VARIATIONAL INEQUALITIES FOR SET-VALUED VECTOR FIELDS ON RIEMANNIAN MANIFOLDS: CONVEXITY OF THE SOLUTION SET AND THE PROXIMAL POINT ALGORITHM

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Abstract. We consider variational inequality problems for set-valued vector fields on general Riemannian manifolds. The existence results of the solution, convexity of the solution set, and the convergence property of the proximal point algorithm for the variational inequality problems for set-valued mappings on Riemannian manifolds are established. Applications to convex optimization problems on Riemannian manifolds are provided.

Key words. variational inequalities, Riemannian manifold, monotone vector fields, proximal point algorithm, convexity of solution set

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1. Introduction. Various problems posed on manifolds arise in many natural contexts. Classical examples are given by some numerical problems such as eigenvalue problems and invariant subspace computations, constrained minimization problems, and boundary value problems on manifolds, etc.; see, for example, [1, 13, 17, 26, 37, 38, 39]. Recent interests are focused on extending some classical and important results for solving these problems on linear spaces to the setting of manifolds. For example, some numerical methods such as Newton’s method, the conjugate gradient method, the trust-region method, and their modifications for optimization problems on linear spaces are extended to the Riemannian manifolds setting [2, 11, 16, 24]. A theory of subdifferential calculus for functions defined on Riemannian manifolds is developed in [3, 20], where these results are applied to show existence and uniqueness of viscosity solutions to Hamilton–Jacobi equations defined on Riemannian manifolds and to study constrained optimization problems and nonclassical problems of calculus of variations on Riemannian manifolds. The important notions of monotonicity in linear spaces were extended to Riemannian manifolds and have been studied extensively in [21, 22, 27, 28, 29, 30, 31, 40], while weak sharp minima for constrained optimization problems on Riemannian manifolds are explored recently in [23], where various notions of weak sharp minima are extended and their complete characterizations are established.

Variational inequalities on \( \mathbb{R}^n \) are powerful tools for studying constrained optimization problems and equilibrium problems, as well as complementary problems, and have been studied extensively; see, for example, the survey [18] and the book [32].

Variational inequality problems on Riemannian manifolds were first introduced and studied in [26] by Németh for univalued vector fields on Hadamard manifolds.
Németh established in [26] some basic results on existence and uniqueness of the solution for variational inequality problems on Hadamard manifolds and proposed an open problem on how to extend the existence and uniqueness results from Hadamard manifolds to Riemannian manifolds. This problem was solved completely in [25] by Li et al. The famous proximal point algorithm for optimization problems and for variational inequality problems on Hilbert spaces were extended to the setting of Hadamard manifolds, respectively, in [14] and [21], where the well-posedness and convergence results of the proximal point algorithm on Hadamard manifolds were established. Another basic and interesting problem for variational inequalities is the convexity of the solution set, which, unlike in the linear cases, seems nontrivial even for univalued vector fields on Hadamard manifolds.

Our interest in the present paper is the study of variational inequality problems for set-valued vector fields on general Riemannian manifolds (not necessarily Hadamard manifolds). The purpose of the present paper is to explore the existence of the solution and the convexity of the solution set, as well as the convergence property of the proximal point algorithm for the variational inequality problems for set-valued mappings on Riemannian manifolds. Compared with the corresponding ones for linear space, the problems considered here for the general Riemannian manifold cases are much more complicated, and most of the known techniques in linear space setting does not work; for example, even in a Hadamard manifold we do not have the following property (which holds trivially in a linear space and plays a crucial role in the study of the convexity problem of the solution set): The image of any linear segment in the tangent space under an exponential map is a geodesic segment in the underlying manifold. Most of the main results obtained in the present paper extend and improve the corresponding ones for univalued vector fields on Hadamard manifolds and/or Riemannian manifolds, while the convexity results of solution sets are completely new for set-valued vector fields on Hadamard and Riemannian manifolds.

The paper is organized as follows. The next section contains some necessary notation, notion, and preliminary results. The existence and uniqueness of the solution set of the variational inequality problems for set-valued mappings on general Riemannian manifolds are established in section 3. In sections 4 and 5, the convexity results of the solution set and the convergence results of the proximal point algorithm of the variational inequality problems are provided for set-valued mappings on Riemannian manifolds of curvature bounded above. Applications of our results on convergence of the proximal point algorithm to convex optimization problems on Riemannian manifolds are presented in section 6.

2. Notation and preliminary results. We begin with some necessary notation, notions, and preliminary results about Riemannian manifolds that will be used in the next sections. The readers are referred to some textbooks for details, for example, [5, 12, 33, 36].

Let $M$ be a finite-dimensional Riemannian manifold with the Levi–Civita connection $\nabla$ on $M$. Let $x \in M$, and let $T_x M$ denote the tangent space at $x$ to $M$. We denote by $\left< \cdot , \cdot \right>_x$ the scalar product on $T_x M$ with the associated norm $\| \cdot \|_x$, where the subscript $x$ is sometimes omitted. For $x, y \in M$, let $c : [0, 1] \to M$ be a piecewise smooth curve joining $x$ to $y$. Then the arc-length of $c$ is defined by $l(c) := \int_0^1 \| \dot{c}(t) \| \, dt$, while the Riemannian distance from $x$ to $y$ is defined by $d(x, y) := \inf_c l(c)$, where the infimum is taken over all piecewise smooth curves $c : [0, 1] \to M$ joining $x$ to $y$. Recall that a curve $c : [0, 1] \to M$ joining $x$ to $y$ is a geodesic if

\begin{equation}
(2.1) \quad c(0) = x, \; c(1) = y, \; \text{and} \; \nabla_c \dot{c} = 0 \; \text{on} \; [0, 1],
\end{equation}
and a geodesic \( c : [0, 1] \to M \) joining \( x \) to \( y \) is minimal if its arc-length equals its Riemannian distance between \( x \) and \( y \). By the Hopf–Rinow theorem (cf. [12]), if \( M \) is additionally complete and connected, then \((M, d)\) is a complete metric space, and there is at least one minimal geodesic joining \( x \) to \( y \). Moreover, the exponential map at \( x \) \( \exp_x : T_x M \to M \) is well-defined on \( T_x M \). Clearly, a curve \( c : [0, 1] \to M \) is a minimal geodesic joining \( x \) to \( y \) if and only if there exists a vector \( v \in T_x M \) such that \( \|v\| = d(x, y) \) and \( c(t) = \exp_x(tv) \) for each \( t \in [0, 1] \).

Let \( \gamma \) be a geodesic. We use \( P_{\gamma, v} \) to denote the parallel transport on the tangent bundle \( TM \) (defined below) along \( \gamma \) with respect to \( \nabla \), which is defined by

\[
P_{\gamma, \gamma(a)}(v) = V(\gamma(b)) \quad \text{for any } a, b \in \mathbb{R} \text{ and } v \in T_{\gamma(a)} M,
\]

where \( V \) is the unique vector field satisfying \( \nabla_{\gamma(t)} V = 0 \) for all \( t \) and \( V(\gamma(a)) = v \). Then, for any \( a, b \in \mathbb{R} \), \( P_{\gamma, \gamma(b)}(\gamma(a)) \) is an isometry from \( T_{\gamma(a)} M \) to \( T_{\gamma(b)} M \). We will write \( P_{\gamma, y, x} \) instead of \( P_{\gamma, y, x} \) in the case when \( \gamma \) is a minimal geodesic joining \( x \) to \( y \) and no confusion arises.

For a Banach space or a Riemannian manifold \( Z \), we use \( B_Z(p, r) \) and \( \overline{B_Z(p, r)} \) to denote, respectively, the open metric ball and the closed metric ball at \( p \) with radius \( r \), that is,

\[
B_Z(p, r) = \{ q \in Z : d(p, q) < r \} \quad \text{and} \quad \overline{B_Z(p, r)} = \{ q \in Z : d(p, q) \leq r \}.
\]

We often omit the subscript \( Z \) if no confusion arises. Recall that, for a point \( x \in M \), the convexity radius at \( x \) is defined by

\[
r_x := \sup \left\{ r > 0 : \text{each ball in } B(x, r) \text{ is strongly convex and each geodesic in } B(x, r) \text{ is minimal} \right\}.
\]

Clearly, if \( M \) is a Hadamard manifold, then \( r_x = +\infty \) for each \( x \in M \). Let

\[
TM := \bigcup_{x \in M} \{x\} \times T_x M
\]

denote the tangent bundle. Then \( TM \) is a Riemannian manifold with the natural differential structure and the Riemannian metric; see, for example, [12]. Thus, the following proposition can be proved by a direct application of the definition of parallel transports (cf. [21, Lemma 2.4] in the case when \( M \) is a Hadamard manifold).

**PROPOSITION 2.1.** Let \( z_0 \in M \), and define the mapping \( P_{z_0} : TM \to T_{z_0} M \) by

\[
P_{z_0}(z, u) := P_{z_0, z} u \quad \text{for each } (z, u) \in TM.
\]

Then \( P_{z_0} \) is continuous on \( \bigcup_{x \in B(z_0, r_{z_0})} \{x\} \times T_x M \).

We denote by \( \Gamma_{x, y} \) the set of all geodesics \( c := \gamma_{xy} : [0, 1] \to M \) satisfying (2.1). Note that each \( c \in \Gamma_{x, y} \) can be extended naturally to a geodesic defined on \( \mathbb{R} \) in the case when \( M \) is complete. Definition 2.2 below presents the notions of the different kinds of convexities, where items (a) and (b) are known in [41], while item (c) is known in [7]; see also [23, 25].

**DEFINITION 2.2.** Let \( A \) be a nonempty subset of \( M \). The set \( A \) is said to be

(a) weakly convex if, for any \( p, q \in A \), there is a minimal geodesic of \( M \) joining \( p \) to \( q \) lying in \( A \);

(b) strongly convex if, for any \( p, q \in A \), there is just one minimal geodesic of \( M \) joining \( p \) to \( q \) and it is in \( A \);
(c) locally convex if, for any \( p \in \overline{A} \), there is a positive \( \varepsilon > 0 \) such that \( A \cap B(p, \varepsilon) \) is strongly convex.

Clearly, the following implications hold for a nonempty set \( A \) in \( M \):

\[
\text{(2.3)} \quad \text{The strong convexity} \implies \text{the weak convexity} \implies \text{the local convexity}.
\]

**Remark 2.1.** Recall (cf. [36]) that \( M \) is a Hadamard manifold if it is a simple connected and complete Riemannian manifold of nonpositive sectional curvature. In a Hadamard manifold, the geodesic between any two points is unique and the exponential map at each point of \( M \) is a global diffeomorphism. Therefore, all convexities in a Hadamard manifold coincide and are simply called the convexity.

Recall from [39, p. 110] (see also [36, Remark V.4.2]) that a point \( o \in M \) is called a pole of \( M \) if \( \exp_o : T_o M \to M \) is a global diffeomorphism, which implies that for each point \( x \in M \), the geodesic of \( M \) joining \( o \) to \( x \) is unique. For the existence result of the solutions of the problem (3.1), the notion of the weak pole of \( A \) in the following definition was introduced in [25].

**Definition 2.3.** A point \( o \in A \) is called a weak pole of \( A \) if for each \( x \in A \), the minimal geodesic of \( M \) joining \( o \) to \( x \) is unique and lies in \( A \).

Clearly, any subset with a weak pole is connected. Let \( p \in A \). For the following proposition, which is known in [36, pp. 169–171], we use \( \epsilon(p) \) to denote the supremum of the radius \( r \) such that the set \( A \cap B(p, r) \) is strongly convex, that is,

\[
\epsilon(p) := \sup\{r > 0 : A \cap B(p, r) \text{ is strongly convex}\}.
\]

**Proposition 2.4.** Suppose that \( M \) is a complete Riemannian manifold. Let \( A \subseteq M \) be a nonempty closed connected and locally convex subset. Then, there exists a connected (embedded) \( k \)-dimensional totally geodesic submanifold \( N \) of \( M \) possessing the following properties:

(i) \( A = \overline{N} \) and \( \gamma_{xy}(0,1) \subseteq N \) for any \( p \in A \), \( x \in B(p, \epsilon(p)) \cap A \), and \( \gamma_{xy} \in \Gamma_{x,y} \),

(ii) \( \gamma_{xy}(t) \notin A \) for any \( y \notin N \) and any \( t \in (1, t_0] \), where \( t_0 > 1 \) is such that \( \gamma_{xy}(0, t_0] \subseteq B(p, \epsilon(p)) \).

Following [36, p. 171], the sets \( \text{int}_R A := N \) and \( \partial_R A := A \setminus N \) are called the (relative) interior and the (relative) boundary of \( A \), respectively. Let \( \text{int} A \) denote the topological interior of \( A \), that is, \( x \in \text{int} A \) if and only if \( B(x, \delta) \subseteq A \) for some \( \delta > 0 \). Some useful properties about the interiors are given in the following proposition. Let \( \bar{x} \in A \). We use \( F_{\bar{x}}A \) to denote the set of all feasible directions by

\[
\text{(2.4)} \quad F_{\bar{x}}A := \{ v \in T_{\bar{x}}M : \text{there exists } \bar{t} > 0 \text{ such that } \exp_{\bar{x}} tv \in A \text{ for any } t \in (0, \bar{t}) \}.
\]

**Remark 2.2.** Assume that \( M \) is complete and \( A \) is locally convex. Following [36, p. 171], we define the set \( C_{(\bar{x})} \) by

\[
C_{(\bar{x})} := \{ v \in T_{\bar{x}}M \setminus \{0\} : \text{there exists } t \in (0, r_{\bar{x}}) \text{ such that } \exp_{\bar{x}} t(v/\|v\|) \in N \} \cup \{0\}.
\]

Then one checks by definition that

\[
\text{(2.5)} \quad C_{(\bar{x})} \subseteq F_{\bar{x}}A \subseteq \overline{C_{(\bar{x})}} \quad \text{for any } \bar{x} \in A
\]

and

\[
\text{(2.6)} \quad C_{(\bar{x})} = F_{\bar{x}}A \quad \text{for any } \bar{x} \in N.
\]
Proposition 2.5. Suppose that $M$ is a complete Riemannian manifold and $A$ is a closed connected and locally convex subset of $M$. Let $N = \text{int}_R A$. Then the following assertions hold:

(i) $\bar{x} \in \text{int}_R A \iff F_x A = T_{\bar{x}} N \iff B(\bar{x}, \delta) \cap \exp_\bar{x}(T_{\bar{x}} N) \subseteq A$ for some $\delta > 0$.

