SKETCHED SOLUTIONS TO EXERCISES OF CHAPTER 3

3.1 One checks that
\[ \delta^{n+1} d^n = (-1)^{n+1} d^{n+1} d^n = 0, \quad d^n [1] \delta^n = (-1) d^{n+1} (-1)^n d^n = 0. \]

Hence \( \delta^{n+1} d^n = d^n [1] \delta^n \), which shows \( \delta \) is a morphism of chains. Define \( s^n : X^n \to X^{[1]^{n-1}} = X^n \) to be the identity or 0 according to \( n \) odd or even. Then
\[ d^{n-1} [1] s^n + s^{n+1} d^n = (-1) \frac{1}{2} (-1)^n d^n + \frac{1}{2} (1 - (-1)^n) d^n = (-1)^n d^n = \delta^n, \]
which shows \( \delta \) is homotopic to 0.

3.2. If \( f \) and \( g \) are homotopic, say \( f^n - g^n = s^{n+1} d_X^n + d_Y^{-1} s^n \), where \( s^n : X^n \to Y^{n-1} \), then we define \( u^n \) by the matrix
\[ u^n = \begin{pmatrix} id & 0 \\ s^{n+1} & id \end{pmatrix} : X^{n+1} \oplus Y^n \to X^{n+1} \oplus Y^n. \]

Using “multiplication” of matrices one verifies that
\[ u^{n+1} d^n_{\text{cone}(f)} = d^n_{\text{cone}(g)} u^n, \]
hence \( u \) gives a morphism \( \text{cone}(f) \to \text{cone}(g) \). It is straightforward to check that \( u \) makes the desired diagram commutative. Conversely if there exists such \( u \) making the diagram commutative, then we may express \( u \) as
\[ u^n = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : X^{n+1} \oplus Y^n \to X^{n+1} \oplus Y^n. \]

From \( u \circ \alpha(f) = \alpha(g) \) we deduce that \( b = 0, d = id \), and from \( \beta(f) = \beta(g) \circ u \) we deduce that \( a = id \). Hence \( u \) is of the form
\[ u^n = \begin{pmatrix} id & 0 \\ s^{n+1} & id \end{pmatrix}, \quad s^{n+1} : X^{n+1} \to Y^n. \]

Then relation \( u^{n+1} d^n_{\text{cone}(f)} = d^n_{\text{cone}(g)} u^n \) implies that \( f^{n+1} - g^{n+1} = s^{n+2} d_X^{n+1} + d_Y s^{n+1} \), i.e. \( f \) and \( g \) are homotopic.

In this case \( u \) is an isomorphism in \( C(\mathcal{C}) \) because it has an inverse \( v \) given by
\[ v^n = \begin{pmatrix} id & 0 \\ -s^{n+1} & id \end{pmatrix}. \]

3.3. (a) \( \Rightarrow \) (b), (c): Let \( f = sd + ds \). Then one can verifies that
\[ s^{n+1} \oplus f^n : \text{cone}(id_X) = X^{n+1} \oplus X^n \to Y \]
defines a morphism, and clearly \( f = (s \oplus f) \circ (id_X) \). Similarly,
\[ f^n \oplus (-s^n) : X^n \to \text{cone}(id_Y)[-1] = Y^n \oplus Y^{n-1} \]
defines a morphism, and \( f = (id_Y)[-1] \circ (f \oplus (-s)) \).

(b), (c) \( \Rightarrow \) (d): We only need to show that \( \text{cone}(id_X) \) is homotopic to zero. Define
\[ s^n = \begin{pmatrix} 0 & id \\ 0 & 0 \end{pmatrix} : X^{n+1} \oplus X^n \to X^n \oplus X^{n-1}. \]
Recall that
\[ d_{\text{cone}(id_X)}^n = \left( \begin{array}{cc} -d & 0 \\ id & d \end{array} \right) : X^{n+1} \oplus X^n \to X^{n+2} \oplus X^{n+1}. \]

Then one can verify that
\[ d_{\text{cone}(id_X)}^{n-1} \circ s^n + s^{n+1} \circ d_{\text{cone}(id_X)}^n = \left( \begin{array}{cc} id & 0 \\ 0 & id \end{array} \right) = \text{id}_{\text{cone}(id_X)^n}. \]

This shows \( \text{cone}(id_X) \) is homotopic to zero. Similarly \( \text{cone}(id_Y)[-1] \) is also homotopic to zero.

