4.1. It is straightforward to verify that if
\[ 0 \to M'_i \to M_i \to M''_i \to 0 \]
is a family of exact sequences of \( A \)-modules, where \( i \in I \), then
\[ 0 \to \prod_{i \in I} M'_i \to \prod_{i \in I} M_i \to \prod_{i \in I} M''_i \to 0 \]
and
\[ 0 \to \bigoplus_{i \in I} M'_i \to \bigoplus_{i \in I} M_i \to \bigoplus_{i \in I} M''_i \to 0 \]
are exact as well.

4.2. Since \( \bigoplus \) is exact, it commutes with kernels and cokernels. Since coimages are defined via cokernels and coimages are isomorphic to images in abelian category, it follows that \( \bigoplus \) also commutes with images. We conclude that for two complexes
\[ 0 \to M'_i \to M_i \to M''_i \to 0 \]
the “if” part: If \( M \oplus K = \text{im}(f_1 + f_2) = \text{im} f_1 \oplus \text{im} f_2 \). Therefore the two sequences are exact if and only if their direct sum is exact.

4.3. (i) Since \( \text{Hom}_A(A^{\langle I \rangle}, M) \cong M^{\langle I \rangle} \) for a set \( I \) and \( \bigoplus \) is exact, it is clear that \( \text{Hom}_A(A^{\langle I \rangle}, -) \) is exact, i.e. \( A^{\langle I \rangle} \) is projective.

(ii) “Only if” part: Let \( A^{\langle P \rangle} = \bigoplus_{p \in P} A_p \) be the free \( A \)-module indexed by \( P \) and \( \psi : A^{\langle P \rangle} \to p \) be the \( A \)-homomorphism such that \( \psi(1_p) = p \). Then \( \psi \) is surjective. If \( P \) is projective, then \( \psi \) has a section, i.e. an \( A \)-homomorphism \( \phi : P \to A^{\langle P \rangle} \) such that \( \psi \circ \phi = id_P \). This shows \( P \) is a direct summand of \( A^{\langle P \rangle} \).

“If” part: If \( P \oplus K \) is free, then by (i) \( P \oplus K \) is projective, hence \( \text{Hom}_A(P \oplus K, -) = \text{Hom}_A(P, -) \oplus \text{Hom}_A(K, -) \) is exact. This implies \( \text{Hom}_A(P, -) \) is exact, hence \( P \) is projective.

(iii) It is easy to show that free modules are flat, hence by (ii) \( - \otimes A (P \oplus K) = (- \otimes A P) \oplus (- \otimes A K) \) is exact, which implies that \( - \otimes A P \) is exact, i.e. \( P \) is flat.

4.4. (i) Let \( M_0 \) be an abelian group and \( M \) be a subgroup. We only need to show that any homomorphism \( \varphi : M \to \mathbb{Q}/\mathbb{Z} \) extends to \( M_0 \). Let \( S \) be the set of pairs \((N, \phi)\) where \( N \) is a subgroup of \( M_0 \) containing \( M \) and \( \phi : N \to \mathbb{Q}/\mathbb{Z} \) is a homomorphism such that \( \phi|_M = \varphi \). Define a partial order on \( S \) such that \((N, \phi) \leq (N', \phi')\) if \( N \subseteq N' \) and \( \phi'|_N = \phi \). Clearly Zorn’s lemma applies and we let \((N, \phi)\) be a maximal element of \( S \). It suffices to show that \( N = M_0 \). Otherwise we may take \( x \in M_0 \setminus N \). Consider \( \mathbb{Z} \bar{x} \subseteq M_0/N \), where \( \bar{x} = x + N \). If \( \mathbb{Z} \bar{x} \cong \mathbb{Z} \), extend \( \phi \) to \( N + \mathbb{Z}x \) by defining \( \phi(x) = 0 \); if \( \mathbb{Z} \bar{x} \cong \mathbb{Z}/n \) for some \( n \in \mathbb{N} \), then we pick up an arbitrary representative \( r \in \mathbb{Q} \) of \( \phi(nx) \in \mathbb{Q}/\mathbb{Z} \) and extend \( \varphi \) to \( N + \mathbb{Z}x \) by defining \( \phi(x) = r/n \mod \mathbb{Z} \). Then one can show that in both cases \( \phi \) is well-defined and extends \( \varphi \) to \( N + \mathbb{Z}x \). But this contradicts the maximality of \( N \). Hence \( N = M_0 \) and the proof is finished.