(ii) $\bar{x} \in \text{int} A \iff F_x A = T_{\bar{x}} M$.

(iii) If $\text{int} A \neq \emptyset$, then $\text{int}_R A = \text{int} A$.

(iv) If $\bar{x} \in \text{int}_R A$ is a weak pole of $A$, then $\gamma_{x\bar{x}}((0, 1]) \subseteq \text{int}_R A$ for any $x \in A$.

Proof. In view of Remark 2.2, one easily sees that assertion (i) holds by [36, pp. 169–171], while assertion (ii) is clear by the definition of the interior. As for assertion (iii), we note that if $\bar{x} \in \text{int} A \neq \emptyset$, then $T_{\bar{x}} N = T_{\bar{x}} M$ by (ii), and so $N$ is an $m$-dimensional totally geodesic submanifold $N$ of $M$, where $m := \dim M$. This means that, for any $x \in N$, $T_x N$ is of dimension $m$, and so $T_x N = T_x M$. Thus (iii) is seen to hold by assertion (ii). Finally, assertion (iv) is known in [25, Proposition 4.3]. 

3. Existence and uniqueness results. Throughout the whole paper, we always assume that $M$ is a complete Riemannian manifold. Let $A \subseteq M$ be a nonempty set, and let $\Gamma_{x,y}^A$ denote the set of all $\gamma_{xy} \in \Gamma_{x,y}$ such that $\gamma_{xy} \subseteq A$. Let $V : A \rightarrow 2^T M$ be a set-valued vector field on $A$, that is, $\emptyset \neq V(x) \subseteq T_x M$ for each $x \in A$. The variational inequality problem considered here on Riemannian manifold $M$ is of finding $\bar{x} \in A$ such that

\begin{equation}
\exists \bar{v} \in V(\bar{x}) \text{ satisfying } \langle \bar{v}, \gamma_{\bar{x}y}(0) \rangle \geq 0 \text{ for each } y \in A \text{ and each } \gamma_{\bar{x}y} \in \Gamma_{\bar{x},y}^A.
\end{equation}

Any point $\bar{x} \in A$ satisfying (3.1) is called a solution of the variational inequality problem (3.1), and the set of all solutions is denoted by $S(V, A)$.

The following theorem on the existence of solutions of the variational problem (3.1) for continuous (univalued) vector fields on $A$ was proved in [25], which plays an important role for the study of this section.

Theorem 3.1. Let $A \subseteq M$ be a compact, locally convex subset of $M$, and let $V$ be a continuous vector field on $A$. Suppose that $A$ has a weak pole $o \in \text{int}_R A$. Then $S(V, A) \neq \emptyset$, that is, the variational inequality problem (3.1) admits at least one solution $\bar{x}$.

Write $A_R := A \cap B(o, R)$, where $R > 0$ and $o \in A$. Consider the following variational inequality problem: Find $x_R \in A_R$ and $v_R \in V(x_R)$ such that

\begin{equation}
\langle v_R, \gamma_{x_R y}(0) \rangle \geq 0 \text{ for each } y \in A_R \text{ and each } \gamma_{x_R y} \in \Gamma_{x_R, y}^{A_R}.
\end{equation}

The following proposition establishes the relationship between the solutions for problems (3.2) and (3.1), which will be used frequently for our study in what follows.

Proposition 3.2. Let $A \subseteq M$ be a nonempty subset, and let $V$ be a set-valued vector field on $A$. Then $\bar{x} \in A$ is a solution of the problem (3.1) if and only if there exist $R > 0$ and a point $o \in A$ such that $\bar{x} \in B(o, R)$ and $\bar{x}$ is a solution of the problem (3.2), that is,

\begin{equation}
\bar{x} \in S(V, A) \iff \exists R > 0, o \in A \text{ such that } \bar{x} \in S(V, A_R) \cap B(o, R).
\end{equation}

Proof. The necessity part is trivial. Hence we need only prove the sufficiency part. To this end, suppose there exist $o \in A$ and $R > 0$ such that $\bar{x} \in S(V, A_R)$, and so $\bar{x} \in B(o, R)$. Let $y \in A$ and $\gamma_{\bar{x}y} \in \Gamma_{\bar{x}, y}^A$. Then there exists $\delta > 0$ such that

$$||\delta \gamma_{\bar{x}y}(0)|| < \min\{r_{\bar{x}}, R - d(o, \bar{x})\},$$

where $r_{\bar{x}}$ is the radius of $B(\bar{x}, R)$.
Recall that $\Gamma$ is upper semicontinuous on any $A$ on space endowed with the product metric $d$ defined by $u$.

Let $\Gamma$ be a set-valued mapping from a field on $\epsilon$.

Since $\bar{\epsilon}$, consequently, $\bar{\epsilon}$.

Below we will extend the existence theorem to the case of set-valued vector fields on $A$. Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. Let $X \times Y$ be the product metric space endowed with the product metric $d$ defined by

$$d((x_1, y_1), (x_2, y_2)) := d_X(x_1, x_2) + d_Y(y_1, y_2) \quad \text{for any } (x_1, y_1), (x_2, y_2) \in X \times Y.$$

Let $\Gamma$ be a set-valued mapping from $X$ to $Y$. Let $\text{gph}(\Gamma)$ denote the graph of $\Gamma$ defined by

$$\text{gph}(\Gamma) := \{(x, y) \in X \times Y : y \in \Gamma(x)\}.$$ 

Recall that $\Gamma$ is upper semicontinuous on $X$ if for any $x_0 \in X$ and any open set $U$ containing $\Gamma(x_0)$, there exists a neighborhood $B(x_0, \delta)$ of $x_0$ such that $\Gamma(x) \subseteq U$ for any $x \in B(x_0, \delta)$.

In the following definition, we extend this notion to set-valued vector fields on manifolds; see, for example, [21].

**Definition 3.3.** Let $A \subseteq M$ be a nonempty subset and $V$ be a set-valued vector field on $A$. Let $x_0 \in A$. $V$ is said to be

(a) upper semicontinuous at $x_0$ if for any open set $W$ satisfying $V(x_0) \subseteq W \subseteq T_{x_0} M$, there exists an open neighborhood $U(x_0)$ of $x_0$ such that $P_{x_0, x} V(x) \subseteq W$ for any $x \in U(x_0) \cap A$.

(b) upper Kuratowski semicontinuous at $x_0$ if for any sequences $\{x_k\} \subseteq A$ and $\{u_k\} \subseteq TM$ with each $u_k \in V(x_k)$, relations $\lim_{k \to \infty} x_k = x_0$ and $\lim_{k \to \infty} u_k = u_0$ imply $u_0 \in V(x_0)$.

(c) upper semicontinuous (resp., upper Kuratowski semicontinuous) on $A$ if it is upper semicontinuous (resp., upper Kuratowski semicontinuous) at each $x_0 \in A$.

**Remark 3.1.** By definition, the following assertions hold:

(i) The upper semicontinuity implies the upper Kuratowski semicontinuity. The converse is also true if $A$ is compact and $V$ is compact-valued.

(ii) A set-valued vector field $V$ on $A$ is upper semicontinuous at $x_0 \in A$ if and only if the set-valued mapping $V : A \to 2^{TM}$ is upper semicontinuous at $x_0$; that is, for any open subset $W$ of $TM$ containing $V(x_0)$, there exists a neighborhood $U(x_0)$ of $x_0$ such that $V(x) \subseteq W$ for all $x \in U(x_0) \cap A$.

The following proposition is known in [4, Theorem 1].

**Proposition 3.4.** Let $X$ be a metric space without isolated points, and let $Y$ be a normed linear space. Let $\Gamma : X \to 2^Y$ be an upper semicontinuous mapping such that $\Gamma(x)$ is compact and convex for each $x \in X$. Then for each $\epsilon > 0$ there exists a continuous function $f_\epsilon : X \to Y$ such that $\text{gph}(f_\epsilon) \subseteq U(\text{gph}(\Gamma), \epsilon)$, where $U(\text{gph}(\Gamma), \epsilon)$ is an $\epsilon$-neighborhood of $\text{gph}(\Gamma)$:

$$U(\text{gph}(\Gamma), \epsilon) := \{(x, y) \in X \times Y : d((x, y), \text{gph}(\Gamma)) < \epsilon\}.$$

Consider a compact subset $A$ of $M$, and let $\{B(x_i, r_i) : i = 1, 2, \ldots, m\}$ be an open cover of $A$, that is, $A \subseteq \bigcup_{i=1}^m B(x_i, r_i)$, where, for each $i = 1, 2, \ldots, m$, $r_i := \frac{r_{x_i}}{2}$ and $r_{x_i}$ is the convexity radius at $x_i$. Fixing such an open cover and $x \in A$, we denote by $I(x)$ the index-set such that $i \in I(x)$ if and only if $x \in B(x_i, r_i)$.
Let $V(A)$ denote the set of all upper Kuratowski semicontinuous set-valued vector fields $V$ satisfying that $V(x)$ is compact and convex for each $x \in A$. By Proposition 3.4, we can verify the following existence result of $\epsilon$-approximations for set-valued vector fields on $A$.

**Lemma 3.5.** Suppose that $A$ is a connected compact subset of $M$ and that $V \in V(A)$. Let $\epsilon > 0$. Then, there exists a continuous vector field $V_\epsilon$ such that for any $x \in M$, there exist a subset $\{x_i^\epsilon : i \in I(x)\} \subseteq A$ and a point $y_i^\epsilon \in \text{conv}\{P_{x,x_i}P_{x_i,x_i}V(x_i^\epsilon) : i \in I(x)\}$ satisfying that

\begin{equation}
\|V_\epsilon(x) - y_i^\epsilon\| < \epsilon \quad \text{and} \quad d(x_i^\epsilon, x) < \epsilon \quad \text{for each } i \in I(x).
\end{equation}

**Proof.** By the well-known finite partition theorem of unity (see [12], for example) and without loss of generality, we may assume that there exist $m$ nonnegative continuous functions $\{\psi_i : A \to \mathbb{R} : i = 1, \ldots, m\}$ such that

\begin{equation}
\text{supp}(\psi_i) := \{x \in A : \psi_i(x) \neq 0\} \subseteq B(x_i, r_i)
\end{equation}

and

\begin{equation}
\sum_{i=1}^{m} \psi_i(x) = 1 \quad \text{for each } x \in A.
\end{equation}

Let $i = 1, 2, \ldots, m$ and consider the mapping $F_i : A \cap \overline{B(x_i, r_i)} \to 2^{T_x M}$ defined by

$$F_i(x) := P_{x,x_i}V(x) \quad \text{for each } x \in A \cap \overline{B(x_i, r_i)}.$$ 

Then $F_i$ is upper semicontinuous by Proposition 2.1 and $F_i(x)$ is compact convex for each $x \in A \cap \overline{B(x_i, r_i)}$. Then by Proposition 3.4, there exists a continuous mapping $f_i^\epsilon : A \cap \overline{B(x_i, r_i)} \to T_{x_i}M$ such that

\begin{equation}
\text{gph}(f_i^\epsilon) \subseteq U(\text{gph}(F_i), \epsilon).
\end{equation}

Let $f_i^\epsilon : A \to TM$ be defined by $f_i^\epsilon(x) := f_i^\epsilon(x)$ for each $x \in A \cap \overline{B(x_i, r_i)}$ and $f_i^\epsilon(x) := 0$ otherwise. Define the vector field $V_\epsilon : A \to M$ by

$$V_\epsilon(x) = \sum_{i=1}^{m} \psi_i(x)P_{x,x_i}f_i^\epsilon(x) \quad \text{for each } x \in A.$$ 

It is easy to check that $V_\epsilon$ is continuous on $A$. Now we prove that $V_\epsilon$ is as desired.

To do this, let $x \in A$ and let $I^+(x)$ be the subset of $\{1, 2, \ldots, m\}$ such that $\psi_i(x) > 0$ for $i \in I^+(x)$. Then $I^+(x) \subseteq I(x)$ by (3.5). Furthermore, we have by (3.7) that, for each $i \in I^+(x)$, there exist $x_i^\epsilon \in A \cap \overline{B(x_i, r_i)}$ and $y_i^\epsilon \in F_i(x_i^\epsilon)$ such that

\begin{equation}
d(x, x_i^\epsilon) + \|f_i^\epsilon(x) - y_i^\epsilon\| < \epsilon.
\end{equation}

Write $y^\epsilon := \sum_{i \in J(x)} \psi_i(x)P_{x,x_i}y_i^\epsilon$. Then $y^\epsilon \in \text{conv}\{P_{x,x_i}P_{x_i,x_i}V(x_i^\epsilon) : i \in I^+(x)\}$ by the definition of $F_i$. Moreover, by (3.8), together with the definitions of $V_\epsilon(x)$ and $y^\epsilon$, one sees that $x_i^\epsilon \in B(x, \epsilon)$ for each $i \in I^+(x)$ and

$$\|V_\epsilon(x) - y^\epsilon\| \leq \sum_{i \in J(x)} \psi_i(x)\|f_i^\epsilon(x) - y_i^\epsilon\| < \epsilon;$$

for each $x \in I^+(x)$.
hence (3.4) is shown and the proof is complete. 

