Frontiers of Computational Mathematics Homework #2
due 2017 NOV 03, 2:55 p.m.

All Definitions, Figures, and Equations in the format of “•.” refer to those in the paper “Boolean algebra on physically meaningful regions in the plane.”

1 The meet operation on two oriented Jordan curves

Utilizing the spadjor-forest representation, we define a meet operation ∧ on two oriented Jordan curves as follows.

Definition 1 (The “meet” operation on two oriented Jordan curves). Let $\sigma_1$ and $\sigma_2$ denote two oriented Jordan curves. The “meet” of $\sigma_1$ and $\sigma_2$, written $\sigma_1 \land \sigma_2$, is a spadjor forest defined as follows.

1. Suppose $\sigma_1 = \sigma_2 = \gamma$. Then

$$\sigma_1 \land \sigma_2 = \begin{cases} \{\gamma\} & \text{if } \sigma_1 \text{ and } \sigma_2 \text{ have the same orientation.} \\ \emptyset & \text{if } \sigma_1 \text{ and } \sigma_2 \text{ have different orientations.} \end{cases}$$

2. Suppose $\sigma_1 \cap \sigma_2 = \emptyset$ or $\sigma_1 \cap \sigma_2 = \{p\}$, i.e., $\sigma_1 \cap \sigma_2$ contains at most one isolated point. Let $q_1 \in \sigma_1$ and $q_2 \in \sigma_2$ be two points distinct from $p$. Then

$$\sigma_1 \land \sigma_2 = \begin{cases} \{\sigma_1\} & \text{if } q_1 \in \text{int}(\sigma_2), \ q_2 \notin \text{int}(\sigma_1) \\ \{\sigma_2\} & \text{if } q_1 \notin \text{int}(\sigma_2), \ q_2 \in \text{int}(\sigma_1) \\ \{\emptyset\} & \text{if } q_1 \notin \text{int}(\sigma_2), \ q_2 \notin \text{int}(\sigma_1) \\ \{\sigma_1, \sigma_2\} & \text{if } q_1 \in \text{int}(\sigma_2), \ q_2 \in \text{int}(\sigma_1) \end{cases}$$

3. Otherwise let $V$ denote the set of isolated points in $\sigma_1 \cap \sigma_2$ and the endpoints of curve segments in $\sigma_1 \cap \sigma_2$. With $V$ being the vertex set, we construct a directed multigraph $G = (V, E)$ as follows.

(a) Removing points in $V$ from $\sigma_1$ yields $\{\beta_i\}$, a set of directed curve segments. For each $\beta_i \subset \sigma_1 \setminus V$, add it to $E$ if $\beta_i \subset \text{int}(\sigma_2)$, or, if there exists $\beta_j \subset \sigma_2 \setminus V$ such that $\beta_j = \beta_i$ and they have the same direction.

(b) For each $\beta_j \subset \sigma_2 \setminus V$, add it to $E$ if $\beta_j \subset \text{int}(\sigma_1)$.

Then we set

$$\sigma_1 \land \sigma_2 = \{\gamma_1, \ldots, \gamma_m\},$$

where $\gamma_1, \ldots, \gamma_m$ are pairwise almost disjoint Jordan curves that partition $E$; they are obtained as follows.

(i) Start with an arbitrary edge $\beta_i \in E$; denote by $p^s_i$ and $p^e_i$ the start and end points of $\beta_i$, respectively.

(ii) If there exists only one edge $\beta_j \subset E$ such that $p^s_j = p^e_i$ is its starting point, then append $\beta_j$ to $\beta_i$ to grow the curve.

(iii) Otherwise there exist two edges $\beta_j, \beta_k \subset E$ such that $p^s_j = p^e_k = p^e_i$. Denote by $\beta^e_i$ the directed arc going from $p^s_i$ to $p^e_i$ such that $\beta^e_i \subset \sigma_2$ if $\beta_j \subset \sigma_1$ and $\beta^e_i \subset \sigma_1$ if $\beta_k \subset \sigma_2$. Construct an oriented Jordan curve $\sigma_i$ by connecting $\beta_i$ and $\beta^e_i$.

- If $\beta_j \subset \text{int}(\sigma_i)$ holds for $\ell = j$ and $\ell = k$, we append $\beta_j$ to $\beta_i$;
- otherwise we append the edge $\beta_{\ell}$ that satisfies $\beta_{\ell} \subseteq \beta_i^e$.

