Archimedean zeta integrals on $U(n, 1)$

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**Abstract**

For a dual pair of unitary groups with equal size, zeta integrals arising from Rallis inner product formula give the central values of certain automorphic $L$-functions. In this paper we explicitly calculate archimedean zeta integrals of this type for $U(n, 1)$, assuming that the corresponding archimedean component of the automorphic representation is a holomorphic discrete series.

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1. Introduction

In order to obtain deep arithmetic applications in the theory of automorphic forms, it is often necessary to have explicit computable results at each place of a number field. This paper is concerned with certain archimedean zeta integrals on unitary groups and central
$L$-values, which arise from theta correspondence of cuspidal automorphic representations. We shall briefly explain the motivation and background of this paper, following [5,6,9].

Let $F^+$ be a totally real number field, $F$ a totally imaginary quadratic extension of $F^+$, $A = A_{F^+}$ the adele ring of $F^+$. Let $V$ (resp. $V'$) be a hermitian (resp. skew-hermitian) vector space of dimension $n + 1$ over $F$, and $W = V \otimes_F V'$, a symplectic space over $F^+$. Fixing an additive character $\psi$ and a complete polarization $W = X \oplus Y$, we have the Schrödinger model of the oscillator representation $\omega_\psi$ of $\hat{Sp}(W)(A)$, realized on the space $L^2(X(A))$ of square integrable functions on $X(A)$. The space of smooth vectors is the space $\mathcal{S}(X(A))$ of Schwartz–Bruhat functions on $X(A)$. Let $G = U(V)$, $G' = U(V')$. By choosing a global splitting character $\chi$ of $\mathbb{A}^\times_F/F^\times$ as in [6], $\omega_\psi$ then defines an oscillator representation $\omega_{V,V',\chi}$ of $G(A) \times G'(A)$ on $L^2(X(A))$. As usual, for $\phi \in \mathcal{S}(X(A))$ we have the theta lifting $f \mapsto \theta_\phi(f)$ for a cusp form $f$ on $G(F^+) \backslash G(A)$.

Let $\pi$ be a cuspidal automorphic representation of $G$, $\pi^\vee$ be the contragredient of $\pi$, $f \in \pi$, $\tilde{f} \in \pi^\vee$. Let $H = U(V \oplus (-V))$, $i_V : G \times G \hookrightarrow H$ be the natural inclusion, following the doubling method. The Piatetski–Shapiro–Rallis zeta integral is then defined by

$$Z(s, f, \tilde{f}, \varphi, \chi) = \int_{(G \times G)(F^+) \backslash (G \times G)(A)} E(i_V(g, \tilde{g}), s, \varphi, \chi)f(g)\tilde{f}(\tilde{g})\chi^{-1}(\det(\tilde{g}))dgd\tilde{g},$$

(1.1)

where $E(\cdot, s, \varphi, \chi)$ is the Eisenstein series on $H(A)$ as in [5, §1], and $\varphi = \varphi(s)$ is a section of a degenerate principal series $I_{n+1}(s, \chi)$ varying in $s$. This integral converges absolutely for $\text{Res} \gg 0$ and admits an Euler expansion if $\varphi$, $f$ and $\tilde{f}$ are factorizable. In this case, for a sufficiently large finite set $S$ of places of $F^+$ including archimedean ones, one has

$$Z(s, f, \tilde{f}, \varphi, \chi) = \prod_{v \in S} Z(s, f_v, \tilde{f}_v, \varphi_v, \chi_v)d_{n+1}^S(s)^{-1}L^S(s + \frac{1}{2}, \pi, St, \chi),$$

where $L^S(s + \frac{1}{2}, \pi, St, \chi)$ is the partial $L$-function of $\pi$ twisted by $\chi$, attached to the $2(n + 1)$-dimensional standard representation of the $L$-group, and $d_{n+1}^S(s)$ is a product of certain partial $L$-functions attached to the extension $F/F^+$ as in [5]. Take $\phi = \otimes_v \phi_v \in \mathcal{S}(X(A))$ and $\varphi = \delta(\phi \otimes \tilde{\phi})$ in the notation of [9, p. 182]. Then after proper normalization the Rallis inner product formula can be written as

$$\langle \theta_\phi(f), \theta_{\tilde{\phi}}(\tilde{f}) \rangle = \prod_{v \in S} Z(0, f_v, \tilde{f}_v, \varphi_v, \chi_v)d_n^S(0)^{-1}L^S(\frac{1}{2}, \pi, St, \chi).$$

(1.2)

The central $L$-value $L(\frac{1}{2}, \pi, St, \chi)$ is of great arithmetic interest and it is quite useful to have explicit local value at each place. In [6] under certain assumptions it is shown that $L^S(\frac{1}{2}, \pi, St, \chi) \geq 0$ for any finite set $S$ of places of $F^+$. As explained in [9, §2], one has
\[
Z(0, f_v, \tilde{f}_v, \varphi_v, \chi_v) = \int_{G(F_v^+)} (\omega_{\chi_v}(g) \phi_v, \phi_v)(\pi_v(g) f_v, f_v) dg,
\]
which integrates matrix coefficient of the oscillator representation against that of \(\pi_v\). From now on we assume that \(v\) is real, \(\pi_v\) is in the discrete series, \(\phi_v\) is in the space of joint harmonics, and we replace \((\pi_v(g) f_v, f_v)\) by a canonical matrix coefficient \(\psi_{\pi_v}(g)\) of \(\pi_v\) (see Section 4). The aim of this paper is to explicitly compute the archimedean zeta integral
\[
\int_{G(F_v^+)} (\omega_{\chi_v}(g) \phi_v, \phi_v) \cdot \psi_{\pi_v}(g) dg \tag{1.3}
\]
in the case that \(G(F_v^+)\) is the real unitary group \(U(n, 1)\) and \(\pi_v\) is a holomorphic discrete series. We mention that the cases \(U(1, 1)\) and \(U(2, 1)\) were solved completely in \([10]\) and \([11]\) respectively. However for \(U(n, 1)\) when \(\pi_v\) is a general discrete series, this problem seems to out of reach at the moment.

