than \( z_0 \). By Möbius conjugation, we assume \( z_0 = \infty \).

Because \( f \) satisfies the BD condition on \( \partial B \), there exists a constant \( \delta > 0 \) such that for any \( u \in \partial B \), any integer \( m \geq 0 \) and any component \( U_m(u) \) of \( f^{-m}(B(u, \delta)) \) that intersects with \( \partial B \), \( U_m(u) \) is simply connected and \( \deg(f^m : U_m(u) \to B(u, \delta)) \leq D \). In fact, we can choose \( \delta \) as the Lebesgue number of the family \( \mathcal{F} = \{ B(u, \epsilon_u); u \in \partial B \} \), which is an open covering of the boundary \( \partial B \).

The proof consists of four steps, as follows:

**Step 1.** Let \( V_m(z) \) be the component of \( f^{-m}(B(z, \delta/2)) \) contained in \( U_m(z) \) and intersecting with \( \partial B \), then

\[
\lim_{m \to \infty} \sup_{z \in \partial B} \text{diam}(V_m(z)) = 0.
\]

Otherwise, there is a constant \( d_0 \geq 0 \) and two sequences \( \{ z_k \} \subset \partial B \) and \( \{ \ell_k \} \) such that \( \text{diam}(V_{\ell_k}(z_k)) \geq d_0 \). For every \( k \geq 1 \), choose a point \( y_k \in f^{-\ell_k}(z_k) \cap V_{\ell_k}(z_k) \). By passing to a subsequence, we assume \( y_k \to y_\infty \in \partial B \) and \( z_k \to z_\infty \in \partial B \). Based on Lemma 6.1, there is a constant \( C(D) \) such that
Shape($V_{\ell_k}(z_k) \ni y_k \mid C(D)) \subseteq (D)Shape(B(z_k, \delta/2), z_k) = C(D).

Because $\text{diam}(V_{\ell_k}(z_k)) \geq d_0$, $V_{\ell_k}(z_k)$ contains a round disk of definite size centered at $y_k$. Therefore, there is a constant $r_0 = r_0(d_0, D)$ such that $V_{\ell_k}(z_k) \supset B(y_\infty, r_0)$ for large $k$. Therefore, $f^{\ell_k}(B(y_\infty, r_0)) \subset B(z_\infty, \delta/2) \subset B(z_\infty, \delta)$. But, this contradicts the fact that $f^{\ell_k}(B(y_\infty, r_0)) \supset J(f)$ when $k$ is large.

**Step 2.** There are two constants $L > 0$ and $v \in (0, 1)$ such that for any $z \in \partial B$ and any $k \geq 1$, $\text{diam}(V_k(z)) \leq L v^k$.

By Step 1, there is an integer $s > 0$ such that $\text{diam}(V_s(z)) < \delta/4$ for all $z \in \partial B$. For each $x \in \partial B$, we take a point $x_{ks} \in V_{ks}(x) \cap f^{-ks}(x)$. (Notice that, in general, $V_{ks}(x) \cap f^{-ks}(x)$ consists of finitely many points, $x_{ks}$ can be either of them.) For $0 \leq j \leq k$, let $x_{js} = f^{(k-j)s}(x_{ks})$ and $U_j$ be the component of $f^{-js}(B(x(k-j)s, \delta/2))$ containing $x_{ks}$. Then,

$$x_{ks} \in V_{ks}(x) = U_0 \subset \cdots \subset U_0 = B(x_{ks}, \delta/2).$$

For every $1 \leq j < k$, $f^{js} : U_j \to B(x(k-j)s, \delta/2)$ is a proper map of degree $\leq D$. Because $f^{js}(U_{j+1})$ is contained in $B(x(k-j)s, \delta/4)$,

$$\text{mod}(U_j \setminus U_{j+1}) \geq \frac{1}{D} \text{mod}(B(x(k-j)s, \delta/2) \setminus f^{js}(U_{j+1})) \geq \log \frac{2}{2\pi D},$$

$$\text{mod}(B(x_{ks}, \delta/2) \setminus V_{ks}(x)) \geq \sum_{0 \leq j \leq k} \text{mod}(U_j \setminus U_{j+1}) \geq \frac{k \log 2}{2\pi D}.$$

We know from the proof of Step 1 that $\text{Shape}(V_{ks}(x), x_{ks}) \leq C(D)$. Therefore, there is a constant $K(D) > 0$ such that $\min_{y \in \partial V_{ks}(x)} |x_{ks} - y| \geq K(D) \text{diam}(V_{ks}(x))$. We have

$$\text{mod}(B(x_{ks}, \delta/2) \setminus V_{ks}(x)) \leq \frac{1}{2\pi} \log \left( \frac{\delta}{2K(D)\text{diam}(V_{ks}(x))} \right).$$

It turns out that $\text{diam}(V_{ks}(x)) \leq \frac{\delta}{2K(D)} 2^{-k/D}$, which implies that there are two constants $L > 0$ and $v \in (0, 1)$ such that $\text{diam}(V_k(x)) \leq L v^k$ for all $k \geq 1$.

**Step 3.** There exists a sequence of Jordan curves $\{\gamma_k : S \to B\}$ such that $\gamma_k$ converges uniformly to a continuous and surjective map $\gamma_\infty : S \to \partial B$, where $S = \mathbb{R}/\mathbb{Z}$ is the unit circle. Hence, $\partial B$ is locally connected.

Recall that the Böttcher map $\phi : B \to \mathbb{C} \setminus \mathbb{D}$ defined by $\phi(z) = \lim_{k \to \infty} (f^k(z))^{d_k}$ is a conformal isomorphism, which satisfies $\phi^{-1}(\sqrt{d} e^{2\pi i t}) = f^{-1}(\sqrt{d} e^{2\pi i t})$ for $(r, t) \in (1, +\infty) \times S$. Let $\ell(R, t) = \phi^{-1}(\sqrt{d} R e^{2\pi i t})$ for $(R, t) \in (1, 2) \times S$. By the boundary behavior of hyperbolic metric, there is a constant $C > 0$ such that for any $(R, t) \in (1, 2) \times S$,

$$\text{Eucl.length} \left( \ell(R, t) \right) \leq C \text{Hyper.length} \left( \ell(R, t) \right) \cdot \text{H.dist} \left( \phi^{-1}(R S), \partial B \right)$$

$$\leq C \log d \cdot \text{H.dist} \left( \phi^{-1}(R S), \partial B \right) \text{ as } R \to 1,$$

where $\text{Hyper.length}$ is the hyperbolic length in $B$ and $\text{H.dist}$ is the Hausdorff distance in the sphere $\mathbb{C}$.

Thus, we can choose $R$ sufficiently close to 1 such that for any $t \in S$, $\ell(R, t) \subset B(z, \delta/2)$ for some $z \in \partial B$. For $k \geq 0$, define a curve $\gamma_k : S \to B$ by $\gamma_k(t) = \phi^{-1}(R^{1/d_k} e^{2\pi i t})$. 


Because $f^k(y_{k+q}(t)) = y_q(d^k t)$ for $q \geq 0$ and $y_0(d^k t), y_1(d^k t) \in \ell(R, d^k t) \subset B(z, \delta/2)$ for some $z \in \partial B$, we conclude that $y_k(t)$ and $y_{k+1}(t)$ lie in the same component of $f^{-k}(B(z, \delta/2))$ that intersect with $\partial B$. Based on Step 2,

$$\max_{t \in S} |y_{k+1}(t) - y_k(t)| = O(\nu^k).$$

So $\{y_k : S \to B\}$ is a Cauchy sequence and hence converges to a continuous map $y_\infty : S \to \partial B$.

To finish, we show $y_\infty$ is surjective. Let $B_k \subset B$ be the disk bounded by $y_k(S)$; then $B_k \Subset B_{k+1}$ and $\bigcup_k B_k = B$. Each point $z \in \partial B$ can therefore be approximated by a sequence of points $\{z_k = y_k(t_k)\}_{k \geq 1}$ with $z_k \in \partial B_k$. There is a subsequence $k_j$ such that $t_{k_j} \to t_\infty \in S$ as $j \to \infty$. We then have $y_\infty(t_\infty) = \lim_j y_{k_j}(t_\infty) = \lim_j y_{k_j}(t_k_j) = z$. It follows that $y_\infty$ is surjective.