**Theorem 3.6.** Let $A \subset M$ be a compact and locally convex subset of $M$. Suppose that $A$ has a weak pole $o \in \text{int}_R A$ and that $V \in \mathcal{V}(A)$. Then the variational inequality problem (3.1) admits at least one solution.

**Proof.** Fix an open cover $\{B(x_i, r_i) : i = 1, 2, \ldots, m\}$ of $A$ and let $n \in \mathbb{N}$. Then, applying Lemma 3.5 to $\frac{1}{n}$ in place of $\epsilon$, we have that there exists a continuous vector field $V_n : A \to TM$ with the properties mentioned in Lemma 3.5 for $V_n$ in place of $V$. Applying Theorem 3.1 to $V_n$, we have that $\mathcal{S}(V_n, A) \neq \emptyset$. Take $\bar{x}_n \in \mathcal{S}(V_n, A)$. Then, we can choose

\[ \{x_i^n : i \in I(\bar{x}_n)\} \subseteq B(\bar{x}_n, \frac{1}{n}) \quad \text{and} \quad v_n \in \text{conv}\{P_{\bar{x}_n, x_i^n}P_{x_i^n, x_i^n}V(x_i^n) : i \in I(\bar{x}_n)\} \]

satisfying

\[ \|V_n(\bar{x}_n) - v_n\| < \frac{1}{n}. \]

Without loss of generality, we may assume that $I(\bar{x}_n) = \{1, 2, \ldots, \tilde{m}\}$ for some natural number $\tilde{m} \leq m$ and $v_n := \sum_{i=1}^{\tilde{m}} t_i^n P_{\bar{x}_n, x_i^n}P_{x_i^n, x_i^n}v_i^n$ with each $v_i^n \in V(\bar{x}_n)$, each $t_i^n \geq 0$, and $\sum_{i=1}^{\tilde{m}} t_i^n = 1$.

Recalling that $A$ is compact, $V$ is upper semicontinuous, and each $V(x)$ is compact, we see that, for each $i = 1, \ldots, \tilde{m}$, the sequence $\{v_i^n : n \in \mathbb{N}\}$ is bounded. Thus, without loss of generality, we may assume that

\[ \bar{x}_n \to \bar{x}, \quad t_i^n \to t_i \quad \text{and} \quad v_i^n \to v_i \quad \text{for each} \quad i = 1, \ldots, \tilde{m} \]

(using subsequences if necessary). This, together with (3.9), implies that

\[ d(x_i^n, \bar{x}) \leq d(x_i^n, \bar{x}_n) + d(\bar{x}_n, \bar{x}) \to 0 \quad \text{for each} \quad i = 1, \ldots, \tilde{m}. \]

It follows from Proposition 2.1 and (3.11) that $P_{\bar{x}_n, x_i^n}P_{x_i^n, x_i^n}v_i^n \to v_i$ for each $i$. Furthermore, we have that $v_i \in V(\bar{x})$ for each $i = 1, \ldots, \tilde{m}$ since $V$ is upper Kuratowski semicontinuous, and so

\[ v := \lim_n v_n \in V(\bar{x}) \]

as $V(\bar{x})$ is closed and convex.

Consider the variational inequality problem (3.2) with $R = r_{\bar{x}}/2$ and $o = \bar{x}$. Since $\bar{x}_n \in \mathcal{S}(V_n, A) \subseteq \mathcal{S}(V_n, A_R)$ and $\bar{x}_n \to \bar{x}$, it follows that, for $n$ large enough,

\[ \langle V_n(\bar{x}_n), \exp_{\bar{x}_n}^{-1}y \rangle \geq 0 \quad \text{for each} \quad y \in A_R. \]

Since $V_n(\bar{x}_n) \to v$ by (3.10) and (3.13), and since $\exp_{\bar{x}_n}^{-1}y \to \exp_{\bar{x}}^{-1}y$ (noting that $\bar{x}_n \to \bar{x}$), we conclude by taking limits that

\[ \langle v, \exp_{\bar{x}}^{-1}y \rangle \geq 0 \quad \text{for each} \quad y \in A_R. \]

This shows that $\bar{x} \in \mathcal{S}(V, A_R)$ as $v \in V(\bar{x})$, and so $\bar{x} \in \mathcal{S}(V, A)$ by Proposition 3.2. The proof is complete. \[ \square \]

Note that if $A$ is a convex subset of a Hadamard manifold $M$, then $\text{int}_R A$ is nonempty and each point of $A$ is a pole. Hence the following corollary, which extends
the corresponding existence result in [26] to the setting of set-valued vector fields, now is a direct consequence of Theorem 3.6.

**Corollary 3.7.** Suppose that $M$ is a Hadamard manifold, and let $A \subseteq M$ be a compact convex set. Let $V \in V(A)$. Then the variational inequality problem (3.1) admits at least one solution.

To extend the existence result on solutions of the variational inequality problem (3.1) to the case when $A$ is not necessarily bounded, we introduce the coerciveness condition for set-valued vector fields on Riemannian manifolds. Recall that $\bar{A}$ denotes the unique minimal geodesic joining $o$ to $x$ when $o$ is a weak pole of $A$ and $x \in A$.

**Definition 3.8.** Let $A \subseteq M$ be a subset of $M$ containing a weak pole $o$ of $A$. Let $V$ be a vector field on $A$. $V$ is said to satisfy the coerciveness condition if

$$\sup_{v_0 \in V(o), v_2 \in V(x)} \frac{\langle v_x, \hat{\gamma}_{x}(0) \rangle - \langle v_0, \hat{\gamma}_{x}(1) \rangle}{d(o, x)} \to -\infty \text{ as } d(o, x) \to +\infty \text{ for } x \in A.$$  

**Definition 3.9.** We say that a locally closed convex set $A$ has the BCC (bounded convex cover) property if there exists $o \in A$ such that, for any $R > 0$, there exists a locally convex compact subset of $M$ containing $A \cap B(o, R)$.

Clearly, a locally closed convex set $A$ has the BCC property if and only if, for any bounded subset $A_0 \subseteq A$, there exists a locally convex compact subset of $M$ containing $A_0$. Moreover, by [25, Proposition 4.5], if $M$ is complete and if the sectional curvature of $M$ is nonnegative everywhere or nonpositive everywhere, then any locally convex closed subset of $M$ has the BCC property.

**Theorem 3.10.** Let $A \subseteq M$ be a locally convex closed set with a weak pole $o \in \text{int}_R A$, and let $V \in V(A)$ satisfy the coerciveness condition. Suppose that $A$ has the BCC property. Then the variational inequality problem (3.1) admits at least a solution.

**Proof.** Take $H > |V(o)| := \inf_{v \in V(o)} \|v\|$. Then, by the assumed coerciveness condition, there is $R > 0$ such that

$$\sup_{v_0 \in V(o), v_2 \in V(x)} \left(\langle v_x, \hat{\gamma}_{x}(0) \rangle - \langle v_0, \hat{\gamma}_{x}(1) \rangle\right) \leq -Hd(o, x) \text{ for each } x \in A \text{ with } d(o, x) \geq R.$$  

Hence, for each $x \in A$ with $d(o, x) \geq R$, one has that

$$\sup_{v_x \in V(x)} \langle v_x, \hat{\gamma}_{x}(0) \rangle \leq -Hd(o, x) + |V(o)| \cdot \|\hat{\gamma}_{x}(1)\| = (|V(o)| - H)d(o, x) < 0.$$  

By assumption, there exists a locally convex compact subset $K_R$ of $M$ containing $A \cap B(o, R)$. Then $A_R := A \cap K_R \subseteq M$ is a locally convex compact set. Moreover, note that, for any $x \in A_R$, the unique minimal geodesic $\bar{\gamma}_{ox}$ joining $o$ to $x$ lies in $B(o, R)$ and thus in $A \cap B(o, R)$ because $o$ is a weak pole of $A$. This implies that $\bar{\gamma}_{ox}$ is in $K_R$ and then in $A_R$. Hence $o \in \text{int}_R A_R$ is a weak pole of $A_R$. Thus Theorem 3.6 is applied (with $A_R$ in place of $A$) to get that $S(V, A_R) \neq \emptyset$.

Since $A_R \subseteq A_R$, it follows that $S(V, A_R) \subseteq S(V, A_R)$. Furthermore, by (3.16), we have that $S(V, A_R) \subseteq B(o, R)$. It follows from Proposition 3.2 that $S(V, A_R) \subseteq S(V, A_R) \subseteq S(V, A)$. This together with the fact that $S(V, A_R) \neq \emptyset$ completes the proof.

**Corollary 3.11.** Let $A \subseteq M$ be a locally convex closed set with a weak pole $o \in \text{int}_R A$, and let $V \in V(A)$ satisfy the coerciveness condition. Suppose that $M$ is
complete and that the sectional curvature of \( M \) is nonnegative everywhere or nonpositive everywhere. Then the variational inequality problem (3.1) admits at least one solution.

Below we consider the uniqueness problem of the solution of the variational inequality problem (3.1). The notions of monotonicity on Riemannian manifolds in the following definition are important tools for the study of the uniqueness problems and have been extensively studied in [8, 9, 10, 15, 19, 27, 28, 29, 30] for univalued vector fields and in [21, 22, 40] for set-valued vector fields.

**Definition 3.12.** Let \( A \subset M \) be a nonempty weakly convex set and let \( V \) be a set-valued vector field on \( A \). The vector field \( V \) is said to be

1. **monotone on** \( A \) if for any \( x, y \in A \) and \( \gamma_{xy} \in \Gamma_{x,y}^A \) the following inequality holds:

\[
\langle v_x, \gamma_{xy}(0) \rangle - \langle v_y, \gamma_{xy}(1) \rangle \leq 0 \quad \text{for any } v_x \in V(x), \; v_y \in V(y);
\]

2. **strictly monotone on** \( A \) if for any \( x, y \in A \) and \( \gamma_{xy} \in \Gamma_{x,y}^A \) the following inequality holds:

\[
\langle v_x, \gamma_{xy}(0) \rangle - \langle v_y, \gamma_{xy}(1) \rangle < 0 \quad \text{for any } v_x \in V(x), \; v_y \in V(y);
\]

3. **strongly monotone on** \( A \) if there exists \( \rho > 0 \) such that for any \( x, y \in A \) and \( \gamma_{xy} \in \Gamma_{x,y}^A \) the following inequality holds:

\[
\langle v_x, \gamma_{xy}(0) \rangle - \langle v_y, \gamma_{xy}(1) \rangle \leq -\rho l^2(\gamma_{xy}) \quad \text{for any } v_x \in V(x), \; v_y \in V(y).
\]

Now we have the following uniqueness result on the solution of problem (3.1).

**Theorem 3.13.** Let \( A_0 \subseteq A \subset M \). Suppose that \( A_0 \) is weakly convex and \( V \) is strictly monotone on \( A_0 \) satisfying \( \mathcal{S}(V, A) \subseteq A_0 \). Then the variational inequality problem (3.1) admits at most one solution. In particular, if \( A \) is weakly convex and \( V \) is strictly monotone on \( A \), then the variational inequality problem (3.1) admits at most one solution.

**Proof.** Suppose on the contrary that problem (3.1) admits two distinct solutions \( \bar{x} \) and \( \bar{y} \). Then \( \bar{x}, \bar{y} \in A_0 \). Since \( A_0 \) is weakly convex, there exists a minimal geodesic \( \gamma_{\bar{x}\bar{y}} \in \Gamma_{\bar{x},\bar{y}}^{A_0} \). Then there exist \( \bar{v} \in V(\bar{x}) \) and \( \bar{u} \in V(\bar{y}) \) such that

\[
\langle \bar{v}, \gamma_{\bar{x}\bar{y}}(0) \rangle \geq 0 \quad \text{and} \quad \langle \bar{u}, -\gamma_{\bar{x}\bar{y}}(1) \rangle \geq 0.
\]

It follows that

\[
\langle \bar{v}, \gamma_{\bar{x}\bar{y}}(0) \rangle - \langle \bar{u}, \gamma_{\bar{x}\bar{y}}(1) \rangle \geq 0.
\]

This contradicts the strict monotonicity of \( V \) on \( A_0 \), and the proof is complete. \( \square \)

It is routine to verify that if \( V \) is strongly monotone, then it satisfies the coerciveness condition. Therefore, the following corollary is straightforward.

**Corollary 3.14.** Let \( A \subset M \) be a closed weakly convex set with a weak pole \( a \in \text{int } A \), and let \( V \in \mathcal{V}(A) \) be a strongly monotone vector field on \( A \). Suppose that \( A \) has the BCC property (e.g., \( M \) is complete and the sectional curvature of \( M \) is nonnegative everywhere or nonpositive everywhere). Then the variational inequality problem (3.1) admits a unique solution.
4. Convexity of solution sets. As assumed in the previous section, let \( A \subseteq M \) be a nonempty closed subset, and let \( V \) be a set-valued vector field on \( A \). This section is devoted to the study of the convexity problem of the solution set of the variational inequality problem. For this purpose, we introduce the notions of \( r \)-convexity and prove some related lemmas.