(ii) Let \( \varphi \in \text{Hom}_Z(M, N) \). By (i) we have the exact sequence
\[ N^\vee \xrightarrow{\varphi^\vee} M^\vee \to (\text{Ker} \varphi)^\vee \to 0. \]
If $\varphi^\vee = 0$, then $M^\vee \cong (\text{Ker } \varphi)^\vee$. Since $\varphi = 0$ on $\text{Ker } \varphi$, it follows that $\varphi = 0$ on $M$ as well. This shows that the map $\varphi \mapsto \varphi^\vee$ is injective.

(iii) One has

$$\text{Hom}_A(-, P^\vee) = \text{Hom}_A(-, \text{Hom}_A(\varphi, \mathbb{Q}/\mathbb{Z})) \cong \text{Hom}_{\mathbb{Z}}(- \otimes_A P, \mathbb{Q}/\mathbb{Z}).$$

$P$ is projective hence flat, together with (i) we see that the last functor above is a composition of two exact functors $- \otimes_A P$ and $\text{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$, hence is exact. This shows $P^\vee$ is left $A$-injective.

(iv) Clearly $Z^\vee = \mathbb{Q}/\mathbb{Z}$. By (ii) $M = \text{Hom}_{\mathbb{Z}}(Z, M) \to \text{Hom}_{\mathbb{Z}}(M^\vee, \mathbb{Q}/\mathbb{Z}) = M^\vee \vee$ is injective. Take a surjective $A^\text{op}$-homomorphism $P \to M^\vee$ where $P$ is projective, which gives an injective $A$-homomorphism $M^\vee \vee \to P^\vee$. By (iii) $P^\vee$ is injective, and $M \to M^\vee \vee \to P^\vee$ is an injection.

4.5. We can make $I$ a filtrant category by giving an arbitrary total order on $I$, which exists by Zorn’s lemma. Define two functors $\alpha, \beta : I \to \mathcal{C}$ such that

$$\alpha(i) = \begin{cases} X_{i_0}, & i = i_0 \\ 0, & \text{otherwise} \end{cases}, \quad \alpha(s) = 0 \text{ for any } s : i \leq j, i \neq j,$$

$$\beta(i) = X_i, \quad \beta(s) = 0 \text{ for any } s : i \leq j, i \neq j.$$

Then the obvious morphism $\alpha \to \beta$ is a monomorphism. Taking direct limit gives a monomorphism $X_{i_0} \to \bigoplus_{i \in I} X_i$.

4.6. (i) “Only if” part is clear since $\text{Hom}_\mathcal{C}(W, -)$ is left exact. For the “if” part, the exactness of the Hom complex implies that

$$\text{Hom}_\mathcal{C}(W, X) \cong \text{Hom}_\mathcal{C}(W, \text{Ker } g)$$

for any $W \in \mathcal{C}$, where $g$ is the morphism $Y \to Z$. It follows from the Yoneda lemma that $X \cong \text{Ker } g$.

(ii) By passing to the opposite category, one can similarly show that $X \to Y \to Z \to 0$ is exact if and only if for any $W \in \mathcal{C}$ the complex

$$0 \to \text{Hom}_\mathcal{C}(Z, W) \to \text{Hom}_\mathcal{C}(Y, W) \to \text{Hom}_\mathcal{C}(X, W)$$

is exact.

4.7. (i) Applying $\text{Hom}_\mathcal{C}(W, -)$ to the diagram for any $W \in \mathcal{C}$, by 4.6 (i) we may reduce to the case $\mathcal{C} = \text{Mod}(\mathbb{Z})$. Let $y \in Y$. Because $f$ is an epimorphism, we may find $x \in X$ such that $f(x) = g(y)$. Hence $(x, y) \in \text{Ker}(f - g)$ and we may find $z \in V$ such that $f'(z) = y$, $g'(z) = x$. Hence $f'$ is an epimorphism.

(ii) By passing to the opposite category, we may reverse the arrows and switch from epimorphism to monomorphism.

4.8. Applying $\text{Hom}_\mathcal{C}(W, -)$ to the diagram for any $W \in \mathcal{C}$, by 4.6 (i) we may reduce to the case $\mathcal{C} = \text{Mod}(\mathbb{Z})$. We need to show the first column is exact.

Exactness at $X'_0$: if $x'_0 \in X'_0$ is mapped to zero in $X'_1$, then it is mapped to zero in $X_1$. But $X'_0 \to X_0 \to X_1$ is injective, hence $x'_0 = 0$.