**Step 4.** If, furthermore, $\partial B$ is a Jordan curve, then $\partial B$ is a quasi-circle.

Because $\partial B$ is a Jordan curve, the Böttcher map $\phi : B \to \hat{C} \setminus \hat{D}$ can be extended to a homeomorphism $\phi : \overline{B} \to \hat{C} \setminus \hat{D}$. Define a map $\psi : S \to \partial B$ by $\psi(\zeta) = \phi^{-1}(\zeta)$ for $\zeta \in S$. Then $f(\psi(\zeta)) = \psi(\zeta^d)$. Let $\varphi = |\phi|_B$ be the inverse of $\psi$. Both $\psi$ and $\varphi$ are uniformly continuous; thus, for any sufficiently small positive number $\epsilon$, there are two small constants $a(\epsilon), b(\epsilon)$ such that

\[
\forall (\zeta_1, \zeta_2) \in S \times S, \quad |\zeta_1 - \zeta_2| < a(\epsilon) \quad \Rightarrow \quad \|\psi(\zeta_1) - \psi(\zeta_2)\| < \epsilon,
\]

\[
\forall (z_1, z_2) \in \partial B \times \partial B, \quad |z_1 - z_2| < b(\epsilon) \quad \Rightarrow \quad \|\varphi(z_1) - \varphi(z_2)\| < a(\epsilon).
\]

Given two points $z_1, z_2 \in \partial B, \partial B \setminus \{z_1, z_2\}$ consists of two components, say $E_1$ and $E_2$. Let $L(z_1, z_2) \in \{E_1, E_2\}$ be a section of $\partial B$ such that $\operatorname{diam}(L(z_1, z_2)) = \min[\operatorname{diam}(E_1), \operatorname{diam}(E_2)]$. Thus, for any positive number $\epsilon \ll \operatorname{diam}(\partial B)$, by uniform continuity we have

$$|z_1 - z_2| < b(\epsilon) \quad \Rightarrow \quad \operatorname{diam}(L(z_1, z_2)) < \epsilon. \quad (1)$$

Based on Alhfors’ characterization of quasi-circles [1], to prove that $\partial B$ is a quasi-circle, it suffices to show that there is a constant $C > 0$ such that for any $z_1, z_2 \in \partial B$ with $z_1 \neq z_2$, $\Delta(L(z_1, z_2); z_1, z_2) \leq C$. In fact, if $|z_1 - z_2| \geq \epsilon$ for some positive constant $\epsilon$, then $\Delta(L(z_1, z_2); z_1, z_2) \leq \operatorname{diam}(\partial B)/\epsilon$. Therefore, we only need to consider the case when $|z_1 - z_2|$ is small. In the following, we assume $\delta \ll \operatorname{diam}(\partial B) \quad \delta/2); \text{it	extquotesingle}s	ext{'}; \text{t}	ext{h}	ext{o}	ext{u}	ext{s} \text{c}	ext{o}	ext{n}	ext{d}	ext{e}	ext{n}	ext{t}	ext{) \text{h}	ext{\text{a}	ext{t}} \text{d}	ext{i}	ext{a}	ext{m}(\text{L}(\text{z}	ext{\text{1}, \text{z}	ext{\text{2}}})) < \delta/2.$

Because $f$ is expanding on $\partial B$, there is an integer $N > 0$ such that $f^k(L(z_1, z_2)) = \partial B$ for all $k \geq N$. We can therefore find a smallest integer $\ell \geq 0$ such that

$$\operatorname{diam}(f^\ell(L(z_1, z_2))) < \delta/2, \quad \operatorname{diam}(f^{\ell+1}(L(z_1, z_2))) \geq \delta/2.$$

On the other hand, there exist two points $w_1, w_2 \in f^\ell(L(z_1, z_2))$ such that

$$\operatorname{diam}(f^{\ell+1}(L(z_1, z_2))) = |f(w_1) - f(w_2)| \leq \int_{[w_1, w_2]} |f'(z)| |dz| \leq M|w_1 - w_2| \leq M \operatorname{diam}(f^\ell(L(z_1, z_2))),$$

where $[w_1, w_2]$ is the straight segment connecting $w_1$ with $w_2$ and
\[ M = \max\{ |f'(z)|; \ \text{Eucl.dist}(z, \partial B) \leq \delta/2 \}. \]

Thus, we have
\[ \frac{\delta}{2M} \leq \text{diam}(\ell(L(z_1, z_2))) = \text{diam}(L(f^\ell(z_1), f^\ell(z_2))) < \frac{\delta}{2}. \]

By (1), there is a constant \( c(\delta, M) > 0 \) such that
\[ |f^\ell(z_1) - f^\ell(z_2)| \geq c(\delta, M). \]

Applying Lemma 6.1 to the situation \((V_1, U_1) = (U_\ell(f^\ell(z_1)), V_\ell(f^\ell(z_1))), (V_2, U_2) = (B(f^\ell(z_1), \delta), B(f^\ell(z_1), \delta/2))\) and \( g = f^\ell \), we conclude that there is a constant \( C(D) > 0 \) such that
\[ \Delta(L(z_1, z_2); z_1, z_2) \leq C(D) \Delta(f^\ell(L(z_1, z_2)); f^\ell(z_1), f^\ell(z_2)) \leq \frac{C(D)\delta}{2c(\delta, M)}. \]

Thus, for any \( x, y \in \partial B \) with \( x \neq y \), the turning \( \Delta(L(x, y); x, y) \) is bounded by
\[ \max \left\{ \frac{\text{diam}(\partial B)}{b(\delta/2)}, \frac{C(D)\delta}{2c(\delta, M)} \right\}. \]

**Remark 6.1.** Using the same argument as [4], one can show further that if \( f \) satisfies BD condition on \( \partial B \), then \( \partial B \) is a John domain.

The following describes an important case in which \( f \) satisfies the BD condition on \( \partial B \).

**Proposition 6.2.** If \( \#(P(f) \cap \partial B) < \infty \) and all periodic points in \( P(f) \cap \partial B \) are repelling, then \( f \) satisfies BD condition on \( \partial B \).

**Proof.** The proof is based on the following claim.

**Claim.** For any \( u \in \partial B \), there is a constant \( \varepsilon_u > 0 \) such that for any \( m \geq 0 \) and any component \( U_m(u) \) of \( f^{-m}(B(u, \varepsilon_u)) \) that intersects with \( \partial B \), \( U_m(u) \) contains at most one critical point of \( f^m \).

The claim implies that \( U_m(u) \) is simply connected by the Riemann–Hurwitz formula. Because the sequence \( U_m(u) \rightarrow f(U_m(u)) \rightarrow \cdots \rightarrow f^{m-1}(U_m(u)) \rightarrow B(u, \varepsilon_u) \) meets every critical point of \( f \) at most once, we conclude that \( \text{deg}(f^m : U_m(u) ightarrow B(u, \varepsilon_u)) \) is bounded by \( D = \prod_{c \in C(f)} \text{deg}(f, c) \).

In the following, we prove the claim.

First, note that every point in \( P(f) \cap \partial B \) is pre-periodic; we can deconstruct \( \partial B \) into three disjoint sets \( X, Y \) and \( Z \), where \( X = \partial B \setminus P(f), Z = \text{the union of all repelling cycles in } P(f) \cap \partial B \) and \( Y = (P(f) \cap \partial B) \setminus Z \).

For any \( x \in X \), choose a small number \( \varepsilon_x > 0 \) such that \( B(x, \varepsilon_x) \cap P(f) = \emptyset \). Then, for any component \( W_m(x) \) of \( f^{-m}(B(x, \varepsilon_x)) \) intersecting with \( \partial B \), \( f^m : W_m(x) \rightarrow B(x, \varepsilon_x) \) is a conformal map.

The set \( Y \) consists of all strictly pre-periodic points. Thus, there is an integer \( q \geq 1 \) such that for any \( y \in Y \), \( f^{-q}(y) \cap P(f) \cap \partial B = \emptyset \). For an open set \( U \) in \( \mathbb{C} \) and a point \( u \in U \), we use \( \text{Comp}_u(U) \) to denote the component of \( U \) that contains \( u \). For every \( y \in Y \), choose \( \varepsilon_y > 0 \)
small enough such that for any \( x \in f^{-q}(y) \cap \partial B \subset X \), \( \text{Comp}_x(f^{-q}(B(y, \varepsilon_y))) \subset B(x, \varepsilon_x) \) and \( \text{Comp}_y(f^{-q}(B(y, \varepsilon_y))) \) contain at most one critical point of \( f^q \).

Finally, we deal with \( Z \). For \( z \in Z \), suppose \( z \) lies in a repelling cycle of period \( p \). Choose \( \varepsilon_z > 0 \) such that

1. \( B(z, \varepsilon_z) \) is contained in the linearizable neighborhood of \( z \) and \( \text{Comp}_z(f^{-p}(B(z, \varepsilon_z))) \) is a subset of \( B(z, \varepsilon_z) \).
2. For every \( u \in (f^{-p}(z) \cap \partial B) \setminus \{ z \} \subset X \cup Y \), \( \text{Comp}_u(f^{-p}(B(z, \varepsilon_z))) \) contains at most one critical point of \( f^p \) and \( \text{Comp}_u(f^{-p}(B(z, \varepsilon_z))) \subset B(u, \varepsilon_u) \).

