ALGEBRAIC TOPOLOGY EXERCISE SOLUTIONS: CHAPTER VIII

Comments can be sent to maliu@zju.edu.cn

2.1 Consider the map $F(x, t) = (1 - t)x + tx/|x|$.

2.2. Fix the embedding $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1}$, $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0)$. Let $P$ be the north pole of $S^n$. For $x \in S^n - \{P\}$, let $\ell_x$ be the line passing through $P$ and $x$. Consider the map $f : S^n - \{P\} \rightarrow \mathbb{R}^n$ such that $f(x)$ is the intersection of $\ell_x$ with the hyperplane $x_{n+1} = 0$. Then $f$ is a homeomorphism, which is called the stereographic projection.

2.3. (a) By Exercise 2.2, their 1-point compactifications are $S^m$ and $S^n$, which are not homeomorphic because they have different homology groups. Hence $\mathbb{R}^m$ and $\mathbb{R}^n$ cannot be homeomorphic either.

(b) The complement of a point in $\mathbb{R}^m$ has the same homotopy type as $S^{m-1}$, hence it cannot be homeomorphic to the complement of a point in $\mathbb{R}^n$.

2.4. If $x$ is an interior point, then $E^n - \{x\}$ has the same homotopy type as $S^{n-1}$; if $x$ is a boundary point, then $E^n - \{x\}$ is contractible. Because $E^n - \{x\}$ and $E^n - \{h(x)\}$ are homeomorphic, $h$ must map the boundary $S^{n-1}$ onto itself.

2.5. If $f$ is not onto, then $f_*$ maps $\tilde{H}_n(S^n)$ into $\tilde{H}_n(S^n - \{x\}) \cong \tilde{H}_n(\mathbb{R}^n) = 0$ for some $x \in S^n$. Hence $f_* = 0$ so that $\deg f = 0$.

2.6. By excision, $H_n(X, X - \{x\}) \cong H_n(N, N - \{x\})$. Since $N$ is contractible, the long homology exact sequence for $(N, N - \{x\})$ shows that $H_n(N, N - \{x\}) \cong \tilde{H}_{n-1}(N - \{x\}) \cong \tilde{H}_{n-1}(B)$, where the last isomorphism follows from the assumption that $B$ is a deformation retract of $N - \{x\}$.

2.7. Using Exercise 2.6, it is easy to see that

$$H_q(E^n, E^n - \{x\}) \cong \tilde{H}_{q-1}(S^{n-1}) \cong \begin{cases} \mathbb{Z}, & q = n \\ 0, & q \neq n \end{cases}$$

if $x$ is an interior point, and $H_q(E^n, E^n - \{x\}) \cong \tilde{H}_{q-1}(S^{n-1} - \{pt\}) \cong \tilde{H}_{q-1}(\mathbb{R}^{n-1}) = 0$ if $x$ is a boundary point.

To prove Exercise 2.4, if $h : E^n \rightarrow E^n$ is a homeomorphism, then $E^n$ has the same local homology groups at $x$ and $h(x)$. Hence our computation shows that $h$ must map the boundary $S^{n-1}$ onto itself.
2.8. The \( q \)-dimensional local homology groups of an \( n \)-dimensional manifold are isomorphic to \( \tilde{H}_{q-1}(S^{n-1}) \), which cannot be equal to \( \tilde{H}_{q-1}(S^{m-1}) \) for all \( q \) unless \( m = n \).

2.9. Denote \( Y = \{ x \in X \mid H_2(X, X - \{ x \}) = 0 \} \). Then \( Y \) is homeomorphic to \( S^1 \) if \( X \) is a Mobius strip; but \( Y \) is homeomorphic to a disjoint union of two \( S^1 \) if \( X \) is the annulus. Hence Mobius strip and annulus are not homeomorphic.

3.1. If \( k = 0 \) then \( f \) is constant hence \( \deg f = 0 \). If \( k \neq 0 \), subdivide \( S^1 \) into \( |k| \) short arcs with equal length. Similar to Example 3.2 one can show that \( \deg f = k \).