**Definition 4.1.** Let \( A \) be a nonempty subset of \( M \) and let \( r \in (0, +\infty) \). The set \( A \) is said to be

(a) weakly \( r \)-convex if, for any \( p, q \in K \) with \( d(p, q) < r \), there is a minimal geodesic of \( M \) joining \( p \) to \( q \) lying in \( A \);

(b) \( r \)-convex if, for any \( p, q \in A \) with \( d(p, q) < r \), there is just one minimal geodesic of \( M \) joining \( p \) to \( q \) and it is in \( A \).

Let \( \kappa \geq 0 \), and assume that \( M \) is of the sectional curvature bounded above by \( \kappa \). The Riemannian manifolds of the sectional curvature bounded above by \( \kappa \) possess some useful properties which are listed in the following proposition. Note by [6, Theorem 1A.6] that any Riemannian manifold of the curvature bounded above by \( \kappa \) is a CAT(\( \kappa \)) space (the reader is referred to [6] for details). Thus Proposition 4.2 below is a direct consequence of [6, Propositions 1.4 and 1.7]. Recall that the model space \( M^n_\kappa \) is the Euclidean space \( \mathbb{E}^n \) if \( \kappa = 0 \) and the Riemannian manifold obtained from the \( n \)-dimensional sphere \( (\mathbb{S}^n, \kappa) \) by multiplying the distance function by the constant \( \sqrt{n} \kappa \) if \( \kappa > 0 \). Then \( M^n_\kappa \) is of the constant curvature \( \kappa \), and the following “law of cosines” in \( M^n_\kappa \) holds:

\[
\begin{align*}
\cos(\sqrt{\kappa}d(\bar{x}, \bar{y})) &= \cos(\sqrt{\kappa}d(\bar{x}, \bar{z})) \cos(\sqrt{\kappa}d(\bar{y}, \bar{z})) + \sin(\sqrt{\kappa}d(\bar{x}, \bar{z})) \sin(\sqrt{\kappa}d(\bar{y}, \bar{z})) & \text{if } \kappa > 0, \\
\cos(\sqrt{n}d(x, y)) &= \cos(\sqrt{n}d(x, z)) \cos(\sqrt{n}d(y, z)) + \sin(\sqrt{n}d(x, z)) \sin(\sqrt{n}d(y, z)) & \text{if } \kappa = 0,
\end{align*}
\]

where \( \bar{x}, \bar{y}, \bar{z} \in M^n_\kappa \) and \( \bar{\alpha} := \angle \bar{z} \bar{x} \bar{y} \) is the angle at \( \bar{z} \) of two geodesics joining \( \bar{z} \) to \( \bar{x}, \bar{y} \); see, for example, [6, p. 24]. By definition, a geodesic triangle \( \Delta(x, y, z) \) in \( M \) consists of three points \( \{x, y, z\} \) in \( M \) (the vertices of \( \Delta(x, y, z) \)) and three geodesic segments joining each pair of vertices (the edges of \( \Delta(x, y, z) \)). Recall that a triangle \( \Delta(\bar{x}, \bar{y}, \bar{z}) \) in the model space \( M^n_\kappa \) is said to be a comparison triangle for \( \Delta(x, y, z) \subset X \) if

\[
d(x, y) = d(\bar{x}, \bar{y}), \ d(x, z) = d(\bar{x}, \bar{z}) \quad \text{and} \quad d(z, y) = d(\bar{z}, \bar{y}).
\]

Define \( D_\kappa := \frac{\pi}{\sqrt{n} \kappa} \) if \( \kappa > 0 \) and \( D_\kappa := +\infty \) if \( \kappa = 0 \).

**Proposition 4.2.** Suppose that \( M \) is of the curvature bounded above by \( \kappa \). Then the following assertions hold:

(i) For any two points \( x, y \in M \) with \( d(x, y) < D_\kappa \), there exists one and only one minimizing geodesic joining \( x \) to \( y \).

(ii) For any \( z \in M \), the distance function \( x \mapsto d(z, x) \) is convex on \( B(z, \frac{D_\kappa}{2}) \); that is, for any two points \( x, y \in B(z, \frac{D_\kappa}{2}) \) and any \( \gamma_{xy} \in \Gamma_{x,y} \), the function \( t \mapsto d(z, \gamma(t)) \) is convex on \([0, 1]\).

(iii) Any ball in \( M \) with radius less than \( \frac{D_\kappa}{2} \) is strongly convex; in particular, we have that \( r_x \geq \frac{D_\kappa}{2} \) for each \( x \in M \), where \( r_x \) is the convexity radius defined by (2.2).

(iv) For any geodesic triangle \( \Delta(x, y, z) \) in \( M \) with \( d(x, y) + d(y, z) + d(z, x) < 2D_\kappa \), there exists a comparison triangle \( \Delta(\bar{x}, \bar{y}, \bar{z}) \) in \( M^n_\kappa \) for \( \Delta(x, y, z) \) such that, for \( \gamma_{yz} \in \Gamma_{y,z} \) and \( \gamma_{zx} \in \Gamma_{z,x} \),

\[
d(\gamma_{yz}(t), x) \leq d(\gamma_{yx}(t), x) \quad \text{for each } t \in [0, 1]
\]

and

\[
\angle xzy \leq \angle \bar{x} \bar{z} \bar{y},
\]
where $\angle xzy$ and $\angle \bar{x}\bar{z}\bar{y}$ denote the inner angles at $z$ and $\bar{z}$ of triangles $\Delta(x,y,z)$ and $\Delta(\bar{x},\bar{y},\bar{z})$, respectively.

Clearly, by assertion (i) of Proposition 4.2, if $M$ is of the curvature bounded above by $\kappa$, then a subset of $M$ is weakly $D_\kappa$-convex if and only if it is $D_\kappa$-convex.

Let $z \in A$ and $\lambda > 0$. Define the vector field $V_{\lambda,z} : A \to 2^{TM}$ by

$$V_{\lambda,z}(x) = \lambda V(x) - E_z(x) \quad \text{for each } x \in A,$$

where $E_z$ is the vector field defined by

$$E_z(x) := \{ u \in \exp_x^{-1} z : \|u\| = d(x,z), \exp_x tu \in A \forall t \in [0,1] \} \quad \text{for each } x \in A.$$

Note that $E_z(x) \neq \emptyset$ for any $z, x \in A$ with $d(x,z) < r$ if $A$ is weakly $r$-convex. Moreover, if $A$ is $r$-convex, then we have that

$$E_z(x) = \exp_x^{-1} z \quad \text{for any } z, x \in A \text{ with } d(x,z) < r.$$

Consider the variational inequality problem (3.1) with $V_{\lambda,z}$ in place of $V$, and recall that $S(V_{\lambda,z}, A)$ is its solution set. Then we define the set-valued map $J_\lambda : M \to 2^A$ by

$$J_\lambda(z) := S(V_{\lambda,z}, A) \quad \text{for each } z \in A,$$

that is, $x \in J_\lambda(z)$ if and only if there exist $v \in V(x)$ and $u \in E_z(x)$ such that

$$\langle \lambda v - u, \dot{\gamma}_{zy}(0) \rangle \geq 0 \quad \text{for each } y \in A \text{ and each } \gamma_{zy} \in \Gamma^A_{z,y}.$$

Clearly, the following equivalence holds for any $z \in A$:

$$z \in S(V, A) \iff z \in J_\lambda(z).$$

Throughout this whole section, we assume that $V$ is monotone on $A$. For a subset $Z$ of some normed space, it would be convenient to use the notion $|Z|$ to denote the distance to the origin from $Z$:

$$|Z| := \begin{cases} \inf_{u \in Z} \|u\| & \text{if } Z \neq \emptyset, \\ 0 & \text{if } Z = \emptyset. \end{cases}$$

**Lemma 4.3.** Let $r \in (0, +\infty)$, and let $A$ be a weakly $r$-convex subset of $M$. Let $\lambda > 0$ and $x \in A$ be such that $\lambda |V(x)| < r$. Then the following assertion holds:

$$B(x,r) \cap J_\lambda(x) \subseteq \overline{B(x,\lambda |V(x)|)}.$$

In particular, if $A$ is weakly convex, then we have that

$$J_\lambda(x) \subseteq \overline{B(x,\lambda |V(x)|)}.$$

**Proof.** Let $z \in B(x,r) \cap J_\lambda(x)$. Since $A$ is weakly $r$-convex, it follows by definition that there exist $v \in V(z)$ and $u \in E_z(x)$ such that (4.4) holds. Let $\gamma_{xz}$ be the geodesic defined by $\gamma_{xz}(t) := \exp_z tu$ for each $t \in [0,1]$. Then by (4.2), $d(x,z) = \|u\|$, $\dot{\gamma}_{xz}(0) = u$, and $\gamma_{xz} \in \Gamma^A_{z,x}$. This, together with (4.4), implies that

$$d^2(x,z) = \langle u, u \rangle = \langle u, \gamma_{xz}(0) \rangle \leq \lambda \langle v, \gamma_{xz}(0) \rangle.$$
Since $V$ is monotone, it follows that
\[
d^2(x, z) \leq \lambda(v, \gamma_{zx}(0)) \leq \lambda(v_x, \gamma_{zx}(1)) \leq \lambda\|v_x\|d(x, z) \quad \text{for each } v_x \in V(x).
\]
Hence $d(z, x) \leq \lambda|V(x)|$, and the proof is complete. \(\square\)

**Lemma 4.4.** Suppose that $M$ is of the sectional curvature bounded above by $\kappa$ and that $A$ is $D_\kappa$-convex. Let $y_0 \in A$ and $\lambda > 0$ be such that $\lambda|V(y_0)| < \frac{D_\kappa}{2}$. Then $J_\lambda(y_0)$ is nonempty.

**Proof.** Let $R > 0$ be such that
\[
\lambda|V(y_0)| < R < r := \frac{D_\kappa}{2}.
\]
Then $A_R = \overline{B(y_0, R)} \cap A$ is compact and locally convex. Take $o \in \text{int}_R A_R$ such that $d(y_0, o) < \frac{R - R}{2}$. Then, for each $x \in A_R$,
\[
d(x, o) \leq d(x, y_0) + d(y_0, o) \leq R + \frac{r - R}{2} < r.
\]
By Proposition 4.2 (i), (ii), the minimal geodesic joining $o$ to $x$ is unique and lies in $A_R$. This means that $o$ is a weak pole of $A_R$. Thus, Theorem 3.6 is applicable to concluding that $S(V_\lambda, A_R) \neq \emptyset$. By Lemma 4.3 (applied to $A_R$ in place of $A$), we have that
\[
d(y_R, y_0) \leq \lambda|V(y_0)| < R \quad \text{for any } y_R \in S(V_\lambda, A_R).
\]
This, together with Proposition 3.2 (applied to $o := y_0$), implies that $S(V_\lambda, A_R) \subseteq S(V_\lambda, A) = J_\lambda(y_0)$. This shows that $J_\lambda(y_0) \neq \emptyset$, and the proof is complete. \(\square\)

**Lemma 4.5.** Suppose that $M$ is of the sectional curvature bounded above by $\kappa$ and that $A$ is $D_\kappa$-convex. Let $y_0 \in A$, $x_0 \in S(V, A)$, and $\lambda > 0$ be such that
\[
\lambda|V(y_0)| + d(y_0, x_0) < \frac{D_\kappa}{2}. \tag{4.9}
\]
Then the following estimates hold for each $z \in J_\lambda(y_0)$:
\[
d(y_0, x_0)^2 \geq d(z, x_0)^2 + d(y_0, z)^2 \cos(\sqrt{\kappa}d(y_0, x_0)) \leq \cos(\sqrt{\kappa}d(z, x_0)) \cos(\sqrt{\kappa}d(y_0, z)) \quad \text{if } \kappa > 0 \tag{4.10}
\]
and
\[
d(z, x_0) \leq d(y_0, x_0). \tag{4.11}
\]

**Proof.** Without loss of generality, we assume that $\kappa = 1$, and thus $D_\kappa = \pi$. Let $z \in J_\lambda(y_0)$. By assumption (4.9), $d(y_0, x_0) < \frac{\pi}{2}$ and $\lambda|V(y_0)| < \frac{\pi}{2}$. Since $A$ is $\pi$-convex, it follows from Lemma 4.3 that
\[
d(y_0, z) \leq \lambda|V(y_0)| < \frac{\pi}{2}. \tag{4.12}
\]
Thus one sees that if (4.10) is shown, then $\cos d(y_0, x_0) \leq \cos d(z, x_0)$ by (4.10); hence (4.11) holds. Thus we need only prove (4.10). To this end, note by (4.12) that
\[
d(z, x_0) \leq d(z, y_0) + d(x_0, y_0) \leq \lambda|V(y_0)| + d(x_0, y_0).
\]
Therefore,
\[(4.13) \quad d(x_0, y_0) + d(y_0, z) + d(z, x_0) \leq 2(\lambda|V(x_0)| + d(x_0, y_0)) \leq \pi.\]

Consider the geodesic triangle \(\Delta(x_0, y_0, z)\) with vertices \(x_0, y_0, z\). By Proposition 4.2(i), the geodesic joining each pair of the vertices of \(\Delta(x_0, y_0, z)\) is unique. Let \(\alpha := \angle x_0z_0y_0\) denote the angle of the geodesic triangle at vertex \(z\), that is, \(\alpha\) satisfies that
\[(4.14) \quad \langle \exp^{-1}_z x_0, \exp^{-1}_z y_0 \rangle = \|\exp^{-1}_z x_0\| \|\exp^{-1}_z y_0\| \cos \alpha.\]