One can easily verify that the collection of neighborhoods \( \{ B(u, \varepsilon_u), u \in \partial B \} \) are just as required.

**Corollary 6.1.** If \( f \) is critically finite, then \( f \) satisfies the BD condition on \( \partial B \).

**Proof.** Because \( f \) is critically finite, every periodic point of \( f \) is either repelling or superattracting, which implies that \( \#(P(f) \cap \partial B) < \infty \) and all periodic points in \( P(f) \cap \partial B \) are repelling. Thus, by Proposition 6.2, \( f \) satisfies the BD condition on \( \partial B \).

7. The boundary \( \partial B_\lambda \) is a Jordan curve

In this section, we will prove Theorem 1.1 and Theorem 1.2. The strategy of the proof is as follows.

First, consider the McMullen maps \( f_\lambda \) with parameter \( \lambda \in \mathcal{H} \). If \( f_\lambda \) is critically finite, then the Julia set is locally connected. Otherwise, by Proposition 4.1, we can find an admissible graph \( \mathbf{G}_\lambda(\theta_1, \ldots, \theta_N) \). With respect to the Yoccoz puzzle induced by this graph, there are two possibilities:

**Case 1.** None of \( T(c) \) with \( c \in C_\lambda \) is periodic. This case is discussed in Section 7.1, and the local connectivity of \( J(f_\lambda) \) follows from Proposition 7.1. The idea of the proof is based on the combinatorial analysis for tableaux introduced by Branner and Hubbard (see [2,21]) and on ‘modified puzzle piece’ techniques.

**Case 2.** Some \( T(c) \) with \( c \in C_\lambda \) is periodic. In this case, the map \( f_\lambda \) is either renormalizable or \(*\)-renormalizable. This case is discussed in Section 7.2. The local connectivity of \( \partial B_\lambda \) follows from Proposition 7.2. The goal of the proof of Proposition 7.2 is to construct a closed curve separating \( \partial B_\lambda \) from the small filled Julia set \( K_c \).

In Section 7.3, we deal with the real parameters \( \lambda \in \mathbb{R}^+ \).

In Section 7.4, we improve the regularity of the boundary \( \partial B_\lambda \). We first include a proof of Devaney that claims that the local connectivity of \( \partial B_\lambda \) implies that \( \partial B_\lambda \) is a Jordan curve. We then show that \( \partial B_\lambda \) is a quasi-circle except in two specific cases.

In Section 7.5, we present some corollaries.

7.1. None of \( T(c) \) with \( c \in C_\lambda \) is periodic

Recall that \( J_0 \) is the set of all points on the Julia set \( J(f_\lambda) \) whose orbits eventually meet the graph \( \mathbf{G}_\lambda(\theta_1, \ldots, \theta_N) \).
Lemma 7.1. Let \( z \in J(f_\lambda) \setminus J_0 \). If \( T(z) \) is non-critical, then \( \text{End}(z) := \bigcap_{d \geq 0} P_d(z) = \{z\} \).

**Proof.** It suffices to prove \( \text{End}(f_\lambda(z)) = \{f_\lambda(z)\} \). Because \( T(z) \) is non-critical, there is an integer \( d_0 \geq 1 \) such that for any \( j > 0 \), the position \( (d_0, j) \) is not critical. Equivalently, for any \( d \geq d_0 \) and any \( j \geq 1 \), the puzzle piece \( P_d(f_\lambda^j(z)) \) contains no critical point. Let \((\hat{P}_{d_0-1}^{(i)} \text{ and } 1 \leq i \leq M)\) be the collection of all modified puzzle pieces of depth \( d_0 - 1 \), numbered so that \( \hat{P}_{d_0-1}^{(1)} = \hat{P}_{d_0-1}(v_\lambda^+) \), \( \hat{P}_{d_0-1}^{(2)} = \hat{P}_{d_0-1}(v_\lambda^-) \), and recall that we use \( \hat{P}_d(w) \) to denote the modified puzzle piece of \( P_d(w) \). Every modified puzzle piece of depth \( \geq d_0 \) is contained in a unique modified puzzle piece \( \hat{P}_{d_0-1}^{(i)} \) of depth \( d_0 - 1 \). Let \( \text{dist}_j(x, y) \) be the Poincaré metric of \( \hat{P}_{d_0-1}^{(i)} \). For \( 2 < i \leq M \), there are exactly \( 2n \) branches of \( f_\lambda^{-1} \) on \( \hat{P}_{d_0-1}^{(i)} \), say \( g_1^i, g_2^i, \ldots, g_{2n}^i \), and each \( g_k^i \) on \( \hat{P}_{d_0-1}^{(i)} \) is univalent and carries \( \hat{P}_{d_0}^{(i)} \) onto a proper subset of some \( \hat{P}_{d_0-1}^{(j)} \). It follows that there is a uniform constant \( 0 < \nu < 1 \) such that

\[
\text{dist}_j(g_k^i(x), g_k^i(y)) \leq \nu \text{dist}_j(x, y)
\]

for any \( x, y \in \hat{P}_{d_0}^{(i)} \) and any \( 2 < i \leq M, 1 \leq k \leq 2n \).

Let \( D \) be the maximum Poincaré diameters of the modified puzzle pieces of depth \( d_0 \). For any integer \( h > 0 \), because the sequence

\[
P_{d_0+h}(f_\lambda(z)) \to P_{d_0+h-1}(f_\lambda^2(z)) \to \cdots \to P_{d_0+1}(f_\lambda^h(z)) \to P_{d_0}(f_\lambda^{h+1}(z))
\]

contains no critical point (this follows from the assumption that \( T(z) \) is non-critical), it follows that

\[
\text{Hyper.diam}(P_{d_0+h}(f_\lambda(z))) \leq D \nu^h
\]

with respect to the Poincaré metric of \( \hat{P}_{d_0-1}(f_\lambda(z)) \). Thus, we have \( \text{End}(f_\lambda(z)) = \{f_\lambda(z)\} \).

**Proposition 7.1.** If \( T(c) \) is not periodic for any \( c \in C_\lambda \), then the Julia set \( J(f_\lambda) \) is locally connected.

**Proof.** Note that \( T(c) \) is either critical or non-critical. First, we prove \( \text{End}(c) = \{c\} \) and \( \text{End}(z) = \{z\} \) for any \( z \in J(f_\lambda) \setminus J_0 \). We then deal with the points that lie in \( J_0 \).

**Case 1.** \( T(c) \) is critical. Because the graph is admissible, we can find a non-degenerate annulus \( A_{d_0}(c) \). Consider the descendents of \( \text{Row}_c(d_0) \). It is obvious that if \( \text{Row}_c(t) \) is a descendent in the \( k \)-th generation of \( \text{Row}_c(d_0) \), the annulus \( A_t(c) \) is non-degenerate with modulus \( \text{mod}(A_{d_0}(c))/2^k \). If \( \text{Row}_c(d_0) \) has at least \( 2^k \) descendents in the \( k \)-th generation for each \( k \geq 1 \), then each of these contributes exactly \( \text{mod}(A_{d_0}(c))/2^k \) to the sum \( \sum_d \text{mod}(A_d(c)) \). Hence, \( \sum_d \text{mod}(A_d(c)) = \infty \), as required. On the other hand, if there are fewer descendents in some generation, then one of them, say \( \text{Row}_c(m) \), must be an only child, hence excellent by Lemma 4.7. Again by Lemma 4.7, we see that \( \sum_d \text{mod}(A_d(c)) = \infty \). Therefore, in either case, \( \text{End}(c) = \{c\} \).

Now consider a point \( z \in J(f_\lambda) \setminus (J_0 \cup C_\lambda) \). If \( T(z) \) is non-critical, then by Lemma 7.1, \( \text{End}(z) = \{z\} \). If \( T(z) \) is critical, then for each \( d \geq 1 \), there is a smallest integer \( l_d \geq 0 \) such
that both \((d, l_1)\) and \((d, l_d + 1)\) are critical positions. It follows that \(f^{l_d}_\lambda : A_{d + l_d}(z) \to A_d(c')\) is a conformal map for some \(c' \in C_\lambda\). In this case, \(\sum_d \text{mod}(A_d(z)) \geq \sum_d \text{mod}(A_d + l_d(z)) = \sum_d \text{mod}(A_d(c)) = \infty\), hence \(\text{End}(z) = \{z\}\).