Exactness at $X'_1$: let $x'_1 \in X'_1$ be mapped to zero in $X'_2$. Then its image $x_1 \in X_1$ is mapped to zero in $X_2$, which implies $x_1$ is the image of $x'_0 \in X_0$ because the second column is exact. Since $x_1$ is mapped to zero in $X'_2$, we see $x_0$ is also mapped to zero in $X'_2$. But $X'_0 \to X'_1 \to X'_2$ is injective, $x_0$ is mapped to zero in $X'_0$. The first row is exact, hence $x_0$ is the image of $x'_0 \in X_0$. Finally $x'_0$ has to be mapped to $x'_1$, because $X'_1 \to X_1$ is injective.

4.9. (i) It is clear that $W/(x_1 W + \cdots + x_{j-1} W) = k[x_i, \partial_i]_{j \leq i \leq n}[\partial_1, \ldots, \partial_{j-1}]$ is a polynomial algebra of $j - 1$ variables over the Weyl algebra $k[x_i, \partial_i]_{j \leq i \leq n} \cong W_{n-j+1}$. The left multiplication
by \(x_j\) is injective on \(k[x_i, \partial_i]_{j \leq i \leq n}\), hence is also injective on \(W/(x_1W + \cdots + x_{j-1}W)\). This shows \(\varphi\) is a regular sequence. It follows that \(H^j(K^\bullet(W, \varphi))\) is concentrated in degree \(n\) and
\[
H^n(K^\bullet(W, \varphi)) \cong W/(x_1W + \cdots + x_nW) \cong k[\partial_1, \ldots, \partial_n],
\]
which is isomorphic to a polynomial ring of \(n\) variables over \(k\).

(ii) Similar to (i), one shows that \(W/(W\partial_1 + \cdots + W\partial_{j-1}) = k[x_i, \partial_i]_{j \leq i \leq n}[x_1, \ldots, x_{j-1}]\) on which the right multiplication by \(\partial_j\) is injective, hence \(\psi\) is a regular sequence. Then \(H^j(K^\bullet(W, \psi))\) is concentrated in degree \(n\) and
\[
H^n(K^\bullet(W, \psi)) \cong W/(W\partial_1 + \cdots + W\partial_n) \cong k[x_1, \ldots, x_n].
\]

(iii) We have \(\text{Ker} \partial_1 \cap \cdots \cap \text{Ker} \partial_{j-1} \cong k[x_j, \ldots, x_n]\), on which \(\partial_j\) is surjective (integral of a polynomial is a polynomial), hence \(\lambda\) is a coregular sequence. Then \(H^j(K^\bullet(O, \lambda))\) is concentrated in degree \(0\) and
\[
H^0(K^\bullet(O, \lambda)) \cong \text{Ker} \partial_1 \cap \cdots \cap \text{Ker} \partial_n \cong k.
\]

4.10. Since \(\varphi_1\) is injective on \(A\) and \(\varphi_2\) is injective on \(A/\varphi_1(A) \cong k[x_2, \partial_2][\partial_1]\), the sequence \(\varphi = (\varphi_1, \varphi_2)\) is regular. Hence \(H^1(K^\bullet(A, \varphi))\) is concentrated in degree \(2\) and \(H^2(K^\bullet(A, \varphi)) = A/(\varphi_1(A) + \varphi_2(A)) = k[\partial_1, x_2]\), which is isomorphic to a polynomial ring of \(2\) variables over \(k\).

4.11. Since \(\varphi_1\) is injective and \(\varphi_2\) is surjective on \(M\), one has
\[
H^0(K^\bullet(M, \varphi)) = 0, \quad H^2(K^\bullet(M, \varphi)) = 0.
\]
To calculate \(H^1(K^\bullet(M, \varphi))\), consider the exact sequence
\[
H^0(K^\bullet(M, \varphi)) \rightarrow H^1(K^\bullet(M, \varphi)) \rightarrow H^1(K^\bullet(M, \varphi_1)) \xrightarrow{\varphi_2} H^1(K^\bullet(M, \varphi_1)).
\]
We compute that \(H^0(K^\bullet(M, \varphi_1)) = 0\), and \(H^1(K^\bullet(M, \varphi_1)) = \bigoplus_{i \geq 1} kt^i\), on which the kernel of \(\varphi_2\) is \(kt \cong k\). It follows that \(H^1(K^\bullet(M, \varphi)) \cong k\).