By Proposition 4.2(iii), there exists a comparison triangle \(\Delta(\bar{x}_0, \bar{y}_0, \bar{z})\) in the model space \(S^2\) such that
\[(4.15) \quad d(\bar{x}_0, \bar{y}_0) = d(x_0, y_0), \quad d(\bar{x}_0, \bar{z}) = d(x_0, z) \quad \text{and} \quad d(\bar{y}_0, \bar{z}) = d(y_0, z).\]

Moreover, we have that \(\alpha \leq \bar{\alpha} := \angle \bar{x}_0\bar{z}\bar{y}_0\), the spherical angle at \(\bar{z}\) of the comparison triangle. By (4.1), we have that
\[(4.16) \quad \cos d(\bar{x}_0, \bar{y}_0) = \cos d(\bar{x}_0, \bar{z}) \cos d(\bar{y}_0, \bar{z}) + \sin d(\bar{x}_0, \bar{z}) \sin d(\bar{y}_0, \bar{z}) \cos \bar{\alpha}.\]

In view of (4.15), to complete the proof, it suffices to show that \(\cos \bar{\alpha} \leq 0\). Note that the unique geodesic \(\gamma_{xz_0}\) joining \(z\) to \(x_0\) is in \(\Gamma^A_z, x_0\), and so \(E_{y_0}(z) = \exp^{-1}_z y_0\) is a singleton. Note also that \(x_0 \in S(V, A) \subset A\) and \(z \in J_A(y_0) \subset A\). It follows from definition that there exist \(v_{x_0} \in V(x_0)\) and \(v_z \in V(z)\) such that
\[\langle v_{x_0}, \exp^{-1}_{x_0} z \rangle \geq 0 \quad \text{and} \quad \langle \lambda v_z - \exp^{-1}_{x_0} y_0, \exp^{-1}_{x_0} x_0 \rangle \geq 0.\]

This, together with the monotonicity of \(V\), implies that
\[\langle \exp^{-1}_z y_0, \exp^{-1}_z x_0 \rangle \leq \lambda \langle v_z, \exp^{-1}_z x_0 \rangle \leq -\lambda \langle v_{x_0}, \exp^{-1}_{x_0} z \rangle \leq 0.\]

Combining this with (4.14) gives that \(\cos \alpha \leq 0\). Since \(\bar{\alpha} \geq \alpha\), we have \(\cos \bar{\alpha} \leq \cos \alpha \leq 0\) and complete the proof.

The main theorem of this section is as follows.

**Theorem 4.6.** Suppose that \(M\) is of the sectional curvature bounded above by \(\kappa\) and that \(A\) is closed and \(D_{\kappa}\)-convex. Then the solution set \(S(V, A)\) is \(D_{\kappa}\)-convex.

**Proof.** Let \(x_1, x_2 \in S(V, A)\) be such that \(d(x_1, x_2) < D_{\kappa}\). Let \(c \in \Gamma_{x_1, x_2}\) be the minimal geodesic. Write \(y_0 := c(\frac{\lambda}{2})\). To complete the proof, it is sufficient to show that \(y_0 \in S(V, A)\). Note that \(d(x_i, y_0) < \frac{D_{\kappa}}{2}\) for each \(i = 1, 2\). We can take \(\lambda > 0\) such that
\[\lambda |V(y_0)| + d(x_i, y_0) < \frac{D_{\kappa}}{2}\quad \text{for each} \quad i = 1, 2.\]

Since \(A\) is \(D_{\kappa}\)-convex, Lemmas 4.4 and 4.5 are applicable to getting that \(J_A(y_0) \neq \emptyset\) and
\[(4.17) \quad d(z, x_i) \leq d(y_0, x_i) \quad \text{for each} \quad z \in J_A(y_0) \text{and} \quad i = 1, 2.\]

Take \(z_0 \in J_A(y_0)\), and let \(\gamma_i\) be the minimal geodesic joining \(z_0\) to \(x_i\) for \(i = 1, 2\). Define
\[\gamma(t) := \begin{cases} \gamma_1(1-2t) & t \in [0, \frac{1}{2}], \\ \gamma_2(2t-1) & t \in \left[\frac{1}{2}, 1\right]. \end{cases}\]
Then applying (4.17), we estimate the length of $\gamma$ by

$$l(\gamma) = l(\gamma_1) + l(\gamma_2) = d(z_0, x_1) + d(z_0, x_2) \leq d(x_1, y_0) + d(y_0, x_2) = d(x_1, x_2) \leq l(\gamma).$$

This implies that $\gamma$ is the minimal geodesic joining $x_1$ and $x_2$. Hence $\gamma = e$ thanks to Proposition 4.2(i). This implies that $z_0 \in c([0, 1])$ and thus $y_0 = z_0 \in J_\lambda(y_0)$ (as $d(z_0, x_1) = d(z_0, x_2)$). Consequently, $y_0 \in S(V, A)$ by Lemma 4.3, and the proof is complete. 

Note that a Hadamard manifold is of the sectional curvature bounded above by $\kappa = 0$ and $D_\kappa = +\infty$. Therefore, from the above theorem we have immediately the following corollary, which was claimed in [26] for any continuous univalued vector field $V$, but the proof provided there is not correct.

**Corollary 4.7.** Suppose that $M$ is a Hadamard manifold and that $A$ is convex. Then the solution set $S(V, A)$ is convex.

For a general Riemannian manifold, we have the following result.

**Theorem 4.8.** Suppose that $M$ is a complete Riemannian manifold and $A$ is locally convex. Then the solution set $S(V, A)$ is locally convex.

**Proof.** Without loss of generality, we assume that $S(V, A)$ is nonempty. Let $\bar{x} \in S(V, A)$. Then there exists $0 < \bar{r} < r_\bar{x}$ such that $B(\bar{x}, \bar{r}) \cap A$ is strongly convex. Since $\bar{B}(\bar{x}, \bar{r})$ is compact, it follows that the sectional curvature on $\bar{B}(\bar{x}, \bar{r})$ is bounded, and so there exists some $\kappa > 0$ such that the sectional curvature on $\bar{B}(\bar{x}, \bar{r})$ is bounded above by $\kappa$. This, together with Proposition 2.4, implies that $N := \bar{B}(\bar{x}, \bar{r})$ is a totally geodesic submanifold of $M$ with the bounded curvature above by $\kappa$. Moreover, by Proposition 2.5, $T_xN = T_\bar{x}M$ for each $x \in N$. This means that the restriction $V|_N$ of $V$ to $N$ is a vector field on $N$.

Let $0 < R < \bar{r}$ and set $A_R := \bar{B}(\bar{x}, R) \cap A$. Then $A_R \subseteq N$ is compact in $N$. Consider the variational inequality problem (3.1) with $N$ and $V|_N$ in place of $M$ and $V$, and let $S_N(V|_N, A_R)$ denote the corresponding solution set. Then

$$S_N(V|_N, A_R) \cap B(\bar{x}, R) = S(V, A_R) \cap B(\bar{x}, R) \supseteq S(V, A) \cap B(\bar{x}, R),$$

where the equality holds because $N$ is a totally geodesic submanifold of $M$. Furthermore, by Proposition 3.2, we have that $S(V, A_R) \cap B(\bar{x}, R) \subseteq S(V, A) \cap B(\bar{x}, R)$; hence $S_N(V, A_R) \cap B(\bar{x}, R) = S(V, A) \cap B(\bar{x}, R)$. Note that $A_R$ is strongly convex in $N$, and thus so is $D_{A_R}$-convex. Thus Theorem 4.6 is applied (to $A_R$ and $N$ in place of $A$ and $M$) to get that $S_N(V, A_R)$ is $D_{A_R}$-convex in $N$. This, together with Proposition 4.2, yields that $S_N(V, A_R) \cap B(\bar{x}, r)$ is strongly convex in $N$ for any $0 < r \leq \frac{\bar{r}}{2}$. Taking $0 < r \leq \min\{R, \frac{\bar{r}}{2}\}$ and noting that $N$ is a totally geodesic submanifold of $M$, we have that $S(V, A) \cap B(\bar{x}, r) \supseteq S_N(V, A_R) \cap B(\bar{x}, r)$ is strongly convex in $M$. Thus we showed that $S(V, A)$ is locally convex as $\bar{x} \in S(V, A)$ is arbitrary. 

**5. Proximal point methods.** As in the previous section, we assume throughout this whole section that $A \subseteq M$ is a nonempty closed subset and $V \in \mathcal{V}(A)$ is monotone on $A$. This section is devoted to the study of convergence of the proximal point algorithm for the variational inequality problem (3.1). Let $x_0 \in D(A)$ and $\{\lambda_n\} \subseteq (0, +\infty)$. The proximal point algorithm (with initial point $x_0$) considered in this section is defined as follows. Letting $n = 0, 1, 2, \ldots$ and having $x_n$, choose $x_{n+1}$ such that

$$x_{n+1} \in J_{\lambda_n}(x_n) \quad \text{for any } n \in \mathbb{N}. \quad (5.1)$$

Let $x \in M$ and $r > 0$. Recall from [12, p. 72] that $B(x, r)$ is a **totally normal ball** around $x$ if there is $\eta > 0$ such that for each $y \in B(x, r)$ we have $\exp_y(B(0, \eta)) \supseteq \ldots$


Proposition 5.1 is applicable to getting that
\[
\langle - \exp_x^{-1} z, \exp_y^{-1} y \rangle - \langle - \exp_y^{-1} z, \exp_y^{-1} x \rangle \leq -2 \sqrt{\kappa} \cot(2 \sqrt{\kappa} r) d^2(x, y).
\]
To show (5.5), note by definition that \( \mathbf{r}(z) \geq \frac{D}{\pi} \) and so \( \mathbf{B}(z, \bar{r}) \subseteq \mathbf{B}(z, \mathbf{r}(z)) \). Thus Proposition 5.1 is applicable to getting that
\[
\left. \left( \frac{d}{ds} d(\exp_x s \exp_x^{-1} z, \exp_y s \exp_y^{-1} z) \right) \right|_{s=0} = \frac{1}{d(x, y)} \left( -\langle \exp_x^{-1} z, \exp_y^{-1} y \rangle + \langle \exp_y^{-1} z, -\exp_y^{-1} x \rangle \right).
\]
On the other hand, let \( r := 2 \sqrt{\kappa} \bar{r} \). Then
\[
\sqrt{\kappa} \max\{d(x, y), d(y, z), d(z, x)\} \leq r < \frac{\pi}{2}.
\]
Then we apply Lemma 5.2 to conclude that
\[
\left. \left( \frac{d}{ds} d(\exp_x s \exp_x^{-1} z, \exp_y s \exp_y^{-1} z) \right) \right|_{s=0} \leq \lim_{s \to 0^+} \frac{\sin(1-s)r}{s} - 1 d(x, y) = (r \cot r) d(x, y).
\]
This, together with (5.6), implies that
\[
\frac{1}{d(x, y)} \left( -\langle \exp^{-1} z, \exp^{-1} y \rangle + \langle \exp^{-1} z, -\exp^{-1} x \rangle \right) \leq -r(cot \theta) d(x, y).
\]
Hence (5.5) holds, and the proof is complete. \( \square \)

The following corollary is useful.

**Corollary 5.4.** Suppose that \( M \) is of the sectional curvature bounded above by \( \kappa \) and that \( A \) is weakly convex. Let \( y_0 \in A \) and \( \lambda > 0 \) be such that \( \lambda |V(y_0)| < \frac{D_1}{4} \).

Then \( J_\lambda(y_0) \) is a singleton.

**Proof.** Let \( A_0 := A \cap B(y_0, \frac{D_1}{4}) \). By Lemmas 4.3 and 4.4, we have that \( \emptyset \neq J_\lambda(y_0) \subseteq A_0 \). Furthermore, \( V_{\lambda, x_0} \) is strictly monotone on \( A_0 \) because \(-E_{x_0}\) is strictly monotone on \( B(x_0, \frac{D_1}{4}) \) by Lemma 5.3, and \( V \) is monotone on \( A \) by assumption. Hence the assumptions of Theorem 3.13 are satisfied, and the conclusion follows. \( \square \)

Recall that the proximal point algorithm (5.1) is well-defined if, for each \( n \in \mathbb{N} \), \( J_{\lambda_n}(x_n) \) is a singleton. For the following proposition, which provides a sufficient condition ensuring the well-posedness of the algorithm, we define inductively the sequence of (set-valued) mappings \( \{J^n\} \) by
\[
J^{n+1} := J_{\lambda_n} \circ J^n \quad \text{for any } n = 0, 1, 2, \ldots,
\]
where \( J^0 := I \), the identity mapping.

**Proposition 5.5.** Suppose that \( M \) is of the sectional curvature bounded above by \( \kappa \) and \( A \) is weakly convex. Let \( x_0 \in A \) be such that
\[
(5.7) \quad \lambda_n |V(J^n(x_0))| \leq \frac{D_1}{4} \quad \text{for each } n = 0, 1, 2, \ldots.
\]

Then algorithm (5.1) is well-defined.