**Case 2.** \(T(c)\) is non-critical. It follows from Lemma 7.1 that \(\text{End}(c) = \{c\}\). For \(z \in J(f_\lambda) \setminus (J_0 \cup C_\lambda)\), we assume \(T(z)\) is critical; otherwise, \(\text{End}(z) = \{z\}\) based on Lemma 7.1. Suppose \(A_{d_0}(c)\) is a non-degenerate annulus and \((d_0 + 1, l_1), (d_0 + 1, l_2), \ldots\) are all critical positions in the \((d_0 + 1)\)-th row of the tableau \(T(z)\). Because all tableaux \(T(c)\) with \(c \in C_\lambda\) are non-critical, there is a constant \(D\) such that \(\text{deg}(f^{l_k}_\lambda : P_{d_0 + l_k}(z) \to P_{d_0}(c)) \leq D\) for all \(k \geq 1\). Thus,

\[
\text{mod}(A_{d_0 + l_k}(z)) \geq D^{-1} \text{mod}(A_{d_0}(c))
\]

for all \(k \geq 1\). Hence, \(\sum_d \text{mod}(A_d(z)) \geq \sum_k \text{mod}(A_{d_0 + l_k}(z)) = \infty\) and \(\text{End}(z) = \{z\}\).

**Points that lie in** \(J_0\). For any \(z \in J_0\), the orbit \(z \mapsto f_\lambda(z) \mapsto f_\lambda^2(z) \mapsto \cdots\) eventually meets the graph \(G_\lambda(\theta_1, \ldots, \theta_N)\). Therefore, the Euclidean distance between the critical set \(C_\lambda\) and the orbit \(\{f^k_\lambda(z)\}_{k \geq 0}\) is bounded below by some positive number \(\epsilon(z)\). In addition, for every \(d\) large enough, \(z\) lies in the common boundary of exactly two puzzle pieces of depth \(d\). We denote these two puzzle pieces by \(P^d_\lambda(z)\) and \(P^\prime_d(z)\). In the previous argument, we have already proved that \(\text{End}(c) = \{c\}\); this implies \(\text{Eucl.diam}(P^d_\lambda) \to 0\) as \(d \to \infty\). Choose a \(d_0\) large enough such that

\[
\text{Eucl.diam}(P_{d_0}(c)) < \epsilon(z) \leq \text{Eucl.dist}(C_\lambda, \{f^k_\lambda(z)\}_{k \geq 0}).
\]

Then, the orbit \(z \mapsto f_\lambda(z) \mapsto f_\lambda^2(z) \mapsto \cdots\) avoids all the critical puzzle pieces of depth \(d_0\). Let \(P^d_\lambda(z) = P^\prime_d(z) \cup P^\prime\prime_d(z)\) for \(d\) large enough. Then, the proof of Lemma 7.1 applies equally well to this situation, and \(\bigcap_d P^d_\lambda(z) = \{z\}\) immediately follows.

**Connectivity of neighborhoods.** Let

\[
P^d_\lambda(z) = \begin{cases} P_{d}(c), & \text{if } z \in J(f_\lambda) \setminus J_0, \\ P_{d}(z) \cup P^\prime_d(z), & \text{if } z \in J_0 \text{ and } d \text{ is large}. \end{cases}
\]

Based on Lemma 4.2, for every \(z \in J(f_\lambda)\) and every large integer \(d\), the intersection \(P^d_\lambda(z) \cap J(f_\lambda)\) is a connected and compact subset of \(J(f_\lambda)\). Thus, \(\{P^d_\lambda(z) \cap J(f_\lambda)\}\) forms a basis of connected neighborhoods of \(z\). Because \(\bigcap(P^d_\lambda(z) \cap J(f_\lambda)) = \{z\}\), the Julia set is locally connected at \(z\). Note that \(z\) is arbitrarily chosen, we conclude that \(J(f_\lambda)\) is locally connected.

### 7.2 Some \(T(c)\) with \(c \in C_\lambda\) is periodic

Suppose some tableau \(T(c)\) with \(c \in C_\lambda\) is \(k\)-periodic for some \(k > 0\). Based on the proof of Lemma 5.1, \(f_\lambda\) is either \(k\)-renormalizable at \(c\) or \(k/2\)-\(\ast\)-renormalizable at \(c\). Let \((\epsilon, f_\lambda^\prime, P_{d_0 + p}(c), P_{d_0}(c))\), where \(d_0\) is a large integer, be the renormalization and

\[
(\epsilon, p) = \begin{cases} (1, k), & \text{if } f_\lambda \text{ is } k\text{-renormalizable at } c, \\ (-1, k/2), & \text{if } f_\lambda \text{ is } k/2\text{-\(\ast\)-renormalizable at } c. \end{cases}
\]
The small filled Julia set of the renormalization \((\epsilon f^p_\lambda, P_{d_0+p}(c), P_{d_0}(c))\) is denoted by \(K_c\). Recall that \(\beta_c\) is the \(\beta\)-fixed point of the renormalization and \(\beta'_c\) is the other preimage of \(\beta_c\) under the map \(\epsilon f^p_\lambda|_{P_{d_0+p}(c)}\).

Assume now that \(K_c \cap \partial B_\lambda \neq \emptyset\); then, based on Lemma 5.3, \(\beta_c \in K_c \cap \partial B_\lambda\) and there is a unique external ray, say \(R_\lambda(\theta)\), landing at \(\beta_c\). The angle \(\theta\) is of the form \(\frac{m}{2^k-1}\). It follows that \(\beta'_c \in K_c \cap \partial T_\lambda\) and there is a unique radial ray \(R_{T_\lambda}(\alpha_\theta)\) in \(T_\lambda\) landing at \(\beta'_c\). The radial ray \(R_{T_\lambda}(\alpha_\theta)\) satisfies \(\epsilon f^p_\lambda(R_{T_\lambda}(\alpha_\theta)) = R_\lambda(\theta)\). Let

\[
K = K_c \cup \overline{R_\lambda(\theta)} \cup \overline{R_{T_\lambda}(\alpha_\theta)} \cup (-K_c) \cup \left((-\overline{R_\lambda(\theta)}) \cup (-\overline{R_{T_\lambda}(\alpha_\theta)})\right).
\]

The set \(K\) is a connected and compact subset of \(\mathbb{C}\). Note that \(-R_{T_\lambda}(\alpha_\theta) = R_{T_\lambda}(\alpha_\theta + 1/2)\). Let \(\Delta_1\) be the component of \(\mathbb{C} \setminus (K \cup \overline{B_\lambda})\) that intersects with \(Q_{T_\lambda}(\alpha_\theta, \alpha_\theta + 1/2)\) and \(\Delta_2\) be the component of \(\mathbb{C} \setminus (K \cup \overline{B_\lambda})\) that intersects with \(Q_{T_\lambda}(\alpha_\theta + 1/2, \alpha_\theta)\), where we use \(Q_{T_\lambda}(\theta_1, \theta_2)\) to denote the set \(\{\phi_{T_\lambda}(re^{i\theta}); 0 < r < 1, \theta_1 \leq t \leq \theta_2\}\). Because \(K \cup \overline{B_\lambda}\) is connected and compact, both \(\Delta_1\) and \(\Delta_2\) are disks. Let \(Z_i\) be the component of \(\mathbb{C} \setminus K\) that contains \(\Delta_i\).

The aim of this section is to prove:

**Proposition 7.2.** Assume that \(K_c \cap \partial B_\lambda \neq \emptyset\), then for \(i \in \{1, 2\}\), there is a curve \(L_i \subset \Delta_i \cup \{0\}\) stemming from \(T_\lambda\) and converging to \(\beta_c\). More precisely, \(L_i\) can be parameterized as \(L_i : [0, +\infty) \to \Delta_i \cup \{0\}\) such that \(L_i(0) = 0, L_i((0, +\infty)) \subset \Delta_i\) and \(\lim_{t \to +\infty} L_i(t) = \beta_c\) (Fig. 13).

**Proof.** Let \(\Gamma = \bigcup_{j \geq 0}(\pm f^j_\lambda(K_c \cup \overline{R_\lambda(\theta)})\). By Lemma 5.3, any two distinct elements in the set \(\{\pm f^j_\lambda(K_c \cup \overline{R_\lambda(\theta)})\}; j \geq 0\) intersect only at the point \(\infty\), which implies that \(U = \overline{\mathbb{C}} \setminus \Gamma\) is a disk.

**Step 1.** There exists \(G_i : U \to U \cap Z_i\), an inverse branch of \(\epsilon f^p_\lambda\) such that the sequence \(\{G^l_i; l \geq 0\}\) converges locally and uniformly in \(U\) to a constant \(z_i \in K_c\).

Because \(U\) has no intersection with the post-critical set of \(f_\lambda\), its preimage \(f_\lambda^{-1}(U)\) has exactly \(2n\) components, say \(V_1, \ldots, V_{2n}\). These components are arranged symmetrically about the origin under the rotation \(z \mapsto e^{\pi i/n}z\). For every \(1 \leq j \leq 2n\), \(f_j : V_j \to U\) is a conformal map. Moreover, \(f_\lambda^{-1}(U) \subset \overline{\mathbb{C}} \setminus K\).