3.2. We just saw in Exercise 3.1 that for any \( k \in \mathbb{Z} \) there exists a continuous map \( f : S^1 \to S^1 \) of degree \( k \). Then \( \Sigma^{n-1} f : S^n \to S^n \) is also of degree \( k \).

3.3. Using stereographic projection, \( V_n \overset{\text{def}}{=} \{ z : |z| > n \} \cup \{ \infty \}, n = 1, 2, \ldots \), form a base of neighborhoods of \( \infty \). Since \( f \) is of positive degree, \( f(r) \to +\infty \) as \( r \in \mathbb{R} \) tends to \( +\infty \). Hence for each \( n \) we have \( \bar{f}^{-1}(V_n) \supset V_m \) for \( m \) large enough. This shows the continuity of \( \bar{f} \) at \( \infty \).

3.4. The stereographic projection maps the upper semisphere onto \( \{ z \in \mathbb{C} : |z| \geq 1 \} \cup \{ \infty \} \), and maps the lower semisphere onto \( \{ z \in \mathbb{C} : |z| \leq 1 \} \). Thus we see that \( \bar{f} : S^2 \to S^2 \) preserves the upper and lower semisphere of \( S^2 \), and its restriction to \( S^1 \) coincides with the map \( f_1 : S^1 \to S^1, z \mapsto z^k \).

Similar to the proof of \( \deg \Sigma f = \deg f \), by the above observations we have a commutative diagram

\[
\begin{array}{ccc}
H_2(S^2) & \xrightarrow{\varphi} & H_1(S^1) \\
\downarrow f_* & & \downarrow f_* \\
H_2(S^2) & \xrightarrow{\varphi} & H_1(S^1)
\end{array}
\]

where \( \varphi \) is the isomorphism constructed in the proof of the induction \( \tilde{H}_{i+1}(S^{n+1}) \cong \tilde{H}_i(S^n) \). It follows that \( \deg \bar{f} = \deg f_1 = k \), where the last equality is given by Exercise 3.1.

3.5. Multiplication by a nonzero complex number induces a homeomorphism of \( S^2 \) unto itself, hence it does no harm to assume that \( f \) is monic. Then the family of maps \( f_t(z) = z^k + (1-t)(f - z^k) \) extends to a homotopy \( \bar{f}_t \) from \( \bar{f} \) to \( z^k : S^2 \to S^2 \). (explain in details why we cannot deform \( f \) to a polynomial of different degree and then extend) Therefore \( \deg \bar{f} = k \) by Exercise 3.4.

3.6. Since \( \deg \bar{f} = k \neq 0 \), by Exercise 2.5 \( \bar{f} \) is onto. Noting that \( \bar{f}(\infty) = \infty \), \( f \) must map some \( z \in \mathbb{C} \) to 0.
3.7. Consider the local homology groups $H_2(X, X - \{x\})$, which we denote by $H_x$ for short. Applying Exercise 2.6 one can show the following:

1. If $x$ is not on any coordinate axis, then $H_x \cong H_1(S^1) \cong \mathbb{Z}$.
2. If $x$ lies on a coordinate axis but is not the origin, then $H_x \cong H_1(Y)$, where $Y$ is a connected graph with 2 vertices and 4 edges joining them. Then $Y$ has Euler characteristic $\chi(Y) = -2$, which implies that rank $H_1(Y) = \text{rank } H_0(Y) - (-2) = 3$, i.e. $H_1(Y) = \mathbb{Z}^3$.
3. If $x$ is the origin, then $H_x \cong H_1(Y)$, where $Y$ is a connected graph with 6 vertices and 12 edges (attached to an octahedron). Then $Y$ has Euler characteristic $\chi(Y) = -6$, hence rank $H_1(Y) = \text{rank } H_0(Y) - (-6) = 7$, i.e. $H_1(Y) = \mathbb{Z}^7$.

In summary, the local homology group at the origin is different from that at any other point. Hence it must be a fixed point of any homeomorphism of $X$ onto itself.

4.1. $H_0(X) = \mathbb{Z}$, $H_1(X) = \mathbb{Z}/3\mathbb{Z}$, $H_q(X) = 0$ for $q \geq 2$.

