SKETCHED SOLUTIONS TO EXERCISES OF CHAPTER 7

7.1. We shall only prove (i) ⇔ (ii), because the equivalence of (i) and (iii) is immediate by passing from $C$ to $C^{op}$.

(ii) ⇒ (i): if $Y^\bullet$ is an injective resolution of $Y$ of length $\leq n$, then for $j \geq n$ one has

$$\Ext^j_C(X,Y) = \Hom_{D(C)}(X,Y[j]) = \Hom_{K(C)}(X,Y^*[j]) = 0.$$ 

(i) ⇒ (ii): By induction on $n$, we shall prove that if $\Ext^i(Y,X) = 0$ for all $Y \in C$ and $i > n$, then $X$ has an injective resolution of length $\leq n$. It is clear if $n = 0$ because in this case $X$ is injective. Assume that it is true for $n = 1$. Take a monomorphism $X \hookrightarrow I$ with $I$ injective. The long exact sequence

$$\cdots \to \Ext^i(Y, I) \to \Ext^i(Y, I/X) \to \Ext^{i+1}(Y, X) \to \cdots$$

shows that $\Ext^i(Y, I/X) = 0$ for $i > n - 1$. Hence by induction hypothesis $I/X$ has an injective resolution $0 \to I/X \to I^0 \to I^1 \to \cdots \to I^{n-1} \to 0$ of length $\leq n - 1$. Then $0 \to X \to I \to I^0 \to I^1 \to \cdots \to I^{n-1} \to 0$ gives an injective resolution of $X$ of length $\leq n$. The induction follows.

7.2. To show the isomorphism for a fixed $k$, it suffices to assume that $X \in D^b(C)$. By 7.1 and Corollary 7.1.7, $X$ is split, i.e.

$$X \cong \bigoplus_{i \in \mathbb{Z}} H^i(X)[-i].$$

Hence after taking a summand and a shift of degree, we may reduce to the case that $X$ is concentrated in degree 0, i.e. $X \in C$. Then there is an injective resolution $0 \to X \to I_0 \to I_1 \to 0$. The required isomorphism

$$H^k(RF(X)) \cong F(H^k(X)) \oplus RF^1(H^{k-1}(X))$$

is trivial if $k \neq 0,1$. We have for $k = 0$

$$H^0(RF(X)) = F(X) = F(H^0(X))$$

and for $k = 1$

$$H^1(RF(X)) = R^1F(X) = R^1F(H^0(X)).$$

(ii) follows from applying (i) to the left exact functor $- \otimes M : \text{Mod}(\mathbb{Z})^{op} \to \text{Mod}(\mathbb{Z})^{op}$. Note that $H^k(X \otimes^L M) = \text{Tor}_k(X,M)$.

7.3. The long exact sequence given by applying the functor $\Hom_C(-, X')$ to the short exact sequence terminates at $\Ext^1_C(X'', X') = 0$, hence $\Hom_C(X, X') \to \Hom_C(X', X')$ is surjective. In particular there is a section $k : X \to X'$ so that the composition $X' \xrightarrow{f} X \xrightarrow{k} X'$ is $\text{id}_{X'}$, hence the short exact sequence splits.

7.4. Similar to the proof of 7.3, one can show that there exists $k : Y \to X$ such that $X \xrightarrow{f} Y \xrightarrow{k} X$ is $\text{id}_X$. Then the result follows from Exercise 6.3 (ii).

7.5. We have a d.t. in $D(C)$

$$H^a(X)[-a] \to \tau^{\geq a}X \to \tau^{\geq a+1}X \xrightarrow{\pm 1}$$

By assumption it is easy to show that $\tau^{a+1}X \cong H^b(X)[-b]$. Since $\Ext^1(H^b(X)[-b], H^a(X)[-a]) = \Ext^{b-a+1}(H^b(X), H^a(X)) = 0$, the result follows from 7.4.
7.6. Let \( \varphi_p \) be the sequence \((x_1, \ldots, x_p, \cdots \partial_{p+1}, \ldots, \partial_n)\), which is regular for \( W \). Hence the Koszul complex \( K^\bullet(W, \varphi_p) \) is a projective resolution of \( W/I_p \). Therefore

\[
\text{RHom}_W(W/I_p, W/I_q) = \text{Hom}_W(K^\bullet(W, \varphi_p), W/I_q) = K^\bullet(W/I_q, \varphi_p).
\]

Similarly

\[
W/J_q \otimes_W^L W/I_p = W/J_q \otimes_W K^\bullet(W, \varphi_p) = K^\bullet(W/J_q, \varphi_p).
\]

The explicit calculation of these two Koszul complexes will be omitted here.