For \(1 \leq j \leq p - 1\), let \(\Omega_j \subset \{V_1, \ldots, V_{2n}\}\) be the component of \(f_\lambda^{-1}(U)\) such that \(\overline{\Omega}_j \cap f^j_\lambda(K_c) \neq \emptyset\) and the inverse of \(f_j : \Omega_j \to U\) is denoted by \(g_j\). For \(j = 0\), let \(\Omega_0^i\) be the component of \(f_\lambda^{-1}(U)\) such that \(\overline{\Omega}_0^i \cap K_c \neq \emptyset\) and \(\Omega_0^i \subset Z_i\). The inverse of \(f_j : \Omega_0^i \to U\) is denoted by \(g_0^i\) for \(i \in \{1, 2\}\).

Now, we define

\[
G_i(z) = \begin{cases} 
  g_0^i \circ g_1 \circ \cdots \circ g_{p-1}(\epsilon z), & z \in U \\
  g_0^i(\epsilon z), & z \in U 
\end{cases} \quad \text{if } p \geq 2,
\]

\[
G_i(z) = \begin{cases} 
  g_0^i(z), & z \in U 
\end{cases} \quad \text{if } p = 1.
\]

Because \((\epsilon f^p_\lambda, P_{d_0+p}(c), P_{d_0}(c))\) is a \(p\)-\((*)\)-renormalization of \(f_\lambda\) at \(c\), we have \(G_i(P_{d_0}(c) \cap U) \subset P_{d_0+p}(c) \cap Z_i\). The map \(G_i : U \to U\) is not surjective; thus, by the Denjoy–Wolff theorem (see [20]), the sequence \(\{G^l_i; l \geq 0\}\) converges locally and uniformly in \(U\) to a constant \(z_i\). It follows from \(G_i(P_{d_0}(c) \cap U) \subset P_{d_0+p}(c) \cap Z_i\) that \(z_i \in K_c\).

**Step 2.** There exists a curve \(C_i \subset U \cap (\Delta_i \cup \{0\})\) connecting \(0\) with \(G_i(0)\) for \(i \in \{1, 2\}\).
Because the graph $G_{\lambda}(\theta_1, \ldots, \theta_N)$ is admissible, the filled Julia set $K_c$ is disjointed from the boundary of any puzzle piece. Thus, for any $\alpha \in \{\tau^s(\theta_j); 1 \leq j \leq N, s \geq 0\}$, $\Gamma$ is disjoint from the cut ray $\Omega^\alpha_{\lambda}$ outside $\infty$. (This is because the external ray $R_{\lambda}(\theta)$ has no intersection with $g_{\lambda}(\theta_1, \ldots, \theta_N)$ outside $\infty$; compare Lemma 5.3.) By Proposition 4.2, for any angle $\alpha \in \{\tau^s(\theta_j); 1 \leq j \leq N, s \geq 0\}$ and any map $g \in \{g_0^1, g_0^2, g_1, \ldots, g_{p-1}\}$, only one curve of $g(\omega_{\lambda}^{\alpha} \setminus \{\infty\}), g(\omega_{\lambda}^{\alpha+1/2} \setminus \{\infty\})$ intersects with $\partial B_{\lambda}$, and the other curve connects $0$ with a preimage of $0$.

Fix an angle $\alpha \in \{\tau^s(\theta_j); 1 \leq j \leq N, s \geq 0\}$; we define a curve family $\mathcal{F}$ by

$$\mathcal{F} = \left\{ \epsilon \omega_{\lambda}^\alpha \setminus \{\infty\}; \epsilon^{2n} = 1 \text{ and } \epsilon \omega_{\lambda}^\alpha \subset \bigcup_{j \in \mathbb{N}} S_j \right\}.$$  

We construct the curve $C_i$ by an inductive procedure, as follows:

First, choose a curve $\xi_{p-1} \in \mathcal{F}$ such that $g_{p-1}(\xi_{p-1}) \cap \partial B_{\lambda} = \emptyset$ and let $\gamma_{p-1} = g_{p-1}(\xi_{p-1})$. Suppose that for some $2 \leq j \leq p - 1$ we have already constructed the curves $\gamma_{p-1}, \ldots, \gamma_j$.

We then choose $\xi_{j-1} \in \mathcal{F}$ such that $g_{j-1}(\xi_{j-1}) \cap \partial B_{\lambda} = \emptyset$ and $\gamma_{j-1} \cap \gamma_j = \emptyset$ and let $\gamma_{j-1} = g_{j-1}(\xi_{j-1} \cup \gamma_j)$. In this way, we can construct a sequence of curves $\gamma_{p-1}, \gamma_{p-2}, \ldots, \gamma_2, \gamma_1$ step by step, and each curve has no intersection with $\partial B_{\lambda}$. These curves connect $0$ with some iterated preimage of $0$. By construction,

$$\gamma_1 = \bigcup_{1 \leq j \leq p-1} g_1 \circ \cdots \circ g_j(\xi_j).$$

We now choose $\xi_0^i \in \mathcal{F}$ such that $g_0^i(\xi_0^i) \cap \partial B_{\lambda} = \emptyset$ and $\xi_0^i \cap \gamma_1 = \emptyset$, and let

$$C_i = \begin{cases} 
  g_0^i(\xi_0^i \cup \gamma_1) \cup \{0\}, & \text{if } p \geq 2, \\
  g_0^i(\xi_0^i) \cup \{0\}, & \text{if } p = 1.
\end{cases}$$

Fig. 13. Constructing two curves $L_1$ and $L_2$ that converge to $\beta_c$, here $n = 3$ and $f_{\lambda}$ is 1-renormalizable at $c = c_0$. 

The curve $C_i$ connects 0 to $G_i(0)$ and $C_i \subset U \cap (\Delta_i \cup \{0\})$, as required.

**Step 3.** The union $L_i = \bigcup_{j \geq 0} G_j^i(C_i)$ is the curve contained in $\Delta_i \cup \{0\}$ and converging to $\beta_c$.

By construction, $G_i(L_i) \subset G_i(C_i) \cup C_i = L_i$ and $L_i \setminus \{0\} \subset \Delta_i$.

To finish, we show $L_i$ converges to $\beta_c$. By step 1, the sequence $\{G_k^i; \ k \geq 0\}$ converges uniformly on any compact subset of $U$ to a constant $z_i \in K_c$. Because $C_i$ is a compact subset of $U$, the curve $L_i$ converges to $z_i \in K_c$ and $G_i(z_i) = z_i$. Because $\epsilon f^p_\lambda(L_i) \supset L_i$, we conclude $z_i = \beta_c$ by Lemma 5.2. □

**Corollary 7.1.** If $T(c)$ is periodic for some $c \in C_\lambda$, then $\partial B_\lambda$ is locally connected.

**Proof.** We can assume that $f_\lambda$ is not geometrically finite; otherwise, the Julia set is locally connected (see [29]). Thus, $f_\lambda$ has no parabolic point.

If $K_c \cap \partial B_\lambda = \emptyset$, then for all $j \geq 0$, $f_\lambda^j(K_c) \cap \partial B_\lambda = \emptyset$. Because $P(f_\lambda)$ is a subset of $(\bigcup_{j \geq 0} f_\lambda^j(\pm f_\lambda(K_c))) \cup \{\infty\}$, we conclude $P(f_\lambda) \cap \partial B_\lambda = \emptyset$. Based on Proposition 6.1 and Proposition 6.2, $\partial B_\lambda$ is locally connected.

If $K_c \cap \partial B_\lambda \neq \emptyset$, then by Proposition 7.2, the closed curve $L = L_1 \cup L_2 \cup \{\beta_c\}$ separates $K_c \setminus \beta_c$ from $\partial B_\lambda \setminus \beta_c$. In this case, for all $j \geq 0$, $f_\lambda^j(K_c) \cap \partial B_\lambda = \{f_\lambda^j(\beta_c)\}$. Thus, $\#(P(f_\lambda) \cap \partial B_\lambda) < \infty$ and all periodic points in $P(f_\lambda) \cap \partial B_\lambda$ are repelling. Again by Proposition 6.1 and Proposition 6.2, $\partial B_\lambda$ is locally connected. □

### 7.3. Real case

In this section, we will deal with real parameters. Due to the symmetry of the parameter plane, we only need to consider $\lambda \in \mathbb{R}^+ = (0, +\infty)$. In this case, the Julia set $J(f_\lambda)$ is symmetric about the real axis. If $C_\lambda \subset A_\lambda$, by ‘The Escape Trichotomy’ (Theorem 2.1), the Julia set $J(f_\lambda)$ is either a Cantor set, a Cantor set of circles or a Sierpinski curve. In the latter two cases, the local connectivity of $\partial B_\lambda$ is already known. In the following discussion, we assume $C_\lambda \cap A_\lambda = \emptyset$.