**Proof.** By assumption (5.7),
\[
\lambda_0 |V(x_0)| = \lambda_0 |V(J^0(x_0))| \leq \frac{D_1}{4}.
\]

Thus applying Corollary 5.4 (to \( \lambda_0 \) and \( x_0 \) in place of \( \lambda \) and \( y_0 \)), we get that \( J_{\lambda_0}(x_0) \) is a singleton. Thus \( x_1 \) is well-defined. Similarly, we have from (5.7) again that
\[
\lambda_1 |V(x_1)| = \lambda_1 |V(J^1(x_0))| \leq \frac{D_1}{4}
\]
and so \( x_2 \) is also well-defined by Corollary 5.4. Thus, the well-definedness of algorithm (5.1) can be established by mathematical induction, and the proof is complete. \( \square \)

**Remark 5.1.** Let \( \{\lambda_n\} \subset (0, +\infty) \). We have the following assertions:

(a) If \( M \) is a Hadamard manifold, algorithm (5.1) is well-defined because \( D_1 = +\infty \) and condition (5.7) holds automatically.

(b) If algorithm (5.1) generates a sequence \( \{x_n\} \) satisfying that
\[
(5.8) \quad \lambda_n |V(x_n)| \leq \frac{D_1}{4} \quad \text{for each } n = 0, 1, 2, \ldots,
\]
then condition (5.7) holds and algorithm (5.1) is well-defined.

Recall that a sequence \( \{x_n\} \) in complete metric space \( X \) is Fejér convergent to \( F \subseteq X \) if
\[
d(x_{n+1}, y) \leq d(x_n, y) \quad \text{for each } y \in F \text{ and each } n = 0, 1, 2, \ldots.
\]
The following proposition provides a local convergence result for the proximal point algorithm.

**Theorem 5.6.** Suppose that $M$ is of the sectional curvature bounded above by $\kappa$ and $A$ is weakly convex. Let $V \in \mathcal{V}(A)$ be a monotone vector field satisfying $\mathcal{S}(V, A) \neq \emptyset$. Let $x_0 \in A$ be such that $d(x_0, \mathcal{S}(V, A)) < \frac{D_\kappa}{4}$ and the algorithm (5.1) is well-defined. Suppose that $\{\lambda_n\}$ satisfies both condition (5.8) and

\[
\liminf_{n \to \infty} \frac{d(x_{n+1}, x_n)}{\lambda_n} = 0.
\]

Then $\{x_n\}$ converges to a solution of the variational inequality problem (3.1).

**Proof.** Write $F := \mathcal{S}(V, A) \cap B(x_0, \frac{D_\kappa}{4})$. Then $F \neq \emptyset$. Let $\hat{x}_0 \in \mathcal{S}(V, A)$ such that $d(x_0, \hat{x}_0) < \frac{D_\kappa}{4}$. We first prove that the following two assertions hold for any $x \in F$ and any $n = 0, 1, \ldots$:

\[
\cos(\sqrt{\kappa}d(x_n, x)) \leq \cos(\sqrt{\kappa}d(x_{n+1}, x)) \cos(\sqrt{\kappa}d(x_{n+1}, x_n)),
\]

\[
d(x_{n+1}, x_0) \leq 2d(\hat{x}_0, x_0) < \frac{D_\kappa}{4} \quad \text{and} \quad d(x_{n+1}, x) \leq d(x_n, x) \leq d(x_0, x).
\]

For this purpose, let $x \in F$. Then

\[
\lambda_0 |V(x_0)| + d(x_0, x) < \frac{D_\kappa}{2}.
\]

Hence, Lemma 4.5 is applicable (with $x_0, x$ in place of $y_0, x_0$). Thus, by (4.10) and (4.11), we obtain that

\[
\cos(\sqrt{\kappa}d(x_0, x)) \leq \cos(\sqrt{\kappa}d(x_1, x)) \cos(\sqrt{\kappa}d(x_1, x_n)) \quad \text{and} \quad d(x_1, x) \leq d(x_0, x).
\]

That is, assertion (5.10) and the second assertion in (5.11) are shown. In particular, we have that $d(\hat{x}_1, \hat{x}_0) < d(\hat{x}_0, x_0)$ because $\hat{x}_0 \in F$ by the choice of $\hat{x}_0$. It follows that

\[
d(x_1, x_0) < d(x_1, \hat{x}_0) + d(\hat{x}_0, x_0) \leq 2d(\hat{x}_0, x_0) < \frac{D_\kappa}{4},
\]

and the first assertion in (5.11) is also proved. Thus assertions (5.10) and (5.11) hold for any $x \in F$ and $n = 0$. Then one can use mathematical induction to show that they hold for any $x \in F$ and any $n = 0, 1, \ldots$.

To verify the convergence of the sequence $\{x_n\}$, we note that $\{x_n\}$ is Fejér convergent to $F$ by (5.11); hence $\{x_n\}$ is bounded. This, together with assumption (5.9), implies that there exists a subsequence $\{n_k\}$ such that $x_{n_k} \to \bar{x}$ for some $\bar{x} \in A$ and

\[
\exp_{x_{n_k+1}}^{-1} x_{n_k} \equiv \frac{d(x_{n_k+1}, x_{n_k})}{\lambda_{n_k}} \to 0.
\]

Then, by the first assertion in (5.11), we have

\[
d(\bar{x}, x_0) = \lim_k d(x_{n_k}, x_0) \leq 2d(\hat{x}_0, x_0) < \frac{D_\kappa}{4};
\]

hence $\bar{x} \in B(x_0, \frac{D_\kappa}{4})$. Moreover, we conclude by (5.10) and (5.11) that

\[
\lim_{n \to \infty} \cos(\sqrt{\kappa}d(x_{n+1}, x_n)) = 1,
\]
and so \( \lim_{n \to \infty} d(x_{n+1}, x_n) = 0 \). This means that \( x_{n+1} \to \bar{x} \). Let \( R := \bar{B} \) and \( o := \bar{x} \). Without loss of generality, we assume that \( \{x_n\}, \{x_{n+1}\} \) are in \( B(\bar{x}, \bar{R}) \). By definition, \( x_{n+1} \in J_{\lambda_n} (x_n) \), and so there exists \( v_{n+1} \in V(x_{n+1}) \) such that

\[
(5.13) \quad \langle \lambda_n v_{n+1} - \exp^{-1} x_{n+1}, \exp^{-1} x_{n+1}, z \rangle \geq 0 \quad \text{for each } z \in A_R := A \cap \bar{B}(o, R).
\]

Since \( x_{n+1} \to \bar{x} \) and each \( V(x_{n+1}) \) is compact, it follows that \( \{v_{n+1}\} \) is bounded. Thus we may assume that \( v_{n+1} \to \tilde{v} \) for some \( \tilde{v} \in T_{\bar{x}} M \) (using subsequences if necessary). Noting that \( V \) is upper Kuratowski semicontinuous at \( \bar{x} \), we have that \( \tilde{v} \in V(\bar{x}) \). Letting \( k \to \infty \), we get from (5.12) and (5.13) that

\[
\langle v, \exp^{-1} z \rangle \geq 0 \quad \text{for each } z \in A_R.
\]

This shows that \( \bar{x} \in S(V, A_R) \). Hence, \( \bar{x} \in S(V, A) \) by Proposition 3.2, and so \( \bar{x} \in F \). Applying the second assertion in (5.11), we obtain that the sequence \( \{d(x_n, \bar{x})\} \) is monotone, which, together with \( \lim_{k} d(x_{nk}, \bar{x}) = 0 \), shows that \( \{x_n\} \) converges to \( x \) and completes the proof. \( \square \)

**Theorem 5.7.** Suppose that \( M \) is of the sectional curvature bounded above by \( \kappa \) and that \( A \) is weakly convex. Let \( V \in \mathcal{V}(A) \) be a monotone vector field satisfying \( S(V, A) \neq \emptyset \). Let \( x_0 \in A \) be such that \( d(x_0, S(V, A)) < \frac{D_n}{4} \) and algorithm (5.1) is well-defined. Suppose that \( \{\lambda_n\} \subset (0, +\infty) \) satisfies

\[
(5.14) \quad \sum_{n=0}^{\infty} \lambda_n^2 = \infty \quad \text{and} \quad \lambda_n |V(x_n)| \leq \frac{D_n}{4} \quad \text{for each } n = 0, 1, \ldots.
\]

Then the sequence \( \{x_n\} \) converges to a solution of the variational inequality problem (3.1).

**Proof.** By Theorem 5.6, it suffices to verify that (5.9) holds. To this end, let \( n \in \mathbb{N} \). We assume that \( \kappa = 1 \) for simplicity. Let \( F = S(V, A) \cap B(x_0, \frac{D_n}{4}) \), and let \( x \in F \). Applying Lemma 4.5, we have that (5.10) and (5.11) hold. Clearly (5.10) is equivalent to the following inequality:

\[
(5.15) \quad \sin^2 \left( \frac{1}{2} d(x_n, x) \right) \geq \sin^2 \left( \frac{1}{2} d(x_{n+1}, x) \right) + \cos d(x_{n+1}, x) \sin^2 \left( \frac{1}{2} d(x_{n+1}, x_n) \right).
\]

By (5.11), \( d(x_{n+1}, x) \leq \frac{\pi}{4} \); hence

\[
\cos d(x_{n+1}, x) \sin^2 \left( \frac{1}{2} d(x_{n+1}, x_n) \right) \geq \frac{\sqrt{2}}{2} \cdot \left( \frac{d(x_{n+1}, x_n)}{\pi} \right)^2 = \frac{\sqrt{2}}{2\pi^2} d(x_{n+1}, x_n)^2.
\]

Combining this with (5.15) yields that

\[
\sum_{n=0}^{\infty} \lambda_n^2 \left( \frac{d(x_{n+1}, x_n)}{\lambda_n} \right)^2 = \sum_{n=0}^{\infty} d(x_{n+1}, x_n)^2
\]

\[
= \sqrt{2\pi^2} \sum_{n=0}^{\infty} \left( \sin^2 \left( \frac{1}{2} d(x_n, x) \right) - \sin^2 \left( \frac{1}{2} d(x_{n+1}, x) \right) \right) < \infty.
\]

Therefore, (5.9) follows because \( \sum_{n=0}^{\infty} \lambda_n^2 = \infty \). \( \square \)

Combining Theorem 5.7 and Proposition 5.5, together with Remark 5.1(b), we immediately get the following corollary.
Corollary 5.8. Suppose that $M$ is of the sectional curvature bounded from above by $\kappa$ and that $A$ is weakly convex. Let $V \in \mathcal{V}(A)$ be a monotone vector field with $\mathcal{S}(V,A) \neq \emptyset$. Let $x_0 \in A$ be such that $d(x_0,\mathcal{S}(V,A)) < \frac{\kappa}{8}$. Suppose that $\{\lambda_n\} \subset (0, +\infty)$ satisfies condition (5.7) and

$$\sum_{n=0}^{\infty} \lambda_n^2 = \infty. \tag{5.16}$$

Then algorithm (5.1) is well-defined, and the generated sequence $\{x_n\}$ converges to a solution of the variational inequality problem (3.1).

In the special case when $M$ is a Hadamard manifold, condition (5.7) and condition $d(x_0,\mathcal{S}(V,A)) < \frac{\kappa}{8}$ hold automatically. Therefore the following corollary is direct and extends [21, Theorem 5.12], which is proved for univalued continuous monotone vector fields $V$ under the stronger assumption that the sequence $\{\lambda_n\}$ satisfies $\inf_n \lambda_n > 0$.

Corollary 5.9. Suppose that $M$ is a Hadamard manifold and that $A$ is convex. Let $V \in \mathcal{V}(A)$ be a monotone vector field with $\mathcal{S}(V,A) \neq \emptyset$. Let $x_0 \in A$, and suppose that $\{\lambda_n\} \subset (0, +\infty)$ satisfies condition (5.16). Then algorithm (5.1) is well-defined, and the generated sequence $\{x_n\}$ converges to a solution of the variational inequality problem (3.1).

Consider the special case when $A = M$. Then the proximal point algorithm (5.1) is reduced to the following for finding a singularity of $V$:

$$0 \in \lambda_n V(x_{n+1}) - E_{x_n}(x_{n+1}) \quad \text{for each } n = 0, 1, 2, \ldots, \tag{5.17}$$

which, in Hadamard manifold $M$, is equivalent to the following:

$$0 \in \lambda_n V(x_{n+1}) - \exp_{x_{n+1}}^{-1} x_n \quad \text{for each } n = 0, 1, 2, \ldots. \tag{5.18}$$

This algorithm was presented and studied in [21] for set-valued vector fields on Hadamard manifolds. The following corollary, which is a direct consequence of Theorem 5.7, extends [21, Corollary 4.8], which was proved on Hadamard manifold $M$ under the stronger assumption that the sequence $\{\lambda_n\}$ satisfies $\inf_n \lambda_n > 0$.

Corollary 5.10. Suppose that $M$ is of the sectional curvatures bounded above by $\kappa$. Let $V \in \mathcal{V}(M)$ be a monotone vector field with $V^{-1}(0) := \{x \in M : 0 \in V(x)\} \neq \emptyset$. Let $x_0 \in M$ be such that $d(x_0, V^{-1}(0)) < \frac{\kappa}{8}$, and let $\{\lambda_n\} \subset (0, +\infty)$ satisfy conditions (5.7) and (5.16). Then the sequence $\{x_n\}$ generated by the algorithm (5.17) is well-defined and converges to a point $\bar{x} \in V^{-1}(0)$.

One natural question is, Does there exist a positive number sequence $\{\lambda_n\}$ satisfying conditions (5.7) and (5.16)? The following remark answers this affirmatively.

Remark 5.2. Let $x_0 \in A$ be such that $d(x_0,\mathcal{S}(V,A)) < \frac{\kappa}{8}$, and set $\Lambda := \sup \{\|V(x)\| : x \in B(x_0, \frac{\kappa}{8})\}$. Let $\{\lambda_k\} \subset (0, \frac{\kappa}{4\Lambda})$ satisfy (5.16). Then, by the proofs for Proposition 5.5 and Theorem 5.6, algorithm (5.1) is well-defined, and the generated sequence $\{x_n\}$ is contained in $B(x_0, \frac{\kappa}{4\Lambda})$. This and the choice of $\{\lambda_k\}$, together with the definition of $\Lambda$, means that conditions (5.7) and (5.16) are satisfied.