The main results of this paper can be formulated as follows. Fix an additive character \(\psi\) of \(\mathbf{R}\). Let \(V\) be an \((n + 1)\)-dimensional complex Hermitian space, and let \(G\) be the unitary group attached to \(V\). For each complex skew-Hermitian space \(V',\) the group \(G\) is a subgroup of the real symplectic group \(\text{Sp}(V \otimes \mathbb{C} V')\) as usual. Define the metaplectic double cover \(\tilde{G}\) of \(G\) to be the double cover of \(G\) induced by the metaplectic double cover \(\tilde{\text{Sp}}(V \otimes \mathbb{C} V') \rightarrow \text{Sp}(V \otimes \mathbb{C} V')\). The cover \(\tilde{G}\) only depends on the parity of \(\text{dim} V'\), and it splits when \(\text{dim} V'\) is even. Let \(\pi_\lambda\) be the genuine discrete series representation of \(\tilde{G}\) with Harish-Chandra parameter \(\lambda := (\lambda_1, \ldots, \lambda_{n+1})\). By theta dichotomy for real unitary groups \([12]\) and a result in \([8]\) on the discrete spectrum of local theta correspondence, up to isometry there exists a unique \((n + 1)\)-dimensional skew-Hermitian space \(V'\) such that \(\pi_\lambda'\) occurs as a subrepresentation of \(\omega_{V', V', \chi}\). Let \(P_\lambda : \omega_{V, V', \chi} \rightarrow \omega_{V, V', \chi}\) be the orthogonal projection to the \(\pi_\lambda'\)-isotypic subspace. Fix a maximal compact subgroup \(K\) of \(G\), which induces a maximal compact subgroup \(\tilde{K}\) of \(\tilde{G}\). Denote by \(\tau_\chi'\) the lowest \(\tilde{K}\)-type of \(\pi_\lambda\). Then there is a positive number \(c_{\psi, V, \lambda}\) such that
\[
\| P_\lambda(\phi) \| = c_{\psi, V, \lambda} \| \phi \|
\]
for all \(\phi\) in the \(\tau_\chi'\)-isotypic subspace of the space of joint harmonics (with respect to \(K\) and an arbitrary maximal compact subgroup of the unitary group attached to \(V'\)). The constant \(c_{\psi, V, \lambda}\) is 1 when either \(V\) or \(V'\) is anisotropic. The main result of this paper is equivalent to an explicit calculation of \(c_{\psi, V, \lambda}\) when \(V\) is of signature \((n, 1)\), \(\psi\) is chosen to be \(\psi_a : t \mapsto e^{2\pi i a t}\) for some \(a > 0\), and \(\pi_\lambda\) is holomorphic. In this case we list the explicit values of \(c_{\psi, V, \lambda}\) below (Corollary 7.3).

**Theorem 1.1.** Follow above notations, assume that \(V\) has signature \((n, 1)\), \(V'\) has signature \((p, q)\) with \(p + q = n + 1\), \(\psi = \psi_a\) for some \(a > 0\) and \(\pi_\lambda\) is a holomorphic discrete series. Let
\[ \Lambda = \lambda + \left(-\frac{n}{2} + 1, -\frac{n}{2} + 2, \ldots, \frac{n}{2}, -\frac{n}{2}\right). \]

Let \( \alpha \)'s, \( \beta \)'s and \( \gamma \) below stand for non-negative integers. Then we have

(i) if \( \Lambda = [(\alpha_1, \ldots, \alpha_n) + \det^{-(1-n)/2}] \otimes [\gamma + \det^{(1-n)/2}] \) with \( \alpha_1 \geq \cdots \geq \alpha_n \geq \gamma + 2 \), in which case \( p = 1, q = n \), then

\[ c_{\psi, V, \lambda}^2 = \prod_{i=1}^{n} \frac{\alpha_i - i + n - 1 - \gamma}{\alpha_i - i + n}; \]

(ii) if \( \Lambda = [(\alpha_1, \ldots, \alpha_{q-1}, -\beta_p, \ldots, -\beta_1) + \det^{-(p-q)/2}] \otimes [-\gamma + \det^{(p-q)/2}] \) with \( \alpha_1 \geq \cdots \geq \alpha_{q-1} \geq -\beta_p \geq \cdots \geq -\beta_1 \geq -\gamma + 2p \), then

\[ c_{\psi, V, \lambda}^2 = \prod_{i=1}^{n} \frac{\gamma + i - \delta_i - 2p}{\gamma + i - p}, \]

where \((\delta_1, \ldots, \delta_n) := (\beta_1, \ldots, \beta_p, -\alpha_{q-1}, \ldots, -\alpha_1)\).

The organization of the paper is as follows. In section 2 we give the pair of weights appearing in the local theta correspondence. Section 3 describes the structure and measure of the real Lie group \( U(n, 1) \). Section 4 deals with the canonical matrix coefficient of a holomorphic discrete series following [3]. Sections 5 and 6 are concerned with the matrix coefficient of oscillator representation, which is calculated using joint harmonics. In section 7 we combine previous results and apply the technique of [3] to evaluate the zeta integral.

In a totally similar way one can get the explicit values of \( c_{\psi, V, \lambda} \) when \( \pi_\lambda \) is an anti-holomorphic discrete series. We also remark that the method of this paper should be applicable to general \( U(p, q) \), at least when one of the components of the lowest \( \tilde{K} \)-type is one-dimensional. This extension will be pursued in a future paper. Furthermore, it also brings us some enlightenment to study certain period integrals for unitary groups.

**Notations.** Let \( 1_n \) and \( 0_n \) be the \( n \times n \) identity matrix and zero matrix respectively. Let \( 1_{p,q} \) stand for the square matrix

\[ \begin{pmatrix} 1_p & 0 \\ 0 & -1_q \end{pmatrix}. \]

In this paper, \( U(p, q) \) is the real unitary group of the hermitian or skew-hermitian form represented by the matrix \( 1_{p,q} \) or \( i1_{p,q} \), where \( i = \sqrt{-1} \), and \( Sp_{2N}(\mathbb{R}) \) is the isometry group of the real symplectic form represented by the matrix

\[ \begin{pmatrix} 0 & 1_N \\ -1_N & 0 \end{pmatrix}. \]

For a complex matrix \( g \), let \( ^t g \) be its transpose, and \( g^* = ^t \bar{g} \) be the complex conjugate transpose. For a field \( k \), \( M_n(k) \) is the set of \( n \times n \) matrices with entries in \( k \). We usually
regard vectors in $k^n$ as column vectors, unless otherwise specified. For $u, v \in k^n$, as usual $u \cdot v$ stands for their dot product, and $|u|^2 = u \cdot \bar{u}$ if $k = \mathbb{R}$ or $\mathbb{C}$.

2. Pair of weights

Let $G = U(n, 1)$ be the unitary group of a complex hermitian space of signature $(n, 1)$. The absolute root system of $G_C = GL(n+1, \mathbb{C})$ is of type $A_n$. Fix the maximal compact subgroup $K = U(n) \times U(1)$, the set of compact positive roots $\Delta^+ = \{ e_i - e_j : 1 \leq i < j \leq n \}$, and the set of positive roots $\Delta^+ = \{ e_i - e_j : 1 \leq i < j \leq n+1 \}$ that contains $\Delta_C^+$.