\[ (d) \Rightarrow (a): \text{ we may decompose } f \text{ as } X \to Z \xrightarrow{id_Z} Z \to Y. \text{ By definition } id_Z \text{ is homotopic to zero. Recall that if } g \text{ is homotopic to zero, then } h \circ g \text{ and } g \circ h \text{ are homotopic to zero for any morphism } h. \text{ From this it follows easily that } f \text{ is homotopic to zero.} \]

3.4. (i) We define an equivalence \( T : \mathcal{C}^{Zd} \to \mathcal{C}^{Zd} \) as follows. If \( F_d : \mathcal{Z}_d \to \mathcal{C} \) is a functor, we define
\[ TF_d : \mathcal{Z}_d \to \mathcal{C}, \quad TF_d(n) = F(n+1), \quad TF_d(id_n) = id_{F(n+1)}. \]

It is an equivalence because it has an obvious inverse.

(ii) If \( F : \mathcal{Z} \to \mathcal{C} \) is a functor, then we may associate a differential object \( (F_d, d_{F_d}) \) of \( \mathcal{C}^{Zd} \) by
\[ F(n) = F(n), \quad d_{F_d} : F_d \to TF_d, \quad d_{F_d}(n) = F(n \to n+1) : F_d(n) = F(n) \to TF_d(n) = F(n+1), \]
where \( n \to n+1 \) is the unique morphism in the ordered set \( \mathbb{Z} \) considered as a category.

Conversely if \( (F_d, d_{F_d}) \) is a differential object of \( \mathcal{C}^{Zd} \), then we have morphisms \( d_{F_d}(n) : F_d(n) \to TF_d(n) = F_d(n+1) \). By composition we obtain a compatible family of morphisms \( F_d(m) \to F_d(n) \) for any \( m \leq n \). Hence we may construct a functor
\[ F : \mathcal{Z} \to \mathcal{C}, \quad F(n) = F_d(n), \quad F(m) = F_d(m) \to F(n) = F_d(n), \quad m \leq n. \]

The two constructions are inverse to each other, so may identity \( \text{Fct}(\mathcal{Z}, \mathcal{C}) \) as the category of differential objects of \( \mathcal{C}^{Zd} \).

(iii) Let \( \mathcal{A} = \mathcal{C}^{Zd} \). Then a differential object \( (X, d_X) \) of \( (\mathcal{A}, T) \) is a sequence of objects \( X^n \overset{\text{def}}{=} X(n) \in \mathcal{C} \) and morphisms \( d_X^n \overset{\text{def}}{=} d_X(n) : X^n = X(n) \to X^{n+1} = TX(n) = X(n+1) \). This is the same as the differential objects given by Definition 3.2.1. Similarly, one checks that the notions of morphism of differential objects also coincide.

Similarly, \( (X, d_X) \) is a complex if \( X \to T(X) \to T^2(X) \) is zero, i.e. the composition
\[ X^n d_X^n T(X^n) = X^{n+1} d_X^{n+1} T^2(X^n) = X^{n+2} \]
is zero. This coincides with the notion of complex given by Definition 3.2.1.

3.5. (1) \( s_j^n \circ s_i^{n+1} = s_i^{n-1} \circ s_j^{n+1} \) for \( 0 \leq j < i \leq n \): both maps are the unique surjective order-preserving map \( f : [0, n+1] \to [0, n-1] \) such that \( f(j) = f(j+1) \) and \( f(i) = f(i+1) \).

\( s_j^{n+1} \circ d_i^n = d_i^{n-1} \circ s_j^{n-1} \) for \( 0 \leq i < j \leq n \): both maps are the unique surjective order-preserving map \( f : [0, n] \to [0, n] \setminus \{i\} \) such that \( f(j) = f(j+1) \).

\( s_j^{n+1} \circ d_i^n = id_{[0,n]} \) for \( 0 \leq i \leq n+1, i = j, j+1 \): the sequence of values of \( d_i^n(k), k = 0, \ldots, n \) is \( 0, \ldots, i-1, i+1, \ldots, n+1 \). For either \( j = i \) or \( j = i-1 \), the map \( s_j^{n+1} \) fixes \( 0, \ldots, i-1 \), \( i+1 \), and \( n+1 \) and maps \( k \) to \( k-1 \) for \( k \geq i+1 \). Hence it is clear that the composition \( s_j^{n+1} \circ d_i^n \) maps \( k \) to \( f(k) = f(i+1) \).