SKETCHED SOLUTIONS TO THE FINAL EXAM

These solutions are brief, but one should be able to easily fill in the details to justify them. Questions can be sent to maliu@zju.edu.cn

1. (Yoneda’s lemma) Let $\mathcal{C}$ be a category, and $\mathcal{C}^\wedge = \text{Fct}(\mathcal{C}^{\text{op}}, \text{Set})$ be the category of functors from $\mathcal{C}^{\text{op}}$ to $\text{Set}$. Let $h : \mathcal{C} \to \mathcal{C}^\wedge$ be the functor $X \mapsto \text{Hom}_\mathcal{C}(\_, X)$.

(i) Let $X$ be an object of $\mathcal{C}$ and $F$ be a functor $\mathcal{C}^{\text{op}} \to \text{Set}$. Construct a bijective map $\text{Hom}_\mathcal{C}^\wedge(h(X), F) \to F(X)$ and its inverse $F(X) \to \text{Hom}_\mathcal{C}^\wedge(h(X), F)$.

(ii) Prove that the functor $h : \mathcal{C} \to \mathcal{C}^\wedge$ is fully faithful.

(iii) Show that a morphism $X \xrightarrow{f} Y$ in $\mathcal{C}$ is an isomorphism if and only if the induced morphism $\text{Hom}_\mathcal{C}(W, X) \xrightarrow{f_*} \text{Hom}_\mathcal{C}(W, Y)$ is bijective for any $W \in \mathcal{C}$.

Solutions: (i) Define the map $\varphi : \text{Hom}_\mathcal{C}^\wedge(h(X), F) \to F(X)$, $f \mapsto f(X)(\text{id}_X)$ and its inverse $\psi : F(X) \to \text{Hom}_\mathcal{C}^\wedge(h(X), F)$, $s \mapsto f_s$, where $f_s$ is the morphism of functors $h(X) \to F$ such that $f_s(Y)(g) = F(g)(s)$ for any $g \in \text{Hom}_\mathcal{C}(Y, X)$. It is straightforward to verify that $\varphi$ and $\psi$ are inverse to each other.

(ii) By (i) one has $\text{Hom}_\mathcal{C}^\wedge(h(X), h(Y)) = h(Y)(X) = \text{Hom}_\mathcal{C}(X, Y)$.

(iii) By (ii) $X \to Y$ is an isomorphism in $\mathcal{C}$ if and only if $h(X) \to h(Y)$ is an isomorphism in $\mathcal{C}^\wedge$ if and only if $h(X)(W) \to h(Y)(W)$ is bijective for any $W \in \mathcal{C}$.

2. In the category of commutative rings with unit,

(i) give an example of epimorphism which is not surjective on the underlying sets.

(ii) show that a monomorphism is always injective on the underlying sets.

Solutions: (i) The natural map $\mathbb{Z} \to \mathbb{Q}$ is an epimorphism of rings but not surjective.

(ii) Let $f : A \to B$ be a monomorphism in the category of rings. If $f(a_1) = f(a_2)$ for some $a_1 \neq a_2 \in A$, define ring homomorphisms $g_i : \mathbb{Z}[X] \to A$, $X \mapsto a_i$, $i = 1, 2$. Then $fg_1 = fg_2$ but $g_1 \neq g_2$, which contradicts the assumption that $f$ is a monomorphism.
Warning: In contrast to module categories, the category of rings with unit is not additive; in particular one cannot talk about the kernel, neither the zero morphism.

3. (i) Let $C$ be an abelian category, and $f : X \to Y$ be a morphism in $C(C)$. Prove that if $f$ is homotopic to 0, then the natural morphism

$$H^n(f) : H^n(X) \to H^n(Y)$$

is 0 for all $n \in \mathbb{Z}$.

(ii) Let $F : C \to C'$ be an exact functor between abelian categories. Show that

$$F(H^n(X)) \cong H^n(F(X))$$

for all $X \in C(C)$ and $n \in \mathbb{Z}$.

Solutions: (i) There are morphisms $s^n : X^n \to Y^{n-1}$ such that $f^n = d_{Y}^{-1}s^n + s^{n+1}d_{X}^n$. Then $f^n$ induces a morphism $\text{Ker} \, d_{X}^n \to \text{Im} \, d_{Y}^{-1}$, which implies that $H^n(f) = 0$.

(ii) $F$ commutes with kernels and cokernels.

4. Let $A$ be a commutative ring. Consider the following exact sequences in $\text{Mod}(A)$

$$0 \to N_1 \to P_1 \to M \to 0$$
$$0 \to N_2 \to P_2 \to M \to 0$$

where $P_1, P_2$ are projective. Show that $N_1 \oplus P_2, N_2 \oplus P_1$ and $\text{Ker}(P_1 \oplus P_2 \to M)$ are isomorphic, where the morphism $P_1 \oplus P_2 \to M$ is the natural one induced from $P_1 \to M$ and $P_2 \to M$.