**Lemma 7.2.** Suppose $\lambda \in \mathbb{R}^+$ and $C_\lambda \cap A_\lambda = \emptyset$; then, $f_\lambda$ is 1-renormalizable at $c_0 = \frac{2a}{\sqrt{\lambda}}$.

**Proof.** Let $U$ be the interior of $(S_0 \cup S_{-(n-1)}) \setminus \{z \in B_\lambda \cup T_\lambda; \ G_\lambda(z) \geq 1\}$ and $V = \overline{C \setminus (\{z \in B_\lambda; \ G_\lambda(z) \geq n\} \cup [-\infty, v_\lambda^-])}$. One can easily verify that $f_\lambda : U \rightarrow V$ is a quadratic-like map.

Because $C_\lambda \cap A_\lambda = \emptyset$, the critical orbit $\{f_\lambda^k(c_0); \ k \geq 0\}$ is contained in $U \cap \mathbb{R}^+$. This implies that $(f_\lambda, U, V)$ is a 1-renormalization of $f_\lambda$ at $c_0$. □

Let $K_{c_0} = \bigcap_{k \geq 0} f_\lambda^{-k}(U)$ be the small filled Julia set of the renormalization $(f_\lambda, U, V)$, $\beta_{c_0}$ be the $\beta$-fixed point and $\beta_{c_0}'$ be the preimage of $\beta_{c_0}$. It is easy to check that $K_{c_0}$ is symmetric about the real axis and $K_{c_0} \cap \mathbb{R}^+$ is a connected and closed interval.

**Proposition 7.3.** $K_{c_0} \cap \partial B_\lambda = \{\beta_{c_0}\}$.

**Proof.** As with the proof of Proposition 7.2, the idea of the proof is to construct a Jordan curve $C$ that separates $K_{c_0} \setminus \{\beta_{c_0}\}$ from $\partial B_\lambda \setminus \{\beta_{c_0}\}$.

We first show that $\beta_{c_0}$ is the landing point of the zero external ray $R_\lambda(0)$. Note that rational external rays (i.e., external rays with a rational angle) always land. Let $z_0$ be the landing point of $R_\lambda(0)$. Obviously, $R_\lambda(z_0) \subset \mathbb{R}^+$ and $z_0$ is a fixed point of $f_\lambda$, which implies that $z_0 \in U \cap \mathbb{R}^+$, and
the orbit of \( z_0 \) does not escape from \( U \). Therefore, \( z_0 \in K_{c_0} \). Because \( R_\lambda(0) \) is an \( f_\lambda \)-invariant ray that lands at \( z_0 \), we conclude \( z_0 = \beta_{c_0} \) based on Lemma 5.2.

Let \( K = K_{c_0} \cup [\beta_{c_0}, +\infty) \cup (\sim K_{c_0}) \cup [\sim \infty, \sim \beta_{c_0}] \). One can easily verify \( f_\lambda^{-1}(K) = \bigcup_{n=1}^{\infty} \omega(K_{c_0} \cup [0, +\infty)) \). The set \( Y = \bar{\mathbb{C}} \setminus K \) is a disk, and its preimage \( f_\lambda^{-1}(Y) \) consists of \( 2n \) components that are symmetric about the origin under the rotation \( z \mapsto e^{i\pi/n}z \). For each component \( X \) of \( f_\lambda^{-1}(Y) \), \( f_\lambda : X \to Y \) is a conformal map. Let \( X_0 \) be the component of \( f_\lambda^{-1}(Y) \) that is contained in \( S_0 \) and \( g \) be the inverse map of \( f_\lambda : X_0 \to Y \). Based on the Denjoy–Wolff theorem, the sequence of maps \( \{g^k; k \geq 0\} \) converges locally and uniformly in \( Y \) to a constant, say \( x \). Because \( g(\mathbb{R} \cap V) \subset X_0 \cap U \), we conclude \( x \in K_{c_0} \).

Let \( \Delta \) be the component of \( \bar{\mathbb{C}} \setminus (\bar{B}_\lambda \cup K_{c_0} \cup (\sim K_{c_0}) \cup \mathbb{R}) \) that intersects with \( T_\lambda \) and lies in the upper half plane.

**Claim.** There is a path \( \mathcal{L} \subset \Delta \cup \{0\} \) stemming from \( T_\lambda \) and converging to \( \beta_{c_0} \). More precisely, \( \mathcal{L} \) can be parameterized as \( \mathcal{L} : [0, +\infty) \to \Delta \cup \{0\} \) such that \( \mathcal{L}(0) = 0, \mathcal{L}((0, +\infty)) \subset \Delta \) and \( \lim_{t \to +\infty} \mathcal{L}(t) = \beta_{c_0} \).

Let \( p_0 = \frac{2\sqrt{-1}}{\lambda} \) be the preimage of 0 that lies in \( S_0 \) and \( \gamma_0 = [0, p_0] \) be the segment connecting 0 with \( p_0 \). Then, \( \gamma_0 \cap (K_{c_0} \cup \partial B_\lambda) = \emptyset \). Indeed, \( \gamma_0 \cap K_{c_0} = \emptyset \) follows from the fact that \( f_\lambda(\gamma_0) \cap K_{c_0} \subset \mathbb{R} \cap K_{c_0} = \emptyset \). In the following, we show that \( \gamma_0 \cap \partial B_\lambda = \emptyset \). It suffices to show that \( B_\lambda \cap D = \emptyset \), where \( D = \{z \in \mathbb{C}; |z| < \frac{2}{\sqrt{-\lambda}}\} \). Otherwise, \( B_\lambda \cap D \neq \emptyset \) implies \( B_\lambda \cap \partial D \neq \emptyset \). Because \( \varphi : z \mapsto \frac{\sqrt{-\lambda}}{z} \) maps \( B_\lambda \) onto \( T_\lambda \) and the restriction \( \varphi|_{\partial D} \) is the identity map, we have \( B_\lambda \cap \partial D = \varphi(B_\lambda \cap \partial D) = T_\lambda \cap \partial D \). But this implies \( B_\lambda \cap T_\lambda \neq \emptyset \), contradiction.

Note that \( g \) maps \( \gamma_0 \) outside \( D \) and \( g(\gamma_0) \) connects \( p_0 \) with a preimage of \( p_0 \) that lies inside \( S_0 \). Let \( \mathcal{L} = \bigcup_{k \geq 0} \varphi^k(\gamma_0) \). By construction, \( \mathcal{L} \cap (K_{c_0} \cup \partial B_\lambda) = \emptyset \), and \( \mathcal{L} \) converges to \( x \in K_{c_0} \). If \( f_\lambda(\mathcal{L}) \subset \mathcal{L} \cup f_\lambda(\gamma_0) \cup \mathcal{L} \), we conclude \( x = \beta_{c_0} \) based on Lemma 5.2.

Let \( \mathcal{C} = \mathcal{L} \cup \mathcal{L}^* \cup \{\beta_{c_0}\} \), where \( \mathcal{L}^* = \{z; z \in \mathcal{L}\} \). \( \mathcal{C} \) is a Jordan curve separating \( K_{c_0} \setminus \{\beta_{c_0}\} \) from \( \partial B_\lambda \setminus \{\beta_{c_0}\} \). The conclusion follows.

**Remark 7.1.** Based on the proof of Proposition 7.3, we conclude

\[
\partial B_\lambda \cap \mathbb{R} = \{\pm \beta_{c_0}\}, \quad K_{c_0} \cap \mathbb{R} = [\beta_{c_0}', \beta_{c_0}], \quad \partial T_\lambda \cap \mathbb{R} = \{\pm \beta_{c_0}'\}.
\]

**Corollary 7.2.** Suppose \( \lambda \in \mathbb{R}^+ \) and \( C_\lambda \cap A_\lambda = \emptyset \); then, \( \partial B_\lambda \) is locally connected.

**Proof.** By Proposition 7.3, if \( n \) is odd, then \( P(f_{\lambda_1}) \cap \partial B_\lambda \subset (\sim K_{c_0} \cup K_{c_0}) \cap \partial B_\lambda \subset \{\pm \beta_{c_0}\} \); if \( n \) is even, then \( P(f_{\lambda_1}) \cap \partial B_\lambda \subset K_{c_0} \cap \partial B_\lambda \subset \{\beta_{c_0}\} \). If \( \beta_{c_0} \) is a parabolic point, then \( f_\lambda \) is geometrically finite, and the local connectivity of \( \partial B_\lambda \) follows from [29]. Otherwise, based on Propositions 6.1 and 6.2, \( \partial B_\lambda \) is also locally connected.

**7.4. Local connectivity implies higher regularity**

At this point, we have already proven that \( \partial B_\lambda \) is locally connected if the Julia set is not a Cantor set. Based on the arguments of Devaney [5], we prove the following proposition, which will lead to Theorem 1.1.