6. Application to convex optimization. As illustrated in [10] and the book [35], lots of nonconvex optimization problems on the Euclidean space can be reformulated as convex ones on some proper Riemannian manifolds. This section is devoted to an application in convex optimization problems on Riemannian manifolds of the convergence results for the proximal point algorithm established in the previous section.
Let $A \subset M$ be a closed weakly convex set, and let $\bar{x} \in A$. Recall that a vector $v \in T_{\bar{x}}M$ is tangent to $A$ if there is a smooth curve $\gamma : [0, \varepsilon] \to A$ such that $\gamma(0) = x$ and $\gamma'(0) = v$. Then the collection $T_{\bar{x}}A$ of all tangent vectors to $A$ at $\bar{x}$ is a convex cone in the space $T_{\bar{x}}M$ and is called the tangent cone of $A$ at $\bar{x}$; see [39, p. 71]). Thus the normal cone of $A$ at $\bar{x}$ is defined by

$$N_A(\bar{x}) := \{ u \in T_{\bar{x}}M : \langle u, v \rangle \leq 0 \text{ for each } v \in T_{\bar{x}}A. \}$$

Recall that $F_{\bar{x}}A$ denotes the set of all feasible directions defined by (2.4), that is,

$$F_{\bar{x}}A = \{ v \in T_{\bar{x}}M : \text{ there exists } t > 0 \text{ such that } \exp_{\bar{x}} tv \in A \text{ for any } t \in (0, \bar{t}) \}.$$

Using the notation $E_{\bar{x}}(\cdot)$ defined by (4.2) and noting (4.3), one sees that

$$F_{\bar{x}}A = \text{cone}(E_{\bar{x}}(A)).$$

Moreover, by [23, Proposition 3.4], $F_{\bar{x}}A \subseteq T_{\bar{x}}A \subseteq \overline{F_{\bar{x}}A}$. Therefore we have that

$$N_A(\bar{x}) = (F_{\bar{x}}A)^\circ = \{ u \in T_{\bar{x}}M : \langle u, v \rangle \leq 0 \text{ for each } v \in E_{\bar{x}}(A) \}.$$

Let $f : M \to \mathbb{R}$ be a proper function, and let $D(f)$ denote its domain, that is,

$$D(f) := \{ x \in M : f(x) \neq \infty \}.$$

We use $\Gamma^f_{x,y}$ to denote the set of all $\gamma_{xy} \in \Gamma_{x,y}$ such that $\gamma_{xy} \subseteq D(f)$. In the following definition, we introduce the notion of convex functions. The notion of the convex function is taken from [39], where it was defined on a totally convex subset.

**DEFINITION 6.1.** Let $f : M \to \mathbb{R}$ be a proper function with weakly convex domain $D(f)$. The function $f$ is said to be

(a) weakly convex if, for any $x, y \in D(f)$, there is a minimal geodesic $\gamma_{xy} \in \Gamma^f_{x,y}$ such that the composition $f \circ \gamma_{xy}$ is convex on $[0, 1]$;

(b) convex if, for any $x, y \in D(f)$ and any geodesic $\gamma_{xy} \in \Gamma^f_{x,y}$, the composition $f \circ \gamma_{xy}$ is convex on $[0, 1]$.

Suppose that $f : M \to \mathbb{R}$ is a proper weakly convex function. Let $x \in D(f)$ and $v \in T_{\bar{x}}M$. The directional derivative in direction $v$ and the subdifferential of $f$ at $x$ are, respectively, defined by

$$f'(x; v) := \lim_{t \to 0^+} \frac{f(\exp_{\bar{x}} tv) - f(x)}{t}$$

and

$$\partial f(x) := \{ u \in T_{\bar{x}}M : \langle u, v \rangle \leq f'(x; v) \ \forall v \in T_{\bar{x}}M \}.$$

Then $f'(x; \cdot)$ is proper and sublinear on $T_{\bar{x}}M$ with its domain

$$D(f'(x; \cdot)) = F_{\bar{x}}(D(f))$$

and

$$\partial f(x) = \partial f'(x; \cdot)(0).$$

In particular, let $\delta_A$ denote the delta function defined by $\delta_A(x) = 0$ if $x \in A$ and $\delta_A(x) = +\infty$ otherwise. Then one has that

$$\partial \delta_A(x) = N_A(x) \text{ for each } x \in A.$$
Furthermore, by Proposition 5.1 (applied to 0 in place of \( v \)), we get the subdifferential formula for the distance function \( d(z, \cdot) \) for some fixed \( z \in M \):

\[
\partial d(z, \cdot)(x) = -\frac{\exp^{-1} z}{d(z, x)} \quad \text{for any } x \in B(z, r(z)) \setminus \{z\}.
\]

The following proposition shows that the subdifferential of a convex function is monotone and upper Kuratowski semicontinuous. The proof for assertion (ii) was given in [39, p. 71], while the proofs for others are by definition and standard.

**Proposition 6.2.** Suppose that \( f : M \to \mathbb{R} \) is a proper weakly convex function.

Then the following assertions hold:

(i) \( \partial f : M \to 2^{TM} \) is upper Kuratowski semicontinuous, and \( \partial f(x) \) is convex for each \( x \in M \).

(ii) If \( x \in \text{int} D(f) \), then \( \partial f(x) \) is nonempty, compact, and convex.

(iii) If \( f \) is convex, then \( \partial f : M \to 2^{TM} \) is monotone.

Consequently, if \( A \subseteq \text{int} D(f) \), then \( \partial f \in \mathcal{V}(A) \).

The following proposition, which was proved in [23, Proposition 4.3], provides some sufficient conditions ensuring the sum rule of subdifferentials.

**Proposition 6.3.** Let \( f, g : M \to \mathbb{R} \) be proper weakly convex functions such that \( f + g \) is weakly convex. Then the following formula holds:

\[
\partial (f + g)(x) = \partial f(x) + \partial g(x) \quad \text{for each } x \in \text{int} D(f) \cap D(g).
\]

Consider the optimization problem

\[
\min_{x \in A} f(x).
\]

The solution set of the optimization problem (6.6) is denoted by \( S_f(A) \). Throughout this whole section, we assume that

\[
A, f, f + \delta_A \text{ are weakly convex and } A \subseteq \text{int} D(f).
\]

Thus, by Proposition 6.3 and (6.3), we have that the following sum rule holds:

\[
\partial (f + \delta_A)(x) = \partial f(x) + N_A(x) \quad \text{for each } x \in A.
\]

The following proposition describes the equivalence between the optimization problem (6.6) and the variational inequality problem (3.1).

**Proposition 6.4.** Under assumption (6.7), we have that \( S_f(A) = S(\partial f, A) \).

**Proof.** We first note the following chain of equivalences:

\[
\bar{x} \in S(\partial f, A) \iff 0 \in \partial f(\bar{x}) + N_A(\bar{x}) \iff 0 \in \partial (f + \delta_A)(\bar{x}) \iff (f + \delta_A)'(\bar{x}, v) \geq 0, \quad \forall v \in T_{\bar{x}}M,
\]

where the first and the last equivalences hold by definition, while the second equivalence does by formula (6.8). Thus to complete the proof, it suffices to show that \( \bar{x} \) is a solution of the optimization problem (6.6) if and only if

\[
(f + \delta_A)'(\bar{x}, v) \geq 0 \quad \text{for each } v \in T_{\bar{x}}M.
\]

The “only if” part is trivial because

\[
(f + \delta_A)(\exp_{\bar{x}} tv) - (f + \delta_A)(\bar{x}) \geq 0
\]
holds for each \( t \in [0, 1] \) and \( v \in T_x M \). To prove the “if” part, let \( x \in A \). Then there exists a geodesic \( \gamma \in \Gamma_{x,x}^{f+\delta_A} \) such that the function \( f \circ \gamma \) is convex on \([0, 1]\). Set \( v := \gamma'(0) \), and it follows that
\[
f(x) - f(\bar{x}) \geq (f + \delta_A)'(\bar{x}, v) \geq 0.
\]
This means that \( \bar{x} \) is a solution of the optimization problem (6.6) and completes the proof. \qed

Consider the following proximal point algorithm with initial point \( x_0 \in A \) for convex optimization problem (6.6):

\[
x_{n+1} \in \arg\min \left\{ f(x) + \frac{1}{2\lambda_n} d(x_n, x)^2 : x \in A \right\} \text{ for each } n \in \mathbb{N}.
\]

**Remark 6.1.** By assumption (6.7), one sees that, for each \( z \in A \) and \( \lambda > 0 \),
\[
\arg\min \left\{ f(x) + \frac{1}{2\lambda} d(z, x)^2 : x \in A \right\} \neq \emptyset.
\]
This means that the algorithm (6.10) generates at least a sequence \( \{x_n\} \).

**Theorem 6.5.** Let \( A \) be a weakly convex subset of \( M \) of the sectional curvature bounded above by \( \kappa \). In addition to assumption (6.7), we further suppose that \( f : M \to \mathbb{R} \) is a convex function such that \( S(f, A) \neq \emptyset \). Let \( x_0 \in A \) be such that \( d(x_0, S(f, A)) < \frac{D_\kappa}{\kappa} \), and let \( \{x_n\} \) be a sequence generated by algorithm (6.10). Suppose that \( \{\lambda_n\} \subset (0, +\infty) \) satisfy
\[
\sum_{n=0}^{\infty} \lambda_n = \infty \quad \text{and} \quad \lambda_n |\partial f(x_n)| < \frac{D_\kappa}{4} \quad \text{for each} \ n = 0, 1, \ldots.
\]
Then the algorithm (6.10) is well-defined, and the sequence \( \{x_n\} \) converges to a solution of the minimization problem (6.6).

**Proof.** Let \( V := \partial f \), and consider the variational inequality problem (3.1). Then \( V \in \mathcal{V}(A) \) is monotone by Proposition 6.2 and \( S(V, A) = S(f, A) \) by Proposition 6.4. Below we will verify that the sequence \( \{x_n\} \) coincides with the sequence generated by algorithm (5.1). To do this, let \( n \in \mathbb{N} \) and consider the function \( g_n : M \to \mathbb{R} \) defined by
\[
g_n(x) := \begin{cases} \frac{1}{2\lambda_n} d(x_n, x)^2 & x \in B(x_n, \frac{D_\kappa}{2}), \\ +\infty & \text{otherwise}. \end{cases}
\]
By Proposition 4.2(ii), \( g_n \) is convex on \( M \) and \( D(g_n) = B(x_n, \frac{D_\kappa}{2}) \) is strongly convex. Moreover, using (6.4) if \( x \neq x_{n+1} \), and by definition if \( x = x_{n+1} \), we have that
\[
\partial g_n(x) = -\exp_{x_{n+1}}^{-1} x_{n+1} \quad \text{for each} \ x \in B \left( x_n, \frac{D_\kappa}{2} \right).
\]
Let \( \gamma_n \in \Gamma_{x_n,x_{n+1}}^{f+\delta_A} \) be defined by \( \gamma_n(t) = \exp_{x_n} t v_n \) for each \( t \in [0, 1] \) and some \( v_n \in T_{x_n} M \) with \( ||v_n|| = d(x_n, x_{n+1}) \) such that \( f \circ \gamma_n \) is convex on \([0, 1]\). Then
\[
f(x_{n+1}) - f(x_n) \geq f'(x_n, v_n) \geq \langle u, v_n \rangle \geq -||u|| ||v_n|| \quad \text{for each} \ u \in \partial f(x_n).
\]
Hence
\[
\lambda_n(f(x_n) - f(x_{n+1})) \leq \lambda_n ||\partial f(x_n)|| ||v_n|| = \lambda_n ||\partial f(x_n)|| d(x_n, x_{n+1}).
\]
By (6.10), one checks that

\[ f(x_{n+1}) + \frac{1}{2\lambda_n}d(x_n, x_{n+1})^2 \leq f(x_n). \]

This, together with (6.13) and (6.11), implies that

\[ d(x_n, x_{n+1}) \leq 2\lambda_n|\partial f(x_n)| \leq \frac{D_n}{2}, \]

that is, \( x_{n+1} \in B(x_n, \frac{D_n}{2}) \). This means that \( x_{n+1} \) is a minimizer of \( f + g_n \) on \( A \). Note by Proposition 4.2 that \( D(f + g) = D(f) \cap D(g) \) is strongly convex. This implies that \( f + g \) is convex. Note also that \( x_n \in D(f) \cap \text{int}(D(g)) \). Then, we can apply Proposition 6.3 and (6.12) to conclude that

\[ \partial(f + g)(x) = \partial f(x) + \partial g(x) = \partial f(x) - \exp^{-1} x_{n+1} \quad \text{for each } x \in B \left( x_n, \frac{D_n}{2} \right). \]

Set \( A_R := A \cap B(o, R) \) with \( o := x_n \) and \( d(x_n, x_{n+1}) < R < \frac{D_n}{2} \). Then it follows from Proposition 6.4 that \( x_{n+1} \in S(V_{\lambda_n}, A_R) \), where \( V_{\lambda_n} \) is defined on \( A_R \) by

\[ V_{\lambda_n}(\cdot) = \partial(f + g)(\cdot) = \partial f(\cdot) - \exp^{-1}_x x_{n+1}. \]

Since \( x_{n+1} \in B(o, R) \), it follows from Proposition 3.2 that \( x_{n+1} \in S(V_{\lambda_n}, A) \), that is \( x_{n+1} \in J_{\lambda_n}(x_n) \). Therefore, the sequence \( \{x_n\} \) coincides with the one generated by algorithm (5.1). Since (5.8) holds by assumption (6.11), it follows from Remark 5.1(b) that algorithm (5.1), and thus algorithm (6.10) is well-defined. It remains to show that the sequence \( \{x_n\} \) converges to a solution of the minimization problem (6.6). Note by (6.14) that

\[ \sum_{n=0}^{+\infty} \lambda_n \left( \frac{d(x_n, x_{n+1})}{\lambda_n} \right)^2 \leq 2 \sum_{n=0}^{+\infty} (f(x_n) - f(x_{n+1})) < +\infty. \]

This, together with (6.11), implies that (5.9) holds. Hence Theorem 5.6 is applicable, and the sequence \( \{x_n\} \) converges to a point in \( S_V = S(f, A) \). Thus the proof is complete.