We consider $\Delta_C^+$-dominant Harish-Chandra parameters of genuine discrete series of $\tilde{G}$, the metaplectic double cover of $G$. Then in fact holomorphic genuine discrete series are parametrized by $\Delta^+$-dominant Harish-Chandra parameters, i.e. by strictly decreasing $(n + 1)$-tuples $\lambda = (\lambda_1, \ldots, \lambda_{n+1})$ of half-integers. The corresponding lowest $\tilde{K}$-type is given by the Blattner parameter

$$\Lambda = \lambda + \rho - 2\rho_c = \lambda + \left( -\frac{n}{2} + 1, -\frac{n}{2} + 2, \ldots, \frac{n}{2}, -\frac{n}{2} \right), \quad (2.1)$$

where $\rho$ (resp. $\rho_c$) is the half sum of all positive (resp. compact positive) roots.

Consider the dual pair $(G, G') = (U(n, 1), U(p, q)) \hookrightarrow \widetilde{Sp}_{2N}(\mathbb{R})$, where $p + q = n + 1$ and $N = (n + 1)^2$. Fix the additive character $\psi : t \mapsto e^{2\pi it}$ of $\mathbb{R}$, and consider the oscillator representation $\omega_\psi$ of $\widetilde{Sp}_{2N}(\mathbb{R})$. Let $\sigma_\Lambda$ be the lowest $\tilde{K}$-type of a holomorphic discrete series $\pi_\Lambda$ of $\tilde{G}$, where $\Lambda$ is the highest weight of $\sigma_\Lambda$. Then $\lambda$ and $\Lambda$ are related by (2.1). Assume that the theta lifting $\pi' = \theta(\pi'_{\Lambda'})$ of the contragredient representation $\pi'_{\Lambda'}$ is a nonzero discrete series of $\tilde{G}'$ with lowest $\tilde{K}'$-type $\sigma_{\Lambda'}$. Let $\sigma_{\Lambda \vee}$ be the lowest $\tilde{K}$-type of $\pi'_{\Lambda'}$, which is the contragradient of $\sigma_\Lambda$. Then we have an irreducible $\tilde{K} \times \tilde{K}'$-module

$$\mathcal{H}_{\Lambda \vee, \Lambda'} \cong \sigma_{\Lambda \vee} \otimes \sigma_{\Lambda'}$$

that occurs in the space of joint harmonics of $\omega_\psi$. It is well-known that $\mathcal{H}_{\Lambda \vee, \Lambda'}$ occurs with multiplicity one, and moreover $\sigma_{\Lambda \vee}$ and $\sigma_{\Lambda'}$ determine each other.

The Harish-Chandra parameter of the anti-holomorphic discrete series $\pi'_\Lambda$ is $\lambda' = (-\lambda_n, \ldots, -\lambda_1, -\lambda_{n+1})$, and one has

$$\Lambda' = \lambda' + \left( -\frac{n}{2}, -\frac{n}{2} + 1, \ldots, \frac{n}{2} - 1, \frac{n}{2} \right).$$

Let $a$ and $b$ be the number of positive entries in $(-\lambda_n, \ldots, -\lambda_1)$ and $(-\lambda_{n+1})$ respectively. Then by [8], above assumption requires that

$$\lambda_n > \lambda_{n+1} \quad \text{and} \quad p = a - b + 1.$$

Let us write
\[ \Lambda^\vee = \left[ (\beta_1, \ldots, \beta_l, 0, \ldots, 0, -\alpha_k, \ldots, -\alpha_1) + \det^{(p-q)/2} \right] \otimes \left[ m + \det^{-(p-q)/2} \right], \quad (2.2) \]

where \( \alpha_1 \geq \cdots \geq \alpha_k > 0, \beta_1 \geq \cdots \geq \beta_l > 0. \) Then

\[ \Lambda = \left[ (\alpha_1, \ldots, \alpha_k, 0, \ldots, 0, -\beta_l, \ldots, -\beta_1) + \det^{-(p-q)/2} \right] \otimes \left[ -m + \det^{(p-q)/2} \right]. \]

We have two cases.

Case (i): \( b = 0. \) Then \( \lambda_n > \lambda_{n+1} > 0, \) which implies that \( a = p - 1 = 0 \) hence \( p = 1, q = n, \) i.e. \( G' = U(1, n) \). The first entry of \( \Lambda^\vee \) is

\[ -\lambda_n - \frac{n}{2} < \frac{p - q}{2} = \frac{1 - n}{2}, \]

which by (2.2) implies that \( l = 0, k = n. \) Let

\[ \gamma := -m = \lambda_{n+1} - \frac{1}{2} \geq 0. \]

By the formulas for the pair of weights \( \Lambda^\vee, \Lambda' \) in [8], we see that

\[ \begin{cases} 
\Lambda^\vee = \left[ (-\alpha_n, \ldots, -\alpha_1) + \det^{(1-n)/2} \right] \otimes \left[ -\gamma + \det^{-(1-n)/2} \right], \\
\Lambda' = \left[ -\gamma + \det^{(n-1)/2} \right] \otimes \left[ (-\alpha_n, \ldots, -\alpha_1) + \det^{-(n-1)/2} \right]. 
\end{cases} \]

The condition \( \lambda_n > \lambda_{n+1} \) reads

\[ \alpha_n \geq \gamma + 2. \]

Case (ii): \( b = 1. \) Then \( \lambda_{n+1} < 0, a = p. \) Let

\[ \gamma := m = -\lambda_{n+1} + \frac{n}{2} + \frac{p - q}{2} > 0. \]

Again by [8] we have

\[ \Lambda' = \left[ (\beta_1, \ldots, \beta_l, 0, \ldots, 0) + \det^{(n-1)/2} \right] \otimes \left[ (\gamma, 0, \ldots, 0, -\alpha_k, \ldots, -\alpha_1) + \det^{(1-n)/2} \right]. \]

Note that the obvious constraints \( l \leq p, k + 1 \leq q \) apply. For convenience let us define \( \beta_i = 0, \alpha_j = 0 \) for \( l < i \leq p \) and \( k < j \leq q - 1, \) so that we may write

\[ \begin{cases} 
\Lambda^\vee = \left[ (\beta_1, \ldots, \beta_p, -\alpha_{q-1}, \ldots, -\alpha_1) + \det^{(p-q)/2} \right] \otimes \left[ \gamma + \det^{-(p-q)/2} \right], \\
\Lambda' = \left[ (\beta_1, \ldots, \beta_p) + \det^{(n-1)/2} \right] \otimes \left[ (\gamma, -\alpha_{q-1}, \ldots, -\alpha_1) + \det^{-(n-1)/2} \right]. 
\end{cases} \]