4.2. Consider an $n$-sided polydisc, whose edges are all given the counter-clockwise orientation. Let $X$ be obtained by identifying the $n$ edges. Then $H_1(X) = \mathbb{Z}/n\mathbb{Z}$.

5.1. Since $\tilde{H}_n(U \cap V) = 0$ for all $n$, the Mayer-Vietoris sequence yields $\tilde{H}_n(X) \cong \tilde{H}_n(U) \oplus \tilde{H}_n(V)$.

5.2. Take $U = A \cup N$, $V = B \cup N$. Then $U, V$ satisfy the assumptions of Exercise 5.1. It is clear that $A, B$ are deformation retracts of $U, V$ respectively. Hence $\tilde{H}_n(X) \cong \tilde{H}_n(U) \oplus \tilde{H}_n(V) \cong \tilde{H}_n(A) \oplus \tilde{H}_n(B)$.

5.3. (a) The assumption $X = A^o \cup B^o$ implies that $\bar{X} - \bar{A} \subset B^o$. Therefore $H_n(A, A \cap B) \cong H_n(X, B)$ by excision. Similarly $H_n(B, A \cap B) \cong H_n(X, A)$.

(b) We have the commutative diagram with exact rows

$$
\begin{array}{cccccc}
0 \to & C_n(A \cap B) \to & C_n(A) \oplus C_n(B) \to & C_n(X, \mathcal{U}) \to & 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 \to & C_n(A \cap B) \to & C_n(A) \to & C_n(A, A \cap B) \to & 0
\end{array}
$$

where $\mathcal{U} = \{A, B\}$, $\psi$ is the natural projection, and $\varphi$ is the natural map $C_n(X, \mathcal{U}) \to C_n(X) \to C_n(X, B) \cong C_n(A, A \cap B)$. The vertical maps also commute with various boundary operators $\partial_n : C_n \to C_{n-1}$. In particular $\varphi$ maps $Z_n(X, \mathcal{U})$ into $Z_n(A, A \cap B)$. Take $\alpha \in Z_n(X, \mathcal{U})$ and a lift $\alpha' \in C_n(A) \oplus C_n(B)$. Then $\psi(\alpha')$ is a lift of $\varphi(\alpha) \in Z_n(A, A \cap B)$. We know that $\partial_n \alpha' = \Phi(\beta) = (i_# \beta, j_# \beta)$ for some $\beta \in Z_{n-1}(A \cap B)$, which implies that $\partial_n \psi(\alpha') = \psi(\partial_n \alpha') = i_# \beta$. It follows that the boundary operator

$\Delta : H_n(X, \mathcal{U}) \to H_{n-1}(A \cap B)$

maps $[\alpha]$ to $[\beta]$, and the boundary operator

$\partial_s : H_n(A, A \cap B) \to H_{n-1}(A \cap B)$
maps \( \varphi_*[\alpha] = [\varphi(\alpha)] \) to \([\beta]\). In other words, \( \Delta \) coincides with \( \partial_* \varphi_* \), where \( \varphi_* : H_n(X, \mathcal{U}) \to H_n(A, A \cap B) \) is the composition

\[
H_n(X, \mathcal{U}) \cong H_n(X) \to H_n(X, B) \cong H_n(A, A \cap B).
\]

This is the required assertion.

5.4. We have the commutative diagram with exact rows

\[
\begin{array}{cccccc}
H_n(A \cap B) & i_* & H_n(A) & j_* & H_n(A, A \cap B) & \partial_* & H_{n-1}(A \cap B) & i_* & H_{n-1}(A) \\
\downarrow k_* & & \downarrow i_* & & \downarrow h_* & & \downarrow k_* & & \downarrow i_* \\
H_n(B) & i' & H_n(X) & j' & H_n(X, B) & \partial' & H_{n-1}(B) & i' & H_{n-1}(X)
\end{array}
\]

The homomorphism \( \Delta : H_n(X) \to H_{n-1}(A \cap B) \) is defined by \( \partial_* \circ h_*^{-1} \circ j_* \). We only need to check the exactness at \( H_n(X) \) and \( H_{n-1}(A \cap B) \) of the Mayer-Vietoris sequence

\[
H_n(A) \oplus H_n(B) \xrightarrow{\psi} H_n(X) \xrightarrow{\Delta} H_{n-1}(A \cap B) \xrightarrow{\varphi} H_{n-1}(A) \oplus H_{n-1}(B)
\]

where \( \psi = (l_* - i_*), \varphi = (i_*, k_*) \).