**Proposition 7.4.** If \( \partial B_\lambda \) is locally connected, then \( \partial B_\lambda \) is a Jordan curve.
Proof. Let $W_0$ be the component of $\mathbb{C} - \overline{B}_\lambda$ containing 0. It is obvious that $\partial W_0 \subset \partial B_\lambda$, $T_\lambda \subset W_0$, $\partial T_\lambda \subset \overline{W}_0$. Based on Lemma 2.1, $e^{i\pi/n} W_0 = W_0$.

Recall that $H_\lambda(z) = \sqrt{\lambda/z}$, so $H_\lambda(\partial W_0) \subset H_\lambda(\partial B_\lambda) = \partial T_\lambda \subset \overline{W}_0$. Because $\partial B_\lambda$ is locally connected, $\partial W_0$ is locally connected. It follows that $H_\lambda(\partial C - W_0) \subset \partial T_\lambda \subset W_0$.

Recall that $H_\lambda(z) = \sqrt{\lambda/z}$, so $H_\lambda(\partial W_0) \subset H_\lambda(\partial B_\lambda) = \partial T_\lambda \subset W_0$. Because $\partial B_\lambda$ is locally connected, $\partial W_0$ is locally connected. It follows that $\overline{C} - W_0$ is connected and $H_\lambda(\overline{C} - W_0) \subset W_0$.

Now, we show that $f_{\lambda}^{-1}(0) \subset W_0$. If not, $f_{\lambda}^{-1}(0) \cap (\overline{C} - W_0) \neq \emptyset$. Based on the symmetry of $f_{\lambda}^{-1}(0)$ and $\overline{C} - W_0$, we have $f_{\lambda}^{-1}(0) \subset \overline{C} - W_0$. This will contradict the fact that $f_{\lambda}^{-1}(0) = H_\lambda(f_{\lambda}^{-1}(0)) \subset H_\lambda(\overline{C} - W_0) \subset W_0$.

Because no point on $\partial W_0$ can be mapped into $W_0$, we have $f_{\lambda}^{-1}(W_0) \subset W_0$ and $f_{\lambda}^{-1}(\overline{W}_0) \subset \overline{W}_0$. Take a point $z \in \partial W_0$; we have $\partial B_\lambda \subset J(f_{\lambda}) = \bigcup_{k \geq 0} f_{\lambda}^{-k}(z) \subset W_0$ and $\partial B_\lambda \subset \partial W_0$. Therefore, $\partial W_0 = \partial B_\lambda$.

Now, we show that $\partial B_\lambda$ is a Jordan curve. If two different external rays, say $R_\lambda(t_1)$ and $R_\lambda(t_2)$, land at the same point $p \in \partial B_\lambda$, then $R_\lambda(t_1) \cup R_\lambda(t_2)$ decomposes $\partial B_\lambda$ into two parts. It turns out that $\partial W_0 \neq \partial B_\lambda$, which is a contradiction. \( \square \)

The aim of this section is to prove Theorem 1.3, as follows:

Proof of Theorem 1.3. By Theorem 1.1 and Proposition 6.1, it suffices to show that $f_\lambda$ satisfies the BD condition on $\partial B_\lambda$. First, we deal with three special cases:

Case 1. The critical orbit escapes to infinity.

Case 2. The parameter $\lambda \in \mathbb{R}^+$ and $\partial B_\lambda$ contains no parabolic point.

Case 3. The map $f_\lambda$ is critically finite.

In Case 1, $P(f_\lambda) \cap \partial B_\lambda = \emptyset$. Based on Proposition 6.2, $f_\lambda$ satisfies the BD condition on $\partial B_\lambda$. For Case 2, by Proposition 7.3, either $P(f_\lambda) \cap \partial B_\lambda = \emptyset$ or $P(f_\lambda) \cap \partial B_\lambda = \{\beta_c\}$ or $P(f_\lambda) \cap \partial B_\lambda = \{\pm \beta_c\}$. In either case, $\beta_c$ is a repelling fixed point of $f_\lambda$. By Proposition 6.1, $f_\lambda$ satisfies the BD condition on $\partial B_\lambda$. For Case 3, $f_\lambda$ satisfies the BD condition on $\partial B_\lambda$ by Corollary 6.2.

In the remaining cases, we can use the Yoccoz puzzle to study the higher regularity of $\partial B_\lambda$. There are two remaining cases:

Case 4. $\partial B_\lambda$ contains no critical point.

Case 5. $C_\lambda \subset \partial B_\lambda$ and all critical points in $C_\lambda$ are non-recurrent.

In either case, by Proposition 4.1, we can find an admissible graph $G_\lambda(\theta_1, \ldots, \theta_N)$. With respect to the Yoccoz puzzle induced by this graph, we consider the critical tableaux. For Case 4, there are two possibilities:

Case 4.1. Some $T(c)$ with $c \in C_\lambda$ is periodic.

Case 4.2. No $T(c)$ with $c \in C_\lambda$ is periodic.

For Case 4.1, we conclude from Proposition 7.2 that $\#(P(f_\lambda) \cap \partial B_\lambda) < \infty$. Because $\partial B_\lambda$ contains no parabolic point, all periodic points in $P(f_\lambda) \cap \partial B_\lambda$ are repelling. Thus, based on Proposition 6.2, $f_\lambda$ satisfies the BD condition on $\partial B_\lambda$. 


For Case 4.2, we have already shown that $\text{End}(c) = \bigcap_{d \geq 0} \overline{P_d(c)} = \{c\}$ for $c \in C_\lambda$ in the proof of Proposition 7.1. Thus, we can choose a $d_0$ large enough such that

$$\text{Eucl. diam}(P_{d_0}(c)) < \text{Eucl. dist}(c, \partial B_\lambda).$$

For $d \geq d_0$, let $U_d$ be the union of all depth pieces of depth $d$ that intersect with $\partial B_\lambda$ and $V_d$ be the interior of $\overline{U_d}$. For every $u \in \partial B_\lambda$, there is a number $\varepsilon_u > 0$ such that $B(u, \varepsilon_u) \subset V_{d_0}$. For any $m \geq 0$ and any component $U_m(u)$ of $f_\lambda^{-m}(B(u, \varepsilon_u))$ intersecting with $\partial B_\lambda$, $U_m(u) \subset V_{d_0+m} \subset V_{d_0}$. By the choice of $d_0$, the sequence $U_m(u) \rightarrow \cdots \rightarrow f_\lambda^{m-1}(U_m(u)) \rightarrow B(u, \varepsilon_u)$ meets no critical point of $f_\lambda$; thus, $f_\lambda^m : U_m(u) \rightarrow B(u, \varepsilon_u)$ is a conformal map. Therefore, in this case, $f_\lambda$ satisfies the BD condition on $\partial B_\lambda$.

In the following, we deal with Case 5. Again, based on Proposition 7.1, $\text{End}(c) = \{c\}$ for $c \in C_\lambda$. Thus, in this case one can verify that $\partial B_\lambda$ contains no recurrent critical point if and only if all tableaux $T(c)$ with $c \in C_\lambda$ are non-critical. Based on Lemma 5.1, $f_\lambda$ is critically finite. It follows from Corollary 6.1 that $f_\lambda$ satisfies the BD condition on $\partial B_\lambda$. □

7.5. Corollaries

In this section, we present some corollaries of Theorem 1.1.

**Proposition 7.5.** If $\partial B_\lambda$ contains a parabolic cycle, then the multiplier of the cycle is 1 and the Julia set $J(f_\lambda)$ contains a quasi-conformal copy of the quadratic Julia set of $z \mapsto z^2 + 1/4$.

**Proof.** Suppose $C = \{z_0, f_\lambda(z_0), \ldots, f_\lambda^q(z_0) = z_0\}$ is a parabolic cycle on $\partial B_\lambda$. We will first consider the case $\lambda \in \mathbb{R}^+$, then deal with the case $\lambda \in \mathcal{H}$.

First, suppose $\lambda \in \mathbb{R}^+$. By Lemma 7.2 and Proposition 7.3, $f_\lambda$ is 1-renormalizable at $c_0$ and $P(f_\lambda) \cap \partial B_\lambda \subset (-K_{c_0} \cup K_{c_0}) \cap \partial B_\lambda = \{\pm \beta_{c_0}\}$. Because a parabolic point must attract a critical point, we conclude that $\beta_{c_0}$ is a parabolic fixed point of $f_\lambda$. Therefore, $(f_\lambda, U, V)$ is quasi-conformally conjugate to a quadratic polynomial $z \mapsto z^2 + \mu$ with a $\beta$-fixed point that is also a parabolic point, thus $\mu = 1/4$. The conclusion follows in this case.