(2.4)

The condition \( \lambda_n > \lambda_{n+1} \) reads

\[ -\beta_1 \geq -\gamma + 2p. \]
3. Structure of $G$

Let $g = u(n, 1)$ be the Lie algebra of $G$, and $g = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition with respect to the Cartan involution $\theta(X) = -X^*$. Let $\mathfrak{a}$ be the maximal abelian subalgebra of $\mathfrak{p}$, which is one-dimensional and spanned by

$$H := E_{1,n+1} + E_{n+1,1},$$

where $E_{ij}$ is the elementary matrix with 1 on the $(i, j)$-entry and 0 everywhere else. Let

$$a_t = \exp(tH) = \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & 1_{n-1} & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix}.$$  

The Cartan decomposition of $G$ is $G = C \cdot K \cong C \times K$, where

$$C = \{ g \in G : g = g^* \text{ is positive-definite hermitian} \}.$$

We normalize the measure on $K = U(n) \times U(1)$ so that the masses of $U(n)$ and $U(1)$ are both equal to 1. The set $C$ can be parametrized by

$$D_{n,1} \to C, \quad z \mapsto h_z = \begin{pmatrix} (1_n - zz^*)^{-1/2} & z(1 - z^*z)^{-1/2} \\ (1 - z^*z)^{-1/2}z^* & (1 - z^*z)^{-1/2} \end{pmatrix}.$$  

(3.1)

where $D_{n,1}$ is the classical domain

$$D_{n,1} = \{ z \in C^n : 1_n - zz^* \text{ is positive definite} \}.$$  

The group $G$ acts on $D_{n,1}$ by generalized fractional linear transformations, and we fix the invariant measure on $D_{n,1}$ to be

$$d^*z = \frac{dz}{(1 - z^*z)^{n+1}} = \frac{dz}{\det(1_n - zz^*)^{n+1}},$$

where $dz$ is the product of the usual additive Haar measures.

One may further parametrize $D_{n,1}$ by $z = xy$, where $x \in U(n)$, $y \in U(1)$, and $x = ^t(r,0,\ldots,0)$ with $-1 < r < 1$. If we write $r = \tanh t$, $t \in \mathbb{R}$, then substituting this parametrization into (3.1) yields

$$h_z = k_z a_t k_z^{-1}, \quad k_z = \begin{pmatrix} x & 0 \\ 0 & y^* \end{pmatrix} \in K.$$  

(3.2)
4. Holomorphic discrete series

We shall briefly review the treatment in [3]. Recall that $\mathfrak{g}$ is the Lie algebra of $G$, and let $\mathfrak{g}_\mathbb{C}$ be its complexification. Let

$$p_+ = \left\{ \begin{pmatrix} 0_n & * \\ 0 & 0 \end{pmatrix} \in \mathfrak{g}_\mathbb{C} \right\}, \quad p_- = \left\{ \begin{pmatrix} 0_n & 0 \\ * & 0 \end{pmatrix} \in \mathfrak{g}_\mathbb{C} \right\}$$

and $N_\pm = \exp p_\pm$. Then one has the Harish-Chandra decomposition

$$G \subset N_+ \cdot K_\mathbb{C} \cdot N_- \subset G_\mathbb{C}.$$ 

Let $\pi_\lambda$ be a holomorphic discrete series of $G$ with lowest $K$-type $\sigma = \sigma_\Lambda$. In [3] it is shown that the canonical $K$-conjugation invariant matrix coefficient of $\pi_\lambda$ is given by

$$\psi_{\pi_\lambda}(g) = \psi_{\pi_\lambda}(n_+ \theta n_-) = \text{tr} \sigma(\theta)$$

if $g = n_+ \theta n_-$ under the Harish-Chandra decomposition. Here we use the holomorphic extension of $\sigma$ to $K_\mathbb{C}$. We remark that $\psi_{\pi_\lambda}$ is equivalent to the canonical matrix coefficient considered in [1,6,8,11].

Recall the Cartan decomposition $g = h_z k$. The Harish-Chandra decomposition $h_z = n_z^+ \theta_z n_z^-$ is

$$h_z = \begin{pmatrix} 1_n & z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (1_n - z z^*)^{1/2} & 0 \\ 0 & (1 - z^* z)^{-1/2} \end{pmatrix} \begin{pmatrix} 1_n & 0 \\ z^* & 1 \end{pmatrix}.$$ 

In particular

$$\theta_z = \begin{pmatrix} (1_n - z z^*)^{1/2} & 0 \\ 0 & (1 - z^* z)^{-1/2} \end{pmatrix}.$$ (4.1)

Then we have

$$\psi_{\pi_\lambda}(g) = \psi_{\pi_\lambda}(n_z^+ \theta_z k k^{-1} n_z^- k) = \text{tr} \sigma(\theta_z k),$$ (4.2)

noting that $K$ normalizes $N_\pm$. Parameterizing $z \in D_{n,1}$ as in Section 3, we may write

$$\theta_z = k_z \theta_t k_z^{-1},$$

where $k_z$ is as in (3.2) and $\theta_t$ is the $K_\mathbb{C}$-component of $a_t$ under Harish-Chandra decomposition, i.e.

$$\theta_t = \begin{pmatrix} (\cosh t)^{-1} & 0 & 0 \\ 0 & 1_n & 0 \\ 0 & 0 & \cosh t \end{pmatrix}.$$ (4.3)
Therefore one may further write

\[ \psi_{\pi_\lambda}(g) = \text{tr} \sigma(k_z \theta_t k_z^{-1} k) = \text{tr} \sigma(\theta_t k_z^{-1} k k_z). \] (4.4)

Finally we remark that by Corollary of [4, Lemma 23.1], the formal degree \( d_{\pi_\lambda} \) of a general discrete series \( \pi_\lambda \) is given by

\[ d_{\pi_\lambda} = C \prod_{1 \leq i < j \leq n+1} |\lambda_i - \lambda_j|, \] (4.5)

where \( C \) is a constant depending on the choice of the Haar measure of \( G \).