Solutions: We shall only prove that $N_1 \oplus P_2 \cong \text{Ker}(P_1 \oplus P_2 \to M)$. Since $P_2$ is projective, one has a commutative diagram

$$
\begin{array}{ccc}
P_2 & \longrightarrow & M \\
\downarrow & & \downarrow \\
P_1 & \longrightarrow & M \\
\end{array}
$$

We have the homomorphism

$$N_1 \oplus P_2 \to \text{Ker}(P_1 \oplus P_2 \to M), \quad (n_1, p_2) \mapsto (n_1 - \varphi(p_2), p_2)$$

and its inverse

$$\text{Ker}(P_1 \oplus P_2 \to M) \to N_1 \oplus P_2, \quad (p_1, p_2) \mapsto (p_1 + \varphi(p_2), p_2).$$

5. Let $A$ be a commutative ring and $M$ be an $A$-module.

(i) Show that $\text{Ext}^n_A(A, M) = 0$ for all $n > 0$.

(ii) Let $x \in A$ be an element of $A$ which is not a zero divisor. Compute $\text{Ext}^n_A(A/(x), M)$ for all $n \geq 0$. 

2
Solutions: (i) Hom$_A(A, -)$ is an exact functor, hence its right derived functor Ext$^n_A(A, -)$ is zero for $n > 0$.

(ii) Since $x$ is not a zero divisor, one has the short exact sequence

$$0 \to A \xrightarrow{x} A \to A/(x) \to 0,$$

which gives a long exact sequence

$$\cdots \to \text{Ext}^n_A(A/(x), M) \to \text{Ext}^n_A(A, M) \to \text{Ext}^n_A(A, M) \to \cdots$$

and shows that

$$\text{Ext}^n_A(A/(x), M) = \begin{cases} 
\text{Hom}_A(A/(x), M) & n = 0, \\
M/xM & n = 1, \\
0 & n \geq 2.
\end{cases}$$

6. (i) Let $A$ be a commutative ring, and $0 \to N \to I \to N' \to 0$ be an exact sequence of $A$-modules with $I$ an injective module. Show that for any $A$-module $M$, Ext$^n_A(M, N) \cong \text{Ext}^{n-1}_A(M, N')$ for $n \geq 2$ and Ext$^1_A(M, N)$ is the cokernel of Hom$_A(M, I) \to$ Hom$_A(M, N')$.

(ii) Baer’s lemma states that an $A$-module $I$ is injective if and only if for any ideal $a$ of $A$, the natural map

$$\text{Hom}_A(A, I) \to \text{Hom}_A(a, I)$$

is surjective. Applying this lemma (you do not need to prove it), show that if $I$ is an injective $\mathbb{Z}$-module and $N \subset I$ is a submodule, then the quotient $I/N$ is an injective $\mathbb{Z}$-module.

(iii) Show that Ext$^2_\mathbb{Z}(M, N) = 0$ for any $\mathbb{Z}$-modules $M, N$ and $n \geq 2$.

Solutions: (i) The assertion follows from the long exact sequence

$$\cdots \to \text{Ext}^n_A(M, N) \to \text{Ext}^n_A(M, I) \to \text{Ext}^n_A(M, N') \to \cdots$$

and the fact that Ext$^n_A(M, I) = 0$ for $n > 0$.

(ii) By Baer’s lemma, it suffices to show that the restriction map

$$\text{Hom}_\mathbb{Z}(\mathbb{Z}, I/N) \to \text{Hom}_\mathbb{Z}(n\mathbb{Z}, I/N)$$

is surjective for any $n \in \mathbb{Z}$. Let $\varphi : n\mathbb{Z} \to I/N$ be a homomorphism. Let $a \in I$ be a lift of $\varphi(n) \in I/N$. Since $I$ is injective, one has a surjective map

$$\text{Hom}_\mathbb{Z}(\mathbb{Z}, I) \to \text{Hom}_\mathbb{Z}(n\mathbb{Z}, I)$$

hence there exists $b \in I$ such that $nb = a$. Define $\tilde{\varphi} : \mathbb{Z} \to I/N$, $1 \mapsto b + N$. Then $\tilde{\varphi}|_{n\mathbb{Z}} = \varphi$.

(iii) Since the category of $\mathbb{Z}$-modules has enough injectives, by (ii) $N$ has an injective resolution $0 \to I \to I/N \to 0 \to \cdots$. Applying Hom$_\mathbb{Z}(M, -)$ to this complex yields immediately that Ext$^n_\mathbb{Z}(M, N) = 0$ for $n \geq 2$. 

3