It is easy to see that \( \Delta \psi = 0, \varphi \Delta = 0 \). It remains to show that

(1) \( \ker \Delta \subset \operatorname{im} \psi \): Take \( \gamma \in H_n(X) \) such that \( \Delta \gamma = 0 \). Then there exists \( \alpha \in H_n(A) \) such that \( h_*^{-1} j_* \gamma = j_* \alpha \). It follows that \( \gamma - l_* \alpha \in \operatorname{im} j'_* \), i.e. \( \gamma \in \operatorname{im} \psi \).

(2) \( \ker \varphi \subset \operatorname{im} \Delta \): Take \( \alpha \in H_{n-1}(A \cap B) \) such that \( \varphi(\alpha) = 0 \), i.e. \( i_* \alpha = 0, k_* \alpha = 0 \). Then \( \alpha = \partial_* \beta \) for some \( \beta \in H_n(A, A \cap B) \). We have \( \partial'_* h_* \beta = k_* \partial_* \beta = 0 \), hence \( h_* \beta = j'_* \gamma \) for some \( \gamma \in H_n(X) \), i.e. \( \alpha = \Delta \gamma \).

6.1. Consider \( \mathbb{R}^n \) as the complement of a point in \( S^n \), so that the Mayer-Vietoris sequence for \( S^n = \mathbb{R}^n \cup (S^n - Y) \) reads

\[
\cdots \to \tilde{H}_i(\mathbb{R}^n - Y) \to \tilde{H}_i(\mathbb{R}^n) \oplus \tilde{H}_i(S^n - Y) \to \tilde{H}_i(S^n) \to \tilde{H}_{i-1}(\mathbb{R}^n - Y) \to \cdots
\]

Since \( \tilde{H}_i(\mathbb{R}^n) = \tilde{H}_i(S^n - Y) = 0 \) for all \( i \), we have \( \tilde{H}_i(\mathbb{R}^n - Y) \cong \tilde{H}_{i+1}(S^n) = \mathbb{Z} \) if \( i = n - 1 \) and 0 otherwise.

6.2. Again think of \( \mathbb{R}^n \) as a complement of a point in \( S^n \), and we have the Mayer-Vietoris sequence

\[
\cdots \to \tilde{H}_i(\mathbb{R}^n - A) \to \tilde{H}_i(\mathbb{R}^n) \oplus \tilde{H}_i(S^n - A) \to \tilde{H}_i(S^n) \to \tilde{H}_{i-1}(\mathbb{R}^n - A) \to \cdots
\]

Since \( \tilde{H}_i(\mathbb{R}^n) = 0 \) for all \( i \), \( \tilde{H}_i(S^n - A) \cong \mathbb{Z} \) for \( i = n - k - 1 \) and 0 otherwise, we have

(1) If \( k = 0 \), then \( \tilde{H}_i(\mathbb{R}^n - A) \cong \mathbb{Z}^2 \) for \( i = n - 1 \) and 0 otherwise.

(2) If \( k > 0 \), then \( \tilde{H}_i(\mathbb{R}^n - A) \cong \mathbb{Z} \) for \( i = n - 1 \) or \( n - k - 1 \), and 0 otherwise.

It follows that \( \mathbb{R}^n - A \) has three components if \( n = 1, k = 0 \); two components if \( n > 1 \) and \( k = n - 1 \); one component otherwise.
6.3. Let $A$ be a subset of $\mathbb{R}^n$ which is homeomorphic to $S^{n-1}$. If $n = 1$, then $\mathbb{R} - A$ has three components and the assertion cannot hold. If $n > 1$, then by the previous two exercises, $\mathbb{R}^n - A$ has two components, and the analog of Lemma 6.2 holds, i.e. $\mathbb{R}^n - Y$ is connected for a subset $Y$ of $\mathbb{R}^n$ which is homeomorphic to $I^{n-1}$. Then the proof of Proposition 6.5 can be carried out word-by-word to this case, which shows that $A$ is the boundary of each component of $\mathbb{R}^n - A$.