5. Fock model

The Fock model of the oscillator representation \( \omega_\psi \) of \( \tilde{Sp}_{2N}(\mathbb{R}) \) can be realized on the Fock space \( \mathcal{F}_N \) of entire functions on \( \mathbb{C}^N \) which are square integrable with respect to the hermitian inner product

\[ \langle f, g \rangle_\omega = \int_{\mathbb{C}^N} f(z \bar{g}(\bar{z})) e^{-\pi |z|^2} dz. \]

The monomials \( \left\{ \sqrt{\frac{\pi^{|\alpha|}}{\alpha!}} z^\alpha, |\alpha| \geq 0 \right\} \) form an orthonormal basis of \( \mathcal{F}_N \). The Harish-Chandra module \( \omega^{HC}_\psi \) can be realized as the subspace \( \mathcal{P}_N \) of polynomials on \( \mathbb{C}^N \). In the rest of the paper we denote by \( \omega \) the projective representation of \( Sp_{2N}(\mathbb{R}) \) such that \( \omega_\psi(g, \epsilon) = \epsilon \omega(g) \), \( \epsilon \in \{ \pm 1 \} \). Without any further comments, the explicit formulas about the action of \( \omega \) will hold only up to a factor of \( \pm 1 \).

Following [2], introduce the linear map

\[ M_{2N}(\mathbb{R}) \to M_{2N}(\mathbb{C}), \]

\[ g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto g^c = \frac{1}{2} \begin{pmatrix} A + D + i(C - B) & A - D + i(C + B) \\ A - D - i(C + B) & A + D - i(C - B) \end{pmatrix}. \] (5.1)

Denote by \( Sp_{2N}^c \) the image of \( Sp_{2N}(\mathbb{R}) \). Let \( \nu \) be the Fock projective representation of \( Sp_{2N}^c \) on \( \mathcal{F}_N \). Then for \( g^c = \begin{pmatrix} P & Q \\ Q & P \end{pmatrix} \in Sp_{2N}^c \), the operator \( \nu(g^c) \) is given by

\[ \nu(g^c)f(z) = \int_{\mathbb{C}^N} K_{g^c}(z, \bar{w}) f(\bar{w}) e^{-\pi |\bar{w}|^2} d\bar{w}, \]

\[ K_{g^c}(z, \bar{w}) = (\det P)^{-\frac{1}{4}} \exp \left[ \frac{\pi}{2} (t z Q P^{-1} z + 2 t \bar{w} P^{-1} \bar{z} - t \bar{w} P^{-1} Q \bar{w}) \right]. \] (5.2)
Let $J = 1_{n,1} \otimes 1_{p,q}$. We have the embedding

$$i_G : G \hookrightarrow Sp_{2N}(\mathbb{R}), \quad X + iY \mapsto \left( \begin{array}{cc} X \otimes 1_{n+1} & (Y \otimes 1_{n+1})J \\ -J(Y \otimes 1_{n+1}) & J(X \otimes 1_{n+1})J \end{array} \right). \quad (5.3)$$

In spirit of $\mathbb{C}^N \cong \mathbb{C}^{n+1} \otimes \mathbb{C}^{n+1} \cong M_{n+1}(\mathbb{C})$, it is more convenient to label the variables $z_1, \ldots, z_N$ as $z_1, z_{n+1}, \ldots, z_{n+1,1}, \ldots, z_{n+1,n+1}$. For instance, if we write the matrix of variables in the block form

$$z = (z_{ij})_{i,j=1,\ldots,n+1} = \left( \begin{array}{cc} A_{n \times p} & B_{n \times q} \\ C_{1 \times p} & D_{1 \times q} \end{array} \right), \quad (5.4)$$

then from (5.1)–(5.3), for $k = (x, y) \in K = U(n) \times U(1)$ one has

$$\omega(k)f(z) = \nu(k^c)f(z) = (\det x)^{(p-q)/2}(\det y)^{-(p-q)/2} f \left( \begin{array}{cc} t_x \mathcal{A} & x^{-1} \mathcal{B} \\ y^{-1} \mathcal{C} & t_y \mathcal{D} \end{array} \right). \quad (5.5)$$

For further computations, it is necessary to know the action of $\omega(a_t)$. We use the notation $\bar{z}_i = (z_{i,1}, \ldots, z_{i,n+1})$, $i = 1, \ldots, n+1$, so that we can write $f(z) = f(\bar{z}_1, \ldots, \bar{z}_{n+1}) \in \mathcal{F}_N$. As a preliminary step we have

**Lemma 5.1.** For $f(z) \in \mathcal{P}_N$,

$$\omega(a_t)f(z) = (\cosh t)^{-n-1} \exp(\pi(\tanh t) \bar{z}_1 \cdot \bar{z}_{n+1})$$

$$\times \int_{\mathbb{C}^{n+1}} f(w_1, \bar{z}_2, \ldots, \bar{z}_n, (\cosh t)^{-1} \bar{z}_{n+1}$$

$$- (\tanh t) \bar{w}_1) \exp(\pi(\cosh t)^{-1} \bar{z}_1 \cdot \bar{w}_1 - \pi |\bar{w}_1|^2) \, dw_1.$$

**Proof.** One can show that

$$i_G(a_t) = \left( \begin{array}{cc} a_t \otimes 1_{n+1} & 0 \\ 0 & a_{-t} \otimes 1_{n+1} \end{array} \right), \quad i_G(a_t)^c = \left( \begin{array}{cc} P_t & Q_t \\ Q_t & P_t \end{array} \right)$$

where

$$P_t = \left( \begin{array}{ccc} \cosh t & 0 & 0 \\ 0 & 1_{n-1} & 0 \\ 0 & 0 & \cosh t \end{array} \right) \otimes 1_{n+1}, \quad Q_t = \left( \begin{array}{ccc} 0 & 0 & \sinh t \\ 0 & 0_{n-1} & 0 \\ \sinh t & 0 & 0 \end{array} \right) \otimes 1_{n+1}.$$

We calculate that

$$^t z Q_t P_t^{-1} z = 2(\tanh t) \bar{z}_1 \cdot \bar{z}_{n+1}, \quad ^t \bar{w} P_t^{-1} Q_t \bar{w} = 2(\tanh t) \bar{w}_1 \cdot \bar{w}_{n+1}$$

$$^t \bar{w} P_t^{-1} z = (\cosh t)^{-1}(\bar{z}_1 \cdot \bar{w}_1 + \bar{z}_{n+1} \cdot \bar{w}_{n+1}) + \sum_{i=2}^n \bar{z}_i \cdot \bar{w}_i.$$
The lemma follows from integrating over \( w_2, \ldots, w_{n+1} \) in (5.2) and applying the following formula, which will be used later as well.

\[
\int_{\mathbb{C}} z^i \bar{z}^j e^{\pi c z - \pi |z|^2} dz = \begin{cases} 
\frac{i!}{(i-j)!} \frac{c^{i-j}}{\pi^j} & \text{if } i \geq j, \\
0 & \text{if } i < j, 
\end{cases} \tag{5.6}
\]

where \( i, j \geq 0 \) are integers and \( c \) is a constant. \( \square \)

6. Joint harmonics

The notion of joint harmonics was introduced in [7]. It is the subspace \( \mathcal{H} \subset \mathcal{P}_N \) annihilated by certain second order differential operators from the centralizers of \( \mathfrak{t} \) and \( \mathfrak{t}' \) in \( \mathfrak{sp} \), under the action of the oscillator representation. We refer the readers to [7] for the precise definition.