7. Conclusion. We have established, by developing a new approach, the results on the existence of the solution, on convexity of the solution set, and on the convergence of the proximal point algorithm for variational inequality problems for set-valued vector fields on general Riemannian manifolds. To the best of our knowledge, all of the known and important works in this direction, as mentioned in the introduction, are done for univalued vector fields and on Hadamard manifolds, except the work in [25], which was done on Riemannian manifolds, but was concerned only with the existence problem for univalued vector fields. In particular, it seems that ours is the first paper to explore the proximal point algorithm on general Riemannian manifolds and/or to establish the convexity results of the solution sets for the variational inequality problems, which seems new even for univalued vector fields on Hadamard manifolds. Below we provide two examples which illustrate that our results in the present paper are applicable, but the results in [14, 26] are not (and neither are those in [21, 25]).
Example 7.1. Let

\[ M = H := \{(y_1, y_2) \in \mathbb{R}^2 | y_2 > 0\} \]

be the Poincaré plane endowed with the Riemannian metric defined by

\[ g_{11} = g_{22} = \frac{1}{y_2^2}, \quad g_{12} = 0 \quad \text{for each } (y_1, y_2) \in H. \]

It is well known that \( H \) is a Hadamard manifold (cf. [39, p. 86]). Taking \( \bar{x} = (\bar{y}_1, \bar{y}_2) \in H \), we get \( T_{\bar{x}}H = \mathbb{R}^2 \). Moreover, the geodesics of the Poincaré plane are the semilines \( C_a : y_1 = a, \ y_2 > 0 \) and the semicircles \( C_{b,r} : (y_1 - b)^2 + y_2^2 = r^2, \ y_2 > 0 \). Let \( A \subseteq H \) and \( V : H \to \mathbb{R}^2 \) be defined, respectively, by

\[
\begin{aligned}
A &:= \{(y_1, y_2) \in \mathbb{R}^2 : y_1 \in [1, 3], \ 2 \leq (y_1 - 2)^2 + y_2^2 \leq 10\} \\
V(y_1, y_2) &:= \left\{ \begin{array}{l}
\{(2y_1y_2, y_2^2 - y_1^2) : t \in [0, 1]\}, \ y_1^2 + y_2^2 > 3, \\
\{(2t(y_1, y_2, t(y_2^2 + \frac{1}{y_2} - y_1^2) - \frac{1}{y_2}) : t \in [0, 1]\}, \ 1 \leq y_1^2 + y_2^2 \leq 3, \\
(0, -\frac{1}{y_2}), \ y_1^2 + y_2^2 < 1.
\end{array} \right.
\end{aligned}
\]

Then \( V(x) \) is compact and convex for each \( x \in H \). Furthermore, \( V \) is upper Kuratowski semicontinuous on \( H \), and thus \( V \in \mathcal{V}(A) \). Note that \( A \) is compact and convex. Thus Corollary 3.7 is applicable to conclude that the variational inequality problem (3.1) admits at least one solution. In fact, one can check by definition that the solution set is

\[ S(V, A) \supseteq \left\{ (1, y_2) \in A : y \in \left[ 1, \frac{\sqrt{2}}{2} \right] \right\}. \]

Since \( V \) is not univalued, the corresponding existence results in [26] or [25] are not applicable.

Next we consider the vector field \( V \in \mathcal{V}(A) \) defined as follows. Let \( V_1 \) be defined by

\[ V_1(y_1, y_2) := \left\{ \begin{array}{l}
(2y_1y_2, y_2^2 - y_1^2), \ y_1^2 + y_2^2 > 2, \\
(2t(y_1, y_2, t(y_2^2 + 2 - y_1^2) - 2) : t \in [0, 1]\}, \ y_1^2 + y_2^2 = 2, \\
(0, -2), \ y_1^2 + y_2^2 < 2.
\end{array} \right. \]

By [39, p. 301], the function \( f \) defined on \( H \) by

\[ f(y_1, y_2) := \max \left\{ \frac{y_1^2 + y_2^2}{y_2}, \frac{2}{y_2} \right\} \quad \text{for any } (y_1, y_2) \in H \]

is convex, and by [39, p. 297], one checks that \( V_1 \) is equal to the subdifferential of \( f \). Hence, by [21, Theorem 5.1], \( V_1 \) is monotone and upper Kuratowski semicontinuous on \( H \). Let \( P_C \) denote the projection on \( C \), where \( C := \{(y_1, y_2) \in H : y_1^2 + y_2^2 \leq 3\} \subseteq H \) is a closed convex subset. Then we define \( V \) by

\[ V(y_1, y_2) := \exp_{(y_1, y_2)}^{-1} P_C(y_1, y_2) + V_1(y_1, y_2) \quad \text{for any } (y_1, y_2) \in H. \]

By [41], \( P_C \) is nonexpansive on \( H \), and it follows from [31] that the vector field \( (y_1, y_2) \mapsto \exp_{(y_1, y_2)}^{-1} P_C(y_1, y_2) \) is monotone and continuous on \( H \). Consequently, we
have that $V \in \mathcal{V}(A)$ is monotone. Consider the variational inequality problem (3.1) with $V$ given as above and $A$ defined by (7.1). Then we conclude from Corollaries 3.7 and 4.7 that $S(V, A)$ is nonempty and convex. Again, neither the results in [26] nor the results in [25] are applicable.

Example 7.2. Let $M = S^2 := \{(y_1, y_2, y_3) \in \mathbb{R}^3 \mid y_1^2 + y_2^2 + y_3^2 = 1 \}$ be the 2-dimensional unit sphere; see [39, p. 84] for more details. Denoting $z_1 := (0, 0, 1)$ and $z_{-1} := (0, 0, -1)$, observe that the manifold $S^2 \setminus \{z_1, z_{-1}\}$ can be parameterized by $\Phi: (0, \pi) \times [0, 2\pi] \subset \mathbb{R}^2 \rightarrow S^2 \setminus \{z_1, z_{-1}\}$ defined as $\Phi(\theta, \varphi) := (y_1, y_2, y_3)^T$ for each $\theta \in (0, \pi)$ and $\varphi \in [0, 2\pi]$ with

$$\begin{align*}
y_1 :&= \sin \theta \cos \varphi, \\
y_2 :&= \sin \theta \sin \varphi, \\
y_3 :&= \cos \theta.
\end{align*}$$

It is easy to see from the above that $(S^2 \setminus \{z_1, z_{-1}\}, \Phi^{-1})$ is a system of coordinates around $x$ whenever $x \in S^2 \setminus \{z_1, z_{-1}\}$. Then the Riemannian metric on $S^2 \setminus \{z_1, z_{-1}\}$ is given by

$$g_{11} = 1, \quad g_{12} = 0, \quad g_{22} = \sin^2 \theta \quad \text{for each} \quad \theta \in (0, \pi) \quad \text{and} \quad \varphi \in [0, 2\pi].$$

The geodesics of $S^2 \setminus \{z_1, z_{-1}\}$ are the greatest circles or semicircles. Furthermore, we have the curvature $\kappa = 1$ and thus $D_\kappa = \pi$. Let $\triangle := \triangle(a, b, c)$ be the geodesic triangle in the first quadrant of $S^2$ with vertices $a := (1, 0, 0)$, $b := (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0)$, $c := (\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2})$ and edges $\gamma_{a,b}$, $\gamma_{b,c}$, $\gamma_{c,a}$. Let $A$ be the set in the first quadrant of $S^2$ with its boundary $\partial A = \triangle$. Clearly, $A$ is closed weakly convex. Define the function $f: S^2 \rightarrow \mathbb{R}$ by

$$f(y) := \begin{cases} -d(y, \triangle) & y \in A, \\ +\infty & \text{otherwise}, \end{cases}$$

where $d(y, \triangle)$ denotes the distance from $y$ to $\triangle$. By [36, Chapter 5, Lemma 3.3], one sees that $f$ is convex on $A$. Set $d := 1 + \sqrt{3} + \sqrt{5}$. Below we show that

$$\begin{align*}
y := \left(\frac{d}{\sqrt{2} + d^2}, \frac{1}{\sqrt{2} + d^2}, \frac{1}{\sqrt{2} + d^2}\right) \in S(f, A).\end{align*}$$

(Indeed, we could further show that $S(f, A) = \{y\}$.) To do this, let $\angle a$ and $\angle b$ denote the corresponding angles of the triangle $\triangle$ at vertices $a$ and $b$, respectively. Then the geodesic segments in $A$ bisection them are $c_a$ and $c_b$, which are, respectively, defined by

$$c_a : y_2 = y_3 \quad \text{and} \quad c_b : y_1 = y_2 + (\sqrt{3} + \sqrt{5})y_3 \quad \text{for any} \quad (y_1, y_2, y_3) \in A.$$

It is routine to check that $y$ is the intersection of the geodesics $c_a$ and $c_b$. Note that the nearest point to $y$ from each edge of the triangle $\triangle$ is in the interior of the edge. Thus, using the law of cosines (4.1) (with $\kappa = 1$), we conclude that the distances from $y$ to all edges of the triangle $\triangle$ are equal, that is, $d(y, \gamma_{a,b}) = d(y, \gamma_{b,c}) = d(y, \gamma_{c,a})$. Let $x \in A$, and let $y_0 \in \gamma_{a,b}$ be the nearest point to $y$ from the edge $\gamma_{a,b}$, that is, $d(y, y_0) = d(y, \gamma_{a,b})$. Without loss of generality, we assume that $x$ is located in
the set bounded by the triangle $\triangle(a, y^0, \hat{y})$. Consider the triangle $\triangle(a, y^0, \hat{y})$, where $\hat{y} \in A$ is the intersection of the geodesic segment joining $y^0$ to $\hat{y}$ and the geodesic line determined by $a, x$. Let $x^0 \in \gamma_{a,b}$ be the point corresponding to $x$ satisfying $\frac{d(a, x^0)}{d(a,y^0)} = \frac{d(a, x)}{d(a, y)}$. We claim that

$$d(x, x^0) \leq d(\hat{y}, y^0).$$

(7.4)

Granting this, we have that

$$d(x, \triangle) \leq d(x, x^0) \leq d(\hat{y}, y^0) < d(\hat{y}, y^0) = d(\hat{y}, \triangle);$$

hence, $f(x) \geq f(\hat{y})$, and (7.2) is proved as $x \in A$ is arbitrary.

To verify (7.4), let $a, b \in (0, \frac{\pi}{2})$ and $\alpha \in [0, \frac{\pi}{2})$. Consider the function $h$ on $[0, 1]$ defined by

$$h(t) := \cos at \cos bt + \sin at \sin bt \cos \alpha \quad \text{for each } t \in [0, 1].$$

(7.5)

Then, by elementary calculus, one checks that $h$ is decreasing (and strictly decreasing if $\alpha \neq 0$) on $[0, 1]$. Now take $a := d(a, y^0)$, $b := d(a, y)$ and let $\alpha$ be the inner angle at vertex $a$ of the triangle $\triangle(a, y^0, \hat{y})$. By the choice of the points $a, b, c$, one sees that $a, b \in (0, \frac{\pi}{2})$ and $\alpha \in [0, \frac{\pi}{2})$. Writing $t := \frac{d(a, x^0)}{d(a, y^0)} = \frac{d(a, x)}{d(a, y)}$ and applying the law of cosines (4.1) with $\kappa = 1$, we conclude that

$$\cos d(x, x^0) = h(t) \geq h(1) = \cos d(\hat{y}, y^0)$$

(noting that $at = d(a, x^0)$ and $bt = d(a, x)$). Thus (7.4) is seen to hold. Moreover, since $f$ is Lipschitz continuous on $A$ with modulus 1 (cf. [23, Proposition 3.1]), we have that $|\partial f(y)| \leq 1$ for each $y \in A$. Let $\{\lambda_n\} \subseteq [0, \frac{\pi}{2}]$ be such that $\sum_{n=0}^{\infty} \lambda_n = \infty$, and let $x_0 \in \operatorname{int} A$ be such that $d(x_0, y) < \frac{\pi}{2}$. Note that there exists a closed convex subset $A_0 \subseteq \operatorname{int} A (= \operatorname{int} D(f))$ such that $x_0, \hat{y} \in A_0$. Thus, Theorem 6.5 is applicable (to $A_0$ in place of $A$) to concluding that the sequence $\{x_n\}$ generated by algorithm (6.10) is well-defined and converges to the unique solution $\hat{y}$. Clearly, $S = S^2$ is not a Hadamard manifold; hence, the results in [14] and [21] cannot be applicable.

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REFERENCES