6.4. Since $A$ is closed and bounded closed sets are compact, a homeomorphism $h : A \to \mathbb{R}^{n-1}$ maps bounded subsets to bounded subsets. It follows that we may add a point to compactify both $\mathbb{R}^n$ and $A$ simultaneously. Therefore $\mathbb{R}^n - A$ is homeomorphic to $S^n - A'$, where $A'$ is a subset of $S^n$ homeomorphic to $S^{n-1}$. Then the assertion follows from the Jordan-Brouwer theorem.

Note that the assertion is not true if $A$ is not closed. For example consider $n = 2$, $A = \{(x,0) : x > 0\}$, then $\mathbb{R}^2 - A$ is connected.

6.5. Let $U$ and $V$ be homeomorphic subsets of $\mathbb{R}^n$. Think of $\mathbb{R}^n$ as the complement of a point of $S^n$, so that it is open in $S^n$. If $U$ is open in $\mathbb{R}^n$, then it is open in $S^n$ as well, hence $V$ is open by Theorem 6.6. It follows directly that if $A, B$ are subsets of $\mathbb{R}^n$ and $h : A \to B$ is a homeomorphism, then $h$ maps interior points onto interior points, and boundary points onto boundary points.

Let $U, V$ be subsets of the $n$-dimensional manifolds $M, N$ respectively, such that $h : U \to V$ is a homeomorphism. Assume that $U$ is open, $h$ maps $x \in U$ to $y \in V$. Take open neighborhoods $U', V'$ of $x, y$ respectively, which are both homeomorphic to $\mathbb{R}^n$. Then $h$ maps $U \cap U' \cap f^{-1}(V')$ onto $V \cap f(U') \cap V'$. Because $U \cap U' \cap f^{-1}(V')$ is open, we have $V \cap f(U') \cap V'$ is open as well. It follows that $V$ is open.

6.6. Take the standard embedding $\mathbb{R}^n \subset \mathbb{R}^m$ by $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0, \ldots, 0)$, where $m > n$. If $\mathbb{R}^m$ is homeomorphic to $\mathbb{R}^n$, then Brouwer’s theorem on invariance of domain for subsets of $\mathbb{R}^n$ (Exercise 6.5) implies that $\mathbb{R}^n$ is open in $\mathbb{R}^m$, which is absurd.

6.7. Take the standard embedding $S^n \subset \mathbb{R}^{n+1} \subset \mathbb{R}^m$. Then $S^n$ has no interior points, but $I^m$ has. Hence by Exercise 6.5 no subset of $S^n$ is homeomorphic to $I^m$.

6.8. We need to show that $f$ is open, i.e. $f$ maps open sets to open sets. Take an open subset $V$ of $U$, $x \in V$, and a closed ball $B(x, r) \subset V$, where $r > 0$. By the closed mapping theorem (which says a continuous map $f$ from a compact space $X$ to a Hausdorff space $Y$ is closed), $f$ gives a homeomorphism from $B(x, r)$ to $f(B(x, r))$. By Exercise 6.5, $f(B(x, r))$ is open. Therefore $f(V)$ is open.

6.9. A proper subset of $S^n$ can be considered as a subset of $\mathbb{R}^n$. But no subset of $\mathbb{R}^n$ can be homeomorphic to $S^n$, by Chapter V, Corollary 9.4.
6.10. This is Chapter V, Corollary 9.3.

6.11. Take an $m$-dimensional closed ball inside $U$. A continuous, one-to-one map $U \to \mathbb{R}^n$ yields a continuous one-to-one map $S^{m-1} \to \mathbb{R}^n \subset \mathbb{R}^{m-1}$, which is impossible.

More generally, if $M, N$ are manifolds of dimension $m > n$ respectively, and $U$ is an open subset of $M$, then there is no continuous, one-to-one map $U \to N$. We omit the proof.