It is known that \( \mathcal{H} \) admits a multiplicity free decomposition

\[ \mathcal{H} \cong \bigoplus \sigma \otimes \sigma' \]

into irreducible \( \tilde{K} \times \tilde{K}' \)-modules such that \( \sigma \) and \( \sigma' \) determine each other. Moreover, the lowest \( \tilde{K} \)- and \( \tilde{K}' \)-types of discrete series correspond to each other under this decomposition.

We consider the subspace of joint harmonics \( \mathcal{H}_{\Lambda^\vee, \Lambda'} \cong \sigma_{\Lambda^\vee} \otimes \sigma_{\Lambda'} \) as in Section 2. The joint highest weight vector of \( \mathcal{H}_{\Lambda^\vee, \Lambda'} \) can be expressed in terms of principal minors. For \( i = 1, \ldots, n \), let

\[
\Delta_i = \det \begin{pmatrix} z_{11} & \cdots & z_{1i} \\ \vdots & \ddots & \vdots \\ z_{i1} & \cdots & z_{ii} \end{pmatrix}, \quad \Delta'_i = \det \begin{pmatrix} z_{n-i+1,n-i+2} & \cdots & z_{n-i+1,n+1} \\ \vdots & \ddots & \vdots \\ z_{n,n-i+2} & \cdots & z_{n,n+1} \end{pmatrix},
\]

which are determinants of \( i \times i \) minors, hence homogeneous polynomials. Then in the two cases of Section 2, we have the following harmonic polynomials of joint highest weight, which are unique up to scalar.

Case (i): We take

\[ \phi(z) = \Delta_1^{\alpha_1-\alpha_2} \Delta_2^{\alpha_2-\alpha_3} \cdots \Delta_n^{\alpha_n-z_{n+1,1}}. \]

For any \( k \in K \), from (5.5) we see that the block \( C \) in (5.4), i.e. \( z_{n+1,1} \), is the only variable of \( z_{n+1} \) that appears in \( \omega(k) \phi \). But the block \( A \), in particular \( z_{11} \), does not show up in \( \omega(k) \phi \). This observation together with Lemma 5.1 and (5.6) gives us

\[
\omega(a_1k) \phi(z) = (\cosh t)^{-n-1} \exp \left( \pi (\tanh t) z_{11} \cdot z_{n+1} \right) \\
\times \omega(k) \phi \left( (\cosh t)^{-1} z_{11}, z_2, \ldots, z_n, (\cosh t)^{-1} z_{n+1} \right) \\
= (\cosh t)^{-n-1} \exp \left( \pi (\tanh t) z_{11} \cdot z_{n+1} \right) \sigma_{\Lambda^\vee} (b_t k) \phi,
\]

where \( i, j \geq 0 \) are integers and \( c \) is a constant. \( \square \)
where we use the extension of $\sigma_{\Lambda^\vee}$ to $K_C$, and
\[
b_t = \begin{pmatrix}
cosh t & 0 & 0 \\
0 & 1_{n-1} & 0 \\
0 & 0 & \cosh t
\end{pmatrix}.
\]
(6.1)

Since the action of $\sigma_{\Lambda^\vee}$ preserves the degree, and monomials are orthogonal basis, we may use Taylor expansion to drop the factor $\exp\left(\pi(tanh t)z_{1+}z_{n+1}\right)$ and obtain
\[
\langle \omega(k'a_t k)\phi, \phi \rangle = (\cosh t)^{-n-1}\langle \sigma_{\Lambda^\vee}(k'b_t k)\phi, \phi \rangle
\]
(6.2)
for any $k, k' \in K$.

Case (ii): We take
\[
\phi(z) = \Delta_1^{\beta_1-\beta_2} \Delta_2^{\beta_2-\beta_3} \cdots \Delta_p^{\beta_p} \Delta_1^{\alpha_1-\alpha_2} \Delta_2^{\alpha_2-\alpha_3} \cdots \Delta_q^{\alpha_{q-1}-\gamma} z_{n+1,p+1}.
\]
The argument is similar to the above. We note that $z_{n+1,p+1}$ is the only variable of $z_{n+1}$ that appears in $\omega(k)\phi$, while the first $p$ rows of the block $B$ in (5.4), in particular $z_{1,p+1}$, do not show up. The same argument as above gives us
\[
\langle \omega(k'a_t k)\phi, \phi \rangle = (\cosh t)^{-n-1}\langle \sigma_{\Lambda^\vee}(k'b_t^{-1} k)\phi, \phi \rangle
\]
(6.3)

We may summarize our results as

**Proposition 6.1.** Under the assumptions of Section 2, for a vector $\phi \in \mathcal{H}_{\Lambda^\vee,\Lambda^\vee}$ of joint highest weight and $k, k' \in K$, one has
\[
\langle \omega(k'a_t k)\phi, \phi \rangle = (\cosh t)^{-n-1}\langle \sigma_{\Lambda^\vee}(k'b_t^{-1} k)\phi, \phi \rangle,
\]
where the $\pm$ sign depends on whether the first (or equivalently, the last) component of $\Lambda^\vee$ is negative or positive.

In particular, by the Harish-Chandra decomposition $g = h_z k = k_z a_t k_z^{-1} k$, one has
\[
\langle \omega(g)\phi, \phi \rangle = (\cosh t)^{-n-1}\langle \sigma_{\Lambda^\vee}(k_z b_t^\pm k_z^{-1} k)\phi, \phi \rangle.
\]

Define
\[
b_z = k_z b_t k_z^{-1} = \begin{pmatrix}
(1_n - zz^*)^{-1/2} & 0 \\
0 & (1 - z^* z)^{-1/2}
\end{pmatrix}
\]
(6.4)
so that
\[
\langle \omega(g)\phi, \phi \rangle = (\cosh t)^{-n-1}\langle \sigma_{\Lambda^\vee}(b_z^\pm k)\phi, \phi \rangle.
\]
(6.5)
7. Zeta integrals

We are ready to compute the archimedean zeta integrals on $U(n,1)$ that involves oscillator representations and holomorphic discrete series, combining the results in the previous sections.

