5.1. $C_n(X)$ is the direct sum of $C_n(A)$ and the free abelian group considered in the question. The assertion follows from the definition $C_n(X, A) = C_n(X)/C_n(A)$.

5.2. (a) The long exact sequence for the pair $(X, A)$ gives the short exact sequences

$$0 \to \ker i_* \to H_n(X, A) \to \coker i_* \to 0.$$  

Hence $i_* : H_n(A) \to H_n(X)$ is an isomorphism for all $n$ if and only if $\ker i_* = \coker i_* = 0$ if and only if $H_n(X, A) = 0$ for all $n$.

(b) Similar to (a), consider the short exact sequences

$$0 \to \ker j_* \to H_n(A) \to \coker j_* \to 0.$$  

(c) By the proof of (a), the short exact sequence at $n$ implies that $H_n(X, A) = 0$ if and only if $i_* : H_n(A) \to H_n(X)$ is an epimorphism and $i_* : H_{n-1}(A) \to H_{n-1}(X)$ is a monomorphism. Therefore, $H_n(X, A) = 0$ for $n \leq q$ if and only if $i_* : H_q(A) \to H_q(X)$ is an epimorphism and $i_* : H_n(A) \to H_n(X)$ for $n < q$ is both an epimorphism and a monomorphism, i.e. an isomorphism.

5.3. One has $H_n(X) \cong \bigoplus \gamma H_n(X_\gamma)$ and $H_n(A) \cong \bigoplus \gamma H_n(X_\gamma \cap A)$, and moreover if $i_\gamma$ is the inclusion $X_\gamma \cap A \to X_\gamma$ then $i_*$ is given by $\bigoplus \gamma i_\gamma$. Therefore the homology exact sequences of $(X, A)$ and $(X_\gamma, X_\gamma \cap A)$ imply that $H_n(X, A) \cong \bigoplus \gamma H_n(X_\gamma, X_\gamma \cap A)$.

If $X_\gamma \cap A \neq \emptyset$, then $H_0(X_\gamma \cap A) \to H_0(X_\gamma)$ is an epimorphism hence $H_0(X_\gamma, X_\gamma \cap A) = 0$. Otherwise, $H_0(X_\gamma, X_\gamma \cap A) \cong H_0(X_\gamma) \cong \mathbb{Z}$.

5.4. We have the natural map $Z_n(X \mod A) \to Z_n(X, A)$, $x \mapsto x + C_n(A)$, which is clearly an epimorphism and has kernel $C_n(A)$. Therefore $Z_n(X, A) \cong Z_n(X \mod A)/C_n(A)$.

Consider the commutative square

$$\begin{array}{ccc}
C_{n+1}(X) & \xrightarrow{\partial} & C_n(X) \\
\downarrow & & \downarrow \\
C_{n+1}(X, A) & \xrightarrow{\partial} & C_n(X, A)
\end{array}$$

Since the left vertical arrow is an epimorphism and $B_n(X, A)$ is the image of the bottom horizontal arrow, it also equals the image of $B_n(X)$ under the right vertical arrow $C_n(X) \to C_n(X, A)$, which is exactly $[B_n(X) + C_n(A)]/C_n(A)$. 

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The last isomorphism follows from the first two:

\[ H_n(X, A) \cong Z_n(X, A)/B_n(X, A) \]
\[ \cong (Z_n(X \mod A)/C_n(A))/([B_n(X) + C_n(A)]/C_n(A)) \]
\[ \cong Z_n(X \mod A)/[B_n(X) + C_n(A)]. \]

5.5. In this case \( H_n(A) = 0 \) for all \( n \) hence by Exercise 5.2. (b) one has \( j_* : H_n(X) \to H_n(X, A) \) is an isomorphism for all \( n \).

5.6. The homology sequence of \((X, A)\) reads
\[ \cdots \to 0 \to \bigoplus_{x \in A} \mathbb{Z} \xrightarrow{i_*} \bigoplus_{x \in X} \mathbb{Z} \xrightarrow{j_*} \bigoplus_{x \in X \setminus A} \mathbb{Z} \to 0, \]
where \( i_* \) and \( j_* \) are the natural inclusion and projection respectively.

6.1. \((X, A)\) and \((Y, B)\) are of the same homotopy type if there are continuous maps \( f : (X, A) \to (Y, B) \), \( g : (Y, B) \to (X, A) \) such that \( gf : (X, A) \to (X, A) \) and \( fg : (Y, B) \to (Y, B) \) are homotopic to the identity maps. Such \( f, g \) are called homotopy equivalences. The analog of Theorem 4.4 is that if \( f : (X, A) \to (Y, B) \) is a homotopy equivalence, then \( f_* : H_n(X, A) \to H_n(Y, B) \), \( n = 0, 1, 2, \ldots \) are isomorphisms. To prove this, note that \( f : X \to Y \) and \( f|_A : A \to B \) are homotopy equivalences as well, hence \( f_* : H_n(X) \to H_n(Y) \) and \( f_* : H_n(A) \to H_n(B) \) are isomorphisms. The assertion then follows from the following commutative diagram and the five-lemma:

\[
\begin{array}{ccccccccc}
\cdots & \to & H_n(A) & \to & H_n(X) & \to & H_n(X, A) & \to & H_{n-1}(A) & \to & \cdots \\
& & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
\cdots & \to & H_n(B) & \to & H_n(Y) & \to & H_n(Y, B) & \to & H_{n-1}(B) & \to & \cdots 
\end{array}
\]

Similarly, consider an inclusion \( i : (Y, B) \to (X, A) \). We say \((Y, B)\) is a retract of \((X, A)\) if there is a continuous map \( r : (X, A) \to (Y, B) \) such that \( r|_Y = \text{id}_Y \). We say \((Y, B)\) is a deformation retract of \((X, A)\) if there is a continuous map \( F : (I \times X, I \times A) \to (X, A) \) such that \( F(0, x) = x \) and \( F(1, x) = r(x) \). The main property of such a deformation retract is that \( i_* : H_n(Y, B) \to H_n(X, A) \), \( n = 0, 1, 2, \ldots \) are isomorphisms.

6.2 It is well-known that such extensions are classified by \( \text{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) \). Take a free (hence projective) resolution of \( \mathbb{Z}/n\mathbb{Z} \):
\[ 0 \to \mathbb{Z} \xrightarrow{n\times} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0. \]
Then \( \text{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) \) is \( H^1 \) of the sequence
\[ 0 \to \text{Hom}(\mathbb{Z}, \mathbb{Z}) \xrightarrow{n\times} \text{Hom}(\mathbb{Z}, \mathbb{Z}) \to 0, \]
which is isomorphic to \( \mathbb{Z}/n\mathbb{Z} \). Hence there are \( n \) isomorphism classes of such extensions.