6.12. (a) In this case we have the Mayer-Vietoris sequence

$$\cdots \to \tilde{H}_i(S^n-(A\cup B)) \to \tilde{H}_i(S^n-A)\oplus \tilde{H}_i(S^n-B) \to \tilde{H}_i(S^n) \to \tilde{H}_{i-1}(S^n-(A\cup B)) \to \cdots$$

Using Theorem 6.3 we deduce that

1. If $p < q$, then $\tilde{H}_i(S^n - (A \cup B)) \cong \mathbb{Z}$ for $i = n - 1, n - p - 1$ or $n - q - 1$, and 0 otherwise.
2. If $p = q$, then $\tilde{H}_i(S^n - (A \cup B)) \cong \mathbb{Z}$ for $i = n - 1$, $\cong \mathbb{Z}^2$ for $i = n - p - 1$, and 0 otherwise.

(b) In this case, identifying $S^n - (A \cap B)$ with $\mathbb{R}^n$, the Mayer-Vietoris sequence reads

$$\cdots \to \tilde{H}_i(S^n-(A\cup B)) \to \tilde{H}_i(S^n-A)\oplus \tilde{H}_i(S^n-B) \to \tilde{H}_i(S^n) \to \tilde{H}_{i-1}(S^n-(A\cup B)) \to \cdots$$

It follows that $\tilde{H}_i(S^n - (A \cup B)) \cong \tilde{H}_i(S^n - A) \oplus \tilde{H}_i(S^n - B)$, which gives us

1. If $p < q$, then $\tilde{H}_i(S^n - (A \cup B)) \cong \mathbb{Z}$ for $i = n - p - 1$ or $n - q - 1$, and 0 otherwise.
2. If $p = q$, then $\tilde{H}_i(S^n - (A \cup B)) \cong \mathbb{Z}^2$ for $i = n - p - 1$, and 0 otherwise.

It follows that if $p = q = n - 1$, then $S^n - (A \cup B)$ has three components in both case (a) and (b).

6.13. It may not be true. For example the exponential map gives a homeomorphism from $A = (-\infty, 0]$ to $B = (0, 1]$. Then $A$ is closed in $\mathbb{R}$, but $B$ is not.

6.14. Following the hint, use induction on the number edges of $X$. If $X = \bar{e}$ has one edge, then $\tilde{H}_i(S^3 - \bar{e}) \cong 0$ for all $i$ by Lemma 6.2, and $H_1(\bar{e}) = 0$ as well.

Take an edge $e$ such that the closure of $X - \bar{e}$ is a connected graph $Y$ having one less edge, so that the assertion holds for $Y$ by induction hypothesis. We have two cases.

1. $Y$ have one vertex in common with $e$. Then the Euler characteristic $\chi(X) = \chi(Y)$, hence rank $H_1(X) = \text{rank } H_1(Y)$. The Mayer-Vietoris sequence

$$\cdots \to H_{i+1}(\mathbb{R}^2) \to H_i(S^3 - X) \to H_i(S^3 - \bar{e}) \oplus H_i(S^3 - Y) \to H_i(\mathbb{R}^2) \to \cdots$$

gives us $H_i(S^3 - X) \cong H_i(S^3 - Y)$ for all $i \geq 1$. Hence the induction follows.

2. $Y$ has two vertices in common with $e$. Then $\chi(X) = \chi(Y) - 1$ hence rank $H_1(X) = \text{rank } H_1(Y) + 1$. The Mayer-Vietoris sequence

$$\cdots \to H_{i+1}(S^3 - S^0) \to H_i(S^3 - X) \to H_i(S^3 - \bar{e}) \oplus H_i(S^3 - Y) \to H_i(S^3 - S^0) \to \cdots$$
together with Theorem 6.3 show that
\[ \text{rank } H_1(S^3 - X) = \text{rank } H_2(S^3 - S^0) + \text{rank } H_1(S^3 - Y) = 1 + \text{rank } H_1(Y) = \text{rank } H_1(X), \]
and \[ H_i(S^3 - X) = 0 \text{ for } i > 1. \] Again the induction follows.

7.1. By Chapter II Exercise 7.5, \( \pi(X, e) \) is abelian hence is equal to its own abelianization, which is isomorphic to \( H_1(X) \).