Algebraic Topology Homework Solutions: Week 9

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6.1. X is a universal covering space of itself, hence there is a homomorphism \( \varphi : (X, \text{id}_X) \to (\tilde{X}, p) \), which implies that \( p \varphi = \text{id}_X \). Then \( p \) is injective hence a homeomorphism.

6.2. \( \pi(S^1) \cong \mathbb{Z} \) is abelian, whose subgroups are \( n\mathbb{Z}, n \geq 1 \) and the trivial subgroup. The corresponding covering spaces are the \( n \)-fold cover \( S^1 \to S^1, z \mapsto z^n \) and the universal cover \( \mathbb{R} \to S^1, t \mapsto e^{2\pi it} \).

\( \pi(P) \cong \mathbb{Z}/2\mathbb{Z} \), whose subgroups are itself and the trivial subgroup. The corresponding covering spaces are \( \text{id}_P : P \to P \) and the 2-fold cover \( S^2 \to P \) which identifies the antipodes of \( S^2 \).

The set \( \{ (x, y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 4 \} \) is homeomorphic to \( X := [1, 2] \times S^1 \). Note that \([1, 2]\) is simply connected. Therefore the covering spaces of \( X \) are of the form \( ([1, 2] \times Y, \text{id} \times p) \), where \((Y, p)\) is a covering space of \( S^1 \).

6.4. Let \( U \subset Z \) be an elementary neighborhood for the covering space \( (X, q) \), and let \( V \) be a path component of \( r^{-1}(U) \). Let \( W \) be a path component of \( p^{-1}(V) \). Then \( p|_W : W \to V \) is a covering space. But \( W \) is a path connected subset of \( q^{-1}(U) \), hence it is mapped homeomorphically by \( q \) into \( U \). It follows that \( p|_W : W \to V \) is injective, hence a homeomorphism. If \( W' \) is the path component of \( q^{-1}(U) \) containing \( W \), then \( p(W') \subset V \), which implies that \( r : V \to U \) is surjective. In fact \( W' = W \). Therefore \( r = q_W \circ (p|_W)^{-1} : V \to W \to U \) is a homeomorphism.

6.5. Let \( (\tilde{X}, p) \) be a universal covering space of \( X \). Then \( \tilde{X} \) is also a universal covering space of \( X_1 \), hence also a covering space of \( X_2 \). The assertion now follows from Exercise 6.4.

7.2. We note that if \( \pi(X) \) is abelian, then \( N[p_* \pi(\tilde{X})] = \pi(X) \) hence \( A(\tilde{X}, p) \cong \pi(X)/p_* \pi(\tilde{X}) \).

Example 2.2: \( A(\tilde{X}) = \mathbb{Z}/n\mathbb{Z} \).
Example 2.4: For the covering space $\tilde{X} = \mathbb{R} \times \mathbb{R}$ which is simply connected, one has $A(\tilde{X}, p) \cong \pi(T) \cong \mathbb{Z}^2$. For $\tilde{X} = \mathbb{R} \times S^1$, one has $A(\tilde{X}, p) \cong \pi(T)/\pi(S^1) \cong \mathbb{Z}^2/\mathbb{Z} \cong \mathbb{Z}$, where the subgroup $\pi(S^1)$ of $\pi(T)$ means the second factor of the direct product $\pi(T) \cong \pi(S^1) \times \pi(S^1)$.

Example 2.8: $\mathbb{C}$ is simply connected, hence $A(\mathbb{C}, \exp) \cong \pi(\mathbb{C}^\times) \cong \mathbb{Z}$.

Example 2.9: $A(\mathbb{C}^\times, p_n) = \mathbb{Z}/n\mathbb{Z}$. 