Since $\pi_\psi^\vee$ occurs in $\omega_\psi$ as a subrepresentation, the matrix coefficient of $\omega_\psi \otimes \pi_\lambda$ on the metaplectic double cover $\tilde{G}$ descends to a function on $G$. On the other hand, by twisting a half-integral power of the determinant character, we may pass from a genuine discrete series of $\tilde{G}$ to a non-genuine discrete series. Hence we may still apply the results in Section 4 in the explicit calculation of zeta integrals.

Keeping in mind the above comments, in terms of the Harish-Chandra decomposition we have by (4.2) and (6.5),

$$\int_G \langle \omega_\psi(g)\phi, \phi \rangle \cdot \psi_{\pi_\lambda}(g) dg = \int_C \int_K \langle \omega_\psi(hz_k)\phi, \phi \rangle \cdot \psi_{\pi_\lambda}(hz_k) dk d^*z$$

$$= \int_C \int_K \langle \sigma_\Lambda^\vee(b^{\pm 1}_z k)\phi, \phi \rangle \cdot \text{tr} \sigma_\Lambda(\theta z_k) \det(1_n - zz^*)^{(n+1)/2} dk d^*z,$$

noting that $\det(1_n - zz^*) = (\cosh t)^{-2}$. We shall follow the strategy in [3] to evaluate the above integral, or more generally the integral

$$I^\pm_s = \int_C \int_K \langle \sigma_\Lambda^\vee(b^{\pm 1}_z k)\phi, \phi \rangle \cdot \text{tr} \sigma_\Lambda(\theta z_k) \det(1_n - zz^*)^s dk d^*z \quad (7.1)$$

which converges absolutely for Re $s \gg 0$. Here the $\pm$ sign is determined by $\Lambda^\vee$ as in Proposition 6.1, i.e. it depends on whether we have Case (i) or (ii).

Our main result is the following

**Theorem 7.1.** Under the assumptions and notations of Sections 2 and 3, for $\phi \in \mathcal{H}_{\Lambda^\vee,\Lambda'}$ one has the zeta integral

Case (i):

$$\int_G \langle \omega_\psi(g)\phi, \phi \rangle \cdot \psi_{\pi_\lambda}(g) dg = \frac{\pi^n}{\dim \sigma_\Lambda} \prod_{i=1}^n \frac{1}{\alpha_i - i + n} \|\phi\|^2;$$

Case (ii):

$$\int_G \langle \omega_\psi(g)\phi, \phi \rangle \cdot \psi_{\pi_\lambda}(g) dg = \frac{\pi^n}{\dim \sigma_\Lambda} \prod_{i=1}^n \frac{1}{\gamma + i - p} \|\phi\|^2,$$

where $\dim \sigma_\Lambda$ is given by the well-known Weyl formula (7.2).
Proof. Since $\sigma_{\Lambda^*}(b_z)^* = \sigma_{\Lambda^*}(b_z^*) = \sigma_{\Lambda^*}(b_z^*)$, $\sigma_{\Lambda^*}(\theta_z^*) = \sigma_{\Lambda^*}(\theta_z^*)$, we have

$$\langle \sigma_{\Lambda^*}(b_{z}^{\pm 1}k)\phi, \phi \rangle \cdot \text{tr} \sigma_{\Lambda}(\theta_z k) = \sum_i \langle \sigma_{\Lambda^*}(k)\phi, \sigma_{\Lambda^*}(b_{z}^{\pm 1})\phi \rangle \cdot \langle \sigma_{\Lambda}(k)x_i, \sigma_{\Lambda}(\theta_z)x_i \rangle,$$

where $\{x_i\}$ is an orthonormal basis of $\sigma_{\Lambda}$. By Schur orthogonality relation, the integration over $K$ leaves us

$$I_{s}^\pm = \frac{1}{\dim \sigma_{\Lambda}} \int_C \langle \phi, x_i \rangle \cdot \langle \sigma_{\Lambda^*}(b_{z}^{\pm 1})\phi, \sigma_{\Lambda}(\theta_z)x_i \rangle \det(1_n - zz^*)^s d^s z$$

$$= \frac{1}{\dim \sigma_{\Lambda}} \int_C \langle \phi, x_i \rangle \cdot \langle \sigma_{\Lambda^*}(\theta_z^{-1}b_{z}^{\pm 1})\phi, x_i \rangle \det(1_n - zz^*)^s d^s z$$

$$= \frac{1}{\dim \sigma_{\Lambda}} \langle \phi, \int_C \sigma_{\Lambda^*}(\theta_z^{-1}b_{z}^{\pm 1}) \det(1_n - zz^*)^s d^s z \phi \rangle.$$

Hence we need to compute the endomorphism

$$T_{s}^\pm = \int_C \sigma_{\Lambda^*}(\theta_z^{-1}b_{z}^{\pm 1}) \det(1_n - zz^*)^s d^s z \in \text{End}_C(\sigma_{\Lambda^*}).$$

We find that

$$\theta_z^{-1}b_z = \begin{pmatrix} (1_n - zz^*)^{-1} & 0 \\ 0 & 1 \end{pmatrix}, \quad \theta_z^{-1}b_z^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 - z^*z \end{pmatrix}.$$

If we decompose $\sigma_{\Lambda^*} \cong \sigma_1 \otimes \sigma_2$ as the outer tensor product of irreducibles $\sigma_1$ of $\tilde{U}(n)$ and $\sigma_2$ of $\tilde{U}(1)$, then

$$\begin{cases} T_{s}^+ = \int_C \sigma_1^{-1}(1_n - zz^*) \det(1_n - zz^*)^s d^s z \in \text{End}_C(\sigma_1), \\ T_{s}^- = \int_C \sigma_2(1 - z^*z) \det(1_n - zz^*)^s d^s z \in \text{End}_C(\sigma_2). \end{cases}$$

By the parametrization of $D_{n,1}$ in Section 3, a change of variables in the above integral shows that $T_{s}^+$ commutes with $\sigma_1(k)$ for any $k \in \tilde{U}(n)$, hence must be a scalar thanks to Schur’s lemma. Similarly $T_{s}^-$ is a scalar as well. In other words, we have

$$I_{s}^\pm = \frac{\overline{T_{s}^\pm}}{\dim \sigma_{\Lambda}} \|\phi\|^2 = \frac{T_{s}^\pm}{\dim \sigma_{\Lambda}} \|\phi\|^2,$$

noting that $T_{s}^\pm$ is real since the integrand is real. Recall the Weyl dimension formula
Then for the respective, variant mistake in Lemma $\text{[3]}$. However the second proposition in [3, §3] was not stated correctly, which caused a mistake in the formulas of the main theorem therein. For reader’s convenience a correct variant form is given in Lemma 7.2 below.