(iv) Repeat (ii) and (iii) to grow the curve until it becomes a Jordan curve $\gamma_1$. Then remove all curve segments in $\gamma_1$ from the edge set $E$.

(v) Repeat steps (i) – (iv) to form other Jordan curves $\gamma_2, \ldots, \gamma_m$ until $E$ becomes an empty set.

The following questions weigh 50 points.

(a) Let $\rho$ denote the boundary-to-interior map in Equations (3.5), (3.6) and Definition 3.14. Define $\rho(\gamma) = \text{int}(\gamma)$ for an oriented Jordan curve. If $\sigma_1$ and $\sigma_2$ are two arbitrary oriented Jordan curves, does the meet operation in Definition 1 satisfies the homomorphism

$$\rho(\sigma_1 \land \sigma_2) = \rho(\sigma_1) \cap \rho(\sigma_2)?$$

(b) If your answer to (a) is no, you must give a counter-example to show the failure of the algorithm in Definition 1 and you must redefine the meet operation of two oriented Jordan curves so that (2) holds.

(c) Prove that the homomorphism (2) holds either for Definition 1 or for your version of the meet operation.
2 Representing a cartoon with a Yin set

For this assignment you need to
(a) Find a black-and-white cartoon figure you like.
(b) Pick characteristic points on the boundary and fit simple closed cubic splines through these points so that your simple closed splines approximate the original boundary curves very well. Save the coordinates of the spline knots in one or multiple pure text files.
   **Hint:** The software Matlab GrabIt may be useful.
(c) Write one or multiple matlab subroutines so that the driver program takes as input the text files in (b) and generate a Yin set, which is supposed to approximate the cartoon figure in (a).
   **Hint:** You can reuse subroutines in Homework #1.
(d) Write a matlab subroutine so that it takes as input a number of pairwise almost disjoint simple closed splines and produces a poset of splines with the binary relation as inclusion in Definition 3.6.
   **Hint:** Your subroutine may output two integer arrays “parent” and “children” that correspond to the input array of splines.
(e) Plot the Hasse diagram of the poset in (d).
   **Important:** Your Hasse diagram must have at least five levels of oriented Jordan curves, i.e. a chain of four covering relations. Hence the cartoon figure you choose in (a) cannot be arbitrary.
(f) Represent the Yin set $\mathcal{Y}$ plotted in (c) as a spadjor forest $\mathcal{F}$, i.e. $\rho(\mathcal{F}) = \mathcal{Y}$. Then expand the expression $\rho(\mathcal{F})$ in terms of interiors of oriented Jordan curves.

See Figure 1 for an example.

The total point for this problem is 30; the above six questions weigh 1, 5, 7, 7, 6, and 4 points, respectively.

Your matlab problem must finish on my computer. Otherwise you get no points for that question. Your Hasse diagram should have the same format as that of Figure 3.9 in the paper. The \LaTeX{} source for that figure is attached for your convenience as follows.

```latex
\documentclass[10pt]{article}
\usepackage{amsmath}
\usepackage{amsfonts}
\usepackage{pstricks,pst-eps}
\usepackage{tikz}
\pagestyle{empty}
\begin{document}
\begin{TeXtoEPS}
\begin{tikzpicture}
\node[circle,draw,fill=yellow](n4) at (0,0) {$\gamma_4^+$};
\node[circle,draw](n6) at (-2,-2) {$\gamma_6^-$};
\node[circle,draw](n5) at (0,-2) {$\gamma_5^-$};
\node[circle,draw](n7) at (2,-2) {$\gamma_7^-$};
\node[circle,draw,fill=yellow](n2) at (-1,-4) {$\gamma_2^+$};
\node[circle,draw,fill=yellow](n3) at (+1,-4) {$\gamma_3^+$};
\node[circle,draw,fill=yellow](n1) at (-4,0) {$\gamma_1^+$};
\draw[line width=2] (n4)--(n6);
\draw[line width=2] (n4)--(n5);
\draw[line width=2] (n4)--(n7);
\draw (n5)--(n2);
\draw (n5)--(n3);
\end{tikzpicture}
\end{TeXtoEPS}
\end{document}
```

3 Literature search

Use ZJU library website to find at least five SCI journal papers on polygon clipping and/or Boolean operations on Nef polygon. Read these papers and write a story out of your reading.

**Important:** You writing must be typeset in \LaTeX{} and the cited references must be in a separate .bib file. At least ten papers should be cited and at least 300 words are required, either in English or in Chinese.

The total point is 20. Your solution will be graded based on the depth and coherence of your writing.