In the following, we deal with the case $\lambda \in \mathcal{H}$. Based on Proposition 4.1, we can find an admissible graph $G_\lambda(\theta_1, \ldots, \theta_N)$. Based on Proposition 3.4, the parabolic cycle $C$ avoids the graph $G_\lambda(\theta_1, \ldots, \theta_N)$. With respect to the Yoccoz puzzle induced by this graph and with an argument similar to that used to prove Corollary 5.1, we conclude that there is a critical point $c \in C_\lambda$ and a point $z \in C$ such that $P_d(z) = P_d(c)$ for all $d \geq 0$. Thus, the tableau $T(c)$ is periodic. Suppose the period of $T(c)$ is $k$. It is obvious that $k$ is a divisor of $q$. By Lemma 5.1, when $d_0$ is large enough, the triple $(\epsilon f_\lambda^p, P_{d_0+p}(c), P_{d_0}(c))$ is either a $k$-renormalization of $f_\lambda$ at $c$ (in this case, $(\epsilon, p) = (1, k)$) or a $k/2$-renormalization of $f_\lambda$ at $c$ (in this case, $(\epsilon, p) = (-1, k/2)$). Moreover, the small filled Julia set $K_c = \text{End}(c) = \bigcap_{d \geq 0} P_d(c)$ and $z \in K_c \cap \partial B_\lambda$.

On the other hand, based on Lemma 5.3, there is a unique external ray $R_\lambda(t)$ landing at $\beta_c$, which is the $\beta$-fixed point of the renormalization $(\epsilon f_\lambda^p, P_{d_0+p}(c), P_{d_0}(c))$. Note that we have already proved that $\partial B_\lambda$ is a Jordan curve; the intersection $\partial B_\lambda \cap \overline{P_d(c)}$ shrinks to a single point as $d \rightarrow \infty$. Thus, we have $K_c \cap \partial B_\lambda = \{\beta_c\}$. By the previous argument, $\beta_c = z$.

Based on the straightening theorem of Douady and Hubbard, $(\epsilon f_\lambda^p, P_{d_0+p}(c), P_{d_0}(c))$ is quasi-conformally conjugate to a quadratic polynomial $p_\mu(z) = z^2 + \mu$ in a neighborhood of the small filled Julia set $K_c$. For this quadratic polynomial, the $\beta$-fixed point is also a parabolic point, thus $\mu = 1/4$. Therefore, the Julia set $J(f_\lambda)$ contains a quasi-conformal copy of the quadratic
Julia set of $z \mapsto z^2 + 1/4$. Because the multiplier of the parabolic point of $z \mapsto z^2 + 1/4$ is 1, it turns out that $(\epsilon f_{\lambda}^k)'(z) = 1$, $(f_{\lambda}^k)'(z) = 1$ and $(f_{\lambda}^k)'(z) = 1$. □

**Proposition 7.6.** Suppose $f_{\lambda}$ has no Siegel disk and the Julia set $J(f_{\lambda})$ is connected, then every Fatou component is a Jordan domain.

**Proof.** By Proposition 7.4 and the fact that $H_{\lambda}(B_{\lambda}) = T_{\lambda}$, we conclude that both $T_{\lambda}$ and $B_{\lambda}$ are Jordan domains.

If the critical orbit tends to $\infty$, then the Julia set is a Sierpinski curve that is locally connected, and all Fatou components are quasi-disks (by Proposition 6.1).

If the critical orbit remains bounded, then for any $U \in \mathcal{P} \setminus \{T_{\lambda}, B_{\lambda}\}$, there is a smallest integer $k \geq 1$ such that $f_{\lambda}^k : U \to T_{\lambda}$ is a conformal map. Thus, if two radial rays $R_U(\theta_1)$ and $R_U(\theta_2)$ land at the same point, then $R_{T_{\lambda}}(\theta_1) = f_{\lambda}^k(R_U(\theta_1))$ and $R_{T_{\lambda}}(\theta_2) = f_{\lambda}^k(R_U(\theta_2))$ also land at the same point. This implies that $U$ is also a Jordan domain. If there are other Fatou components, then they are eventually mapped to a parabolic basin or an attracting basin. By Proposition 5.1, the map is either renormalizable or $*$-renormalizable. It is known that every bounded Fatou component of a quadratic polynomial without a Siegel disk is a Jordan disk; it turns out that all Fatou components of $f_{\lambda}$ are Jordan disks in this case. □

**Proposition 7.7.** If $f_{\lambda}$ has a Cremer point, then the Cremer point cannot lie on the boundary of any Fatou component. In other words, all Cremer points are buried on the Julia set.

**Proof.** Suppose $f_{\lambda}$ has a Cremer point $z$, then the Fatou set $F(f_{\lambda}) = \bigcup_{k \geq 0} f_{\lambda}^{-k}(B_{\lambda})$. If $z$ lies on the boundary of some Fatou component, then after iterations, one sees that $z \in \partial B_{\lambda}$. By Theorem 1, there is a periodic external ray $R_{\alpha}(t)$ landing at $z$. But this is a contradiction because, by the Snail Lemma, every periodic external ray can only land at a parabolic point or a repelling point (see [20]). □

8. Local connectivity of the Julia set $J(f_{\lambda})$

In this section, we study the local connectivity of the Julia set $J(f_{\lambda})$. We will prove Theorem 1.3.

The proof is based on the ‘Characterization of Local Connectivity’ (Proposition 8.1 (see [31])) and the ‘Shrinking Lemma’ (Proposition 8.2 (see [29] or [17])), as follows.

**Proposition 8.1.** A connected and compact set $X \subset \hat{\mathbb{C}}$ is locally connected if and only if it satisfies the following conditions:

1. Every component of $\hat{\mathbb{C}} \setminus X$ is locally connected.
2. For any $\epsilon > 0$, there are only a finite number of components of $\hat{\mathbb{C}} \setminus X$ with spherical diameter greater than $\epsilon$.

**Proposition 8.2.** Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map and $D$ be a topological disk whose closure $\overline{D}$ has no intersection with the post-critical set $P(f)$. Then, either $\overline{D}$ is contained in a Siegel disk or a Herman ring or for any $\epsilon > 0$ there are at most finitely many iterated preimages of $D$ with spherical diameter greater than $\epsilon$. 

Proof of Theorem 1.3. 1. If $f_\lambda$ is geometrically finite, then $J(f_\lambda)$ is locally connected (see [29]). Otherwise, the Fatou set $F(f_\lambda) = \bigcup_{k \geq 0} f_\lambda^{-k}(B_\lambda)$. Because $B_\lambda \cap P(f_\lambda) = \emptyset$, we conclude base on Shrinking Lemma that for any $\epsilon > 0$, there are at most finitely many iterated preimages of $B_\lambda$ with spherical diameter greater than $\epsilon$. Based on Proposition 8.1, $J(f_\lambda)$ is locally connected.

2. If $f_\lambda$ is neither renormalizable nor $\ast$-renormalizable, then the parameter $\lambda \in \mathcal{H}$ by Lemma 7.2. We can assume that $f_\lambda$ is not critically finite; otherwise, the Julia set is locally connected. Thus, based on Proposition 4.1, we can find an admissible graph. By Lemma 5.1, none of the tableaux $T(c)$ with $c \in C_\lambda$ are periodic. The local connectivity of $J(f_\lambda)$ follows from Proposition 7.1.

3. (The notations here are the same as in Section 7.3.) We need only consider the case when $f_\lambda$ is not geometrically finite. In this case, the Fatou set $F(f_\lambda) = \bigcup_{k \geq 0} f_\lambda^{-k}(B_\lambda)$. Note that for any $z > 0$, $f_\lambda(z) \geq 2\sqrt{z^n \cdot \frac{1}{z^n}} = 2\sqrt{\lambda} = v_\lambda^+$. Thus, $\{f_\lambda^k(v_\lambda^+); k \geq 0\} \subset [v_\lambda^+, \beta_\epsilon(0)]$.

If $v_\lambda^+ = \beta_\epsilon(0)$, one can easily verify that the triple $(f_\lambda, U, V)$ is quasi-conformally conjugate to the quadratic polynomial $z \mapsto z^2 - 2$, which is critically finite. Therefore, $f_\lambda$ is also critically finite, and the Julia set is locally connected.

If $v_\lambda^+ > \beta_\epsilon(0)$, then $T_\lambda \cap [v_\lambda^+, \beta_\epsilon(0)] = \emptyset$ by Remark 7.1. Because $P(f_\lambda) \subset [-\beta_\epsilon(0), v_\lambda^-] \cup [v_\lambda^+, \beta_\epsilon(0)] \cup \{\infty\}$, we have $T_\lambda \cap P(f_\lambda) = \emptyset$. Based on Proposition 8.2, for any $\epsilon > 0$ there are at most finitely many iterated preimages of $T_\lambda$ with spherical diameter greater than $\epsilon$. Based on Proposition 8.1, the Julia set is locally connected. \qed

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