Applying this lemma for the representations $\sigma_1 \otimes 1$ and $1 \otimes \sigma_2$ of $(U(n) \times U(1))^\sim$ respectively, we obtain

Case (i): $\Lambda^\sim = [(-\alpha_n, \ldots, -\alpha_1) + \det(1-n)/2] \otimes [-\gamma + \det-(1-n)/2],

\[ T^+_s = S_{\sigma_1 \otimes 1, s} = \pi^n \prod_{i=1}^n \frac{1}{\alpha_i - i + s - (1-n)/2}, \]

Case (ii): $\Lambda^\sim = [(\beta_1, \ldots, \beta_p, -\alpha_{q-1}, \ldots, -\alpha_1) + \det(p-q)/2] \otimes [\gamma + \det-(p-q)/2],

\[ T^-_s = S_{1 \otimes \sigma_2, s} = \pi^n \prod_{i=1}^n \frac{1}{\gamma - i + s - (p-q)/2}. \]

The theorem follows from specializing $s = (n+1)/2$. \hfill \Box

**Lemma 7.2.** ([3]) Let $\sigma = \sigma_1 \otimes \sigma_2$ be an irreducible representation of $(U(p) \times U(q))^\sim$, where $\sigma_1$ and $\sigma_2$ have highest weights $(\kappa_1, \ldots, \kappa_p)$ and $(\iota_1, \ldots, \iota_q)$ respectively. Define

\[ S_{\sigma, s} = \int_{D_{p,q}} \sigma_1^{-1}(1_p - zz^*) \otimes \sigma_2(1_q - z^*z) \det(1_p - zz^*)^s d^*z \in \text{End}_C(\sigma) \]

for $\text{Re } s \gg 0$, where

\[ D_{p,q} = \{ z \in C^{p \times q} : 1_p - zz^* \text{ is positive definite} \}, \quad d^*z = \frac{dz}{\det(1_p - zz^*)^{p+q}}. \]

Then $S_{\sigma, s}$ is a scalar and one has

(i) if $\sigma_2 = \text{det}^t$ is one-dimensional, then

\[ S_{\sigma, s} = \pi^{pq} \prod_{i=1}^p \frac{\Gamma(\iota - \kappa_i - (p + q - i) + s)}{\Gamma(\iota - \kappa_i - (p - i) + s)} = \pi^{pq} \prod_{i=1}^p \left(\frac{1}{(\iota - \kappa_i - (p + q - i) + s) \cdots (\iota - \kappa_i - (p + 1 - i) + s)}\right). \]
(ii) if $\sigma_1 = \det^\kappa$ is one-dimensional, then
\[
S_{\sigma,s} = \pi^{pq} \prod_{i=1}^{q} \frac{\Gamma(t_i - \kappa - (p + i - 1) + s)}{\Gamma(t_i - \kappa - (i - 1) + s)}
= \pi^{pq} \prod_{i=1}^{q} \frac{1}{(t_i - \kappa - (p + i - 1) + s) \cdots (t_i - \kappa - i + s)};
\]

(iii) if $\sigma$ is the lowest $\tilde{K}$-type of an anti-holomorphic discrete series $\pi_\lambda$ of $\tilde{U}(p,q)$, then under the above measure the formal degree $d_{\pi_\lambda}$ of $\pi_\lambda$ is given by
\[
\frac{1}{d_{\pi_\lambda}} = \frac{S_{\sigma,0}}{\dim \sigma}.
\]

We remark that the formulation of [3] is in terms of holomorphic discrete series, and our reformulation here about anti-holomorphic case is just for convenience. Recall from [6] that
\[
\int_G \langle \omega_\psi(g) \phi, \phi \rangle \cdot \psi_{\pi_\lambda}(g) dg = \frac{c_{\psi,\pi_\lambda}^2 \|\phi\|_2}{d_{\pi_\lambda}},
\]
where $c_{\psi,\pi_\lambda}$ is the positive number such that
\[
\|P_{\psi,\pi_\lambda}(\phi)\| = c_{\psi,\pi_\lambda} \|\phi\|
\]
for $\phi \in \mathcal{H}_{\Lambda^\vee,\Lambda'}$ and $P_{\psi,\pi_\lambda}$ is the orthogonal projection from $\omega_\psi$ onto the closed subspace $\sigma_{\Lambda^\vee} \otimes \pi'$. We are interested in the explicit value of $c_{\psi,\pi_\lambda}$. The formal degree $d_{\pi_\lambda}$ is given by (4.5), which depends on the measure of $G$. Instead of specifying the explicit dependence, we may compare our zeta integral with the formal degree given by Lemma 7.2 (iii). This will enable us to find out $c_{\psi,\pi_\lambda}$.

**Corollary 7.3.** The explicit value of $c_{\psi,\pi_\lambda}$ is given by

**Case (i):**
\[
c_{\psi,\pi_\lambda}^2 = \prod_{i=1}^{n} \frac{\alpha_i - i + n - 1 - \gamma}{\alpha_i - i + n};
\]

**Case (ii):**
\[
c_{\psi,\pi_\lambda}^2 = \prod_{i=1}^{n} \frac{\gamma + i - \delta_i - 2p}{\gamma + i - p},
\]
where $(\delta_1, \ldots, \delta_n) := (\beta_1, \ldots, \beta_p, -\alpha_q - 1, \ldots, -\alpha_1)$. 

**Proof.** The proof of Theorem 7.1 shows that

\[ \frac{c^2_{\psi, \pi_{\lambda}}}{d_{\pi_{\lambda}}} = \frac{T_{(n+1)/2}^{\pm}}{\dim \sigma_{\lambda}}. \]

On the other hand, by Lemma 7.2 we have

\[ \frac{1}{d_{\pi_{\lambda}}} = \frac{S_{\sigma_{\lambda \nu}, 0}}{\dim \sigma_{\lambda}}. \]

Comparison of the last two equations yields

\[ c^2_{\psi, \pi_{\lambda}} = \frac{T_{(n+1)/2}^{\pm}}{S_{\sigma_{\lambda \nu}, 0}} = \frac{S_{\sigma_1 \otimes 1_0, (n+1)/2}}{S_{\sigma_1 \otimes \sigma_0, 0}} \quad \text{or} \quad \frac{S_{1 \otimes \sigma_2, (n+1)/2}}{S_{\sigma_1 \otimes \sigma_2, 0}} \]

in Case (i) or (ii) respectively. Plugging in the parameter $\Lambda'$ gives the corollary. \( \square \)

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