SKETCHED SOLUTIONS TO EXERCISES OF CHAPTER 1

1.1 Following the Hint, let $F$ be such an equivalence so that $F(\emptyset) \cong \{pt\}$ and $F(\{pt\}) \cong \emptyset$. Let $X$ be a set with two elements. Since $F$ is fully faithful, $F(X)$ is not empty. Then $\text{Hom}_\text{Set}(\{pt\}, X)$ is a set of two elements, but $\text{Hom}_\text{Set}(F(\{pt\}), F(X)) \cong \text{Hom}_\text{Set}(F(X), \emptyset)$ is empty, so that $\text{Hom}_\text{Set}(\{pt\}, X) \not\cong \text{Hom}_\text{Set}(F(\{pt\}), F(X))$, a contradiction. This argument also shows that $\text{Set}^f$ and $(\text{Set}^f)^{op}$ are not equivalent.

1.2 (i) First assume that $F(f)$ is a monomorphism. If $f \circ g_1 = f \circ g_2$, then $F(f) \circ F(g_1) = F(f) \circ F(g_2)$, which implies that $F(g_1) = F(g_2)$. Faithfulness of $F$ shows that $g_1 = g_2$. Hence $f$ is a monomorphism as well. Similarly one can prove that if $F(f)$ is an epimorphism then so is $f$.

(ii) Let $g'$ be the morphism in $C'$ so that $F(f) \circ g' = id$, $g' \circ F(f) = id$. Since $F$ is full, $g' \cong F(g)$ for some morphism $g$ in $\mathcal{C}$. Then $F(f \circ g) = id$ and $F(g \circ f) = id$. Since $F$ preserves identity morphisms, faithfulness of $F$ forces that $f \circ g = id$, $g \circ f = id$, i.e. $f$ is an isomorphism.

Warning: since monomorphism+epimorphism $\not\Rightarrow$ isomorphism, (ii) does not follow from (i). That is, faithful functors are in general not conservative.

1.3 It is faithful and conservative, but not full. The only nontrivial thing is conservativeness (if you took notes in my lecture, you will find a proof).

1.4 (i) For a finite abelian group $A$, let $A^\vee := \text{Hom}(A, S^1)$ be its Pontryagin dual, i.e. the set of group homomorphisms from $A$ to the unit circle $S^1$. Then $A^\vee$ is a finite abelian group as well. The obvious functor $\mathcal{C} \to \mathcal{C}^{op}$, $A \to A^\vee$ is an equivalence of categories, using the two facts $(A^\vee)^\vee \cong A$ and $\text{Hom}_\mathcal{C}(A, B) \cong \text{Hom}_\mathcal{C}(B^{op}, A^{\vee}) = \text{Hom}_{\mathcal{C}^{op}}(A^{\vee}, B^{\vee})$.

(ii) Applying the obvious transpose $\text{Hom}_\mathcal{C}(X, Y) = \mathcal{P}(X \times Y) \cong \mathcal{P}(Y \times X) = \text{Hom}_{\mathcal{C}^{op}}(X, Y)$.

1.5 (i) First assume that $f : X \to Y$ is a monomorphism. Assume that $f(x_1) = f(x_2)$. Let $g_1 : \{pt\} \to X$ be the morphism with image $\{x_i\}$, $i = 1, 2$. Then $f \circ g_1 = f \circ g_2$, which implies that $g_1 = g_2$ hence $x_1 = x_2$. Therefore $f$ is injective. Conversely, assume that $f$ is injective and $f \circ g_1 = f \circ g_2$. Then $f \circ g_1(z) = f \circ g_2(z), \forall z$, hence $g_1(z) = g_2(z), \forall z$, i.e. $g_1 = g_2$. It follows that $f$ is a monomorphism.

Similarly one can prove that $f$ is an epimorphism $\iff f$ is surjective.

(ii) Recall that we only consider commutative rings with unity. The desired conclusion is equivalent to that for any ring $A$ there exists at most one ring homomorphism $\mathbb{Q} \to A$. If $f$ is such a homomorphism, then $f(p/q) = pf(1/q)$, and $qf(1/q) = f(1) = 1$ shows that $f(1/q)$ is the inverse of $q$ in $A$, which is of course unique.

(iii) As I did in the lecture, let $X$ be any set with cardinality greater than 1. Then the identity map $(X, \text{discrete topology}) \to (X, \text{trivial topology})$ is both a monomorphism and an epimorphism, but is not an isomorphism, since it does not admit a continuous inverse.

1.6 Let $\theta_1, \theta_2 \in \text{End}(id_\mathcal{C})$. We need to show that $\theta_1(X) \circ \theta_2(X) = \theta_2(X) \circ \theta_1(X) \in \text{End}_\mathcal{C}(X)$ for any $X \in \mathcal{C}$. By definition of morphism of functors, we know that for any morphism $f : X \to Y$ in
$\mathcal{C}$ the following diagram commutes:

\[
\begin{array}{ccc}
X & \xrightarrow{\theta_1(X)} & X \\
f & \downarrow & \downarrow f \\
Y & \xrightarrow{\theta_1(Y)} & Y
\end{array}
\]

Take $Y = X$ and $f = \theta_2(X)$ in the diagram.