On Generalized Donating Regions: Classifying Lagrangian Fluxing Particles through a Fixed Curve in the Plane

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Abstract

For a fixed simple curve in a nonautonomous flow, the fluxing index of a passively-advected Lagrangian particle is the total number of times it goes across the curve within a given time interval. Such indices naturally induce flux sets, equivalence classes of the particles at the initial time. This line of research mainly concerns donating regions, the explicit construction of the flux sets from a sufficiently continuous velocity field. In the author’s previous paper [Q. Zhang, SIAM Review, 55(3), 2013], the flux sets with indices ±1 were constructed from characteristic curves of the flow field. This work generalizes the notion of donating regions and shows that flux sets are index-by-index equivalent to the generalized donating regions for any finite integer index, provided that the two backward streaklines seeded at the two endpoints of the simple curve neither intersect nor self-intersect. All Lagrangian particles marked by their initial positions are thus classified by their fluxing indices.

Keywords: flux set, generalized donating regions, streakline, winding number, Hopf theorem, cycle decomposition.

1. Introduction

The nonautonomous ordinary differential equation (ODE)

\[ \frac{dx}{dt} = u(x, t) \]  \hspace{1cm} (1)

admits a unique solution for any given initial time \( t_0 \) and initial position \( p(t_0) \) if the time-dependent velocity field \( u(x,t) \) is continuous in time and Lipschitz continuous in space. This gives rise to a flow map \( \phi : \mathbb{R}^D \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^D \) that maps the initial position \( p(t_0) \) of a Lagrangian particle \( p \), the initial time \( t_0 \),

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and the time increment \( \tau \) to \( p(t_0 + \tau) \), the position of \( p \) at time \( t_0 + \tau \). The shorthand notations \( \overrightarrow{p} = \phi_{t_0}^\tau(p(t_0)) \) and \( \overleftarrow{p} = \phi_{t_0}^{-\tau}(p(t_0 + \tau)) \) are used if \( t_0 \) and \( \tau \) are clear from the context. The flow map also applies to point sets in a straightforward way, i.e. \( \phi_{t_0}^\tau(M) = \{ \phi_{t_0}^\tau(p) : p \in M \} \).

One well-known characteristic curve of the flow map is a pathline, the curve in the phase space generated by following a single particle,

\[
\Phi_{t_0}^{\pm k}(p) := \{ \phi_{t_0}^{\pm \tau}(p) : \tau \in (0, k) \}.
\]  

(2)

**Definition 1.1 (Flux [13]).** A fluxing particle to a fixed simple curve \( \tilde{L}\tilde{N} \) over the time interval \((t_0, t_0 + k)\) is a particle \( p \) whose pathline \( \Phi_{t_0}^{\pm k}(p) \) properly intersects \( \tilde{L}\tilde{N} \) at least once. See [13, Def. 4.1] for a definition of proper intersections. Note that an intersection must be proper for the particle to go across the curve.

**Definition 1.2 (Fluxing index).** Consider the proper intersections of a fluxing particle to a fixed simple curve \( \tilde{L}\tilde{N} \) within a time interval. Let \( \dot{\gamma} \) denote a Jordan curve satisfying \( \dot{\gamma} \supseteq \tilde{L}\tilde{N} \). A proper intersection is called an out-flux with respect to \( \dot{\gamma} \) if the fluxing particle crosses \( \dot{\gamma} \) from the bounded complement of \( \dot{\gamma} \) into the unbounded complement of \( \dot{\gamma} \) in a sufficiently small local neighborhood of the intersection point; otherwise it is called an in-flux with respect to \( \dot{\gamma} \). The fluxing index of a fluxing particle to \( \tilde{L}\tilde{N} \) with respect to \( \dot{\gamma} \supseteq \tilde{L}\tilde{N} \) is the number of its out-fluxes minus that of its in-fluxes.

See Figure 1 for three examples of fluxing indices. In the original paper [13, Def. 4.2], the dot product of the normal vector of \( \tilde{L}\tilde{N} \) and the velocity vector of the fluxing particle is used in determining the fluxing indices; this definition does not apply if the normal vector of \( \tilde{L}\tilde{N} \) does not exist, or, the velocity vector of the particle at a proper intersection aligns with the tangent vector of \( \tilde{L}\tilde{N} \) at
the same intersection. For these cases, Definition 1.2 still holds, and hence it is more general than the original definition in [13].

Given a fixed simple curve \( \tilde{L}N \), a Jordan curve \( \dot{\gamma} \supseteq \tilde{L}N \), a sufficiently continuous velocity field \( \mathbf{u}(x,t) \), and a time interval \((t_0, t_0 + k)\), the fluxing index defines an equivalence class on the points in the plane as follows.

**Definition 1.3 (Flux set).** The flux set of index \( n \) through a simple curve \( \tilde{L}N \) with respect to a Jordan curve \( \dot{\gamma} \supseteq \tilde{L}N \) over the time interval \((t_0, t_0 + k)\), denoted \( F_{\tilde{L}N}^n(t_0, k) \), is the loci of all the fluxing particles of index \( n \) at time \( t_0 \).

The necessity of having a Jordan curve in the above definition can be seen by considering fluxes through faces of fixed control volumes: a cell face belongs to the two neighboring control volumes as part of the cell boundaries. Clearly, the flux set of index +1 through the common cell face with respect to the boundary of one control volume is that of index −1 with respect to the boundary of the other control volume. In this work, the Jordan curve \( \dot{\gamma} \) of the flux sets is clear or irrelevant for the exposition and hence will not be explicitly stated; the only exception is Theorem 4.9, where the choice of \( \dot{\gamma} \) determines one of the two equivalence possibilities between the flux sets and generalized donating regions.

Clearly the flux sets are pairwise disjoint and partition \( \mathbb{R}^2 \). For the case that all Lagrangian particles have their fluxing indices \(|n| \leq 1\), the problem of finding the flux sets \( F_{\tilde{L}N}^n(t_0, k) \) has been solved in [13], where a curvilinear polygon \( D_{\tilde{L}N} \) called the donating region (DR) was defined from the characteristic curves of the ODE (1) and shown to be equivalent to \( F_{\tilde{L}N}^{\pm 1}(t_0, k) \). The DR theory has been applied to analyzing a family of interface tracking methods [12] in multiphase-flow simulations.

In the context of numerically solving the partial differential equations with finite volume methods, fluxes through cell edges are of fundamental importance. Let \( f(x,t) : \mathbb{R}^D \times \mathbb{R} \to \mathbb{R} \) denote a scalar function conserved by the velocity field,

\[
\frac{\partial f}{\partial t} + \nabla \cdot (f \mathbf{u}) = 0,
\]

then the flux of the scalar through \( \tilde{L}N \) within the time interval \((t_0, t_0 + k)\) can be converted to a spatial integral over the DR at the initial time \( t_0 \),

\[
\int_{\tilde{L}N} \int_{t_0}^{t_0+k} f(x,t) \mathbf{u}(x,t) \cdot n_{\tilde{L}N} \, dt \, dx = \int_{D_{\tilde{L}N}} f(x,t_0) \, dx,
\]

which is intuitive for incompressible flows: the value of \( f \) for a Lagrangian particle \( p \) remains the same as \( f(p(t_0)) \) while the particle goes across \( \tilde{L}N \) and contributes to the total flux. This intuitive argument is the only justification of (4) in the original paper [13]. A later work by the author [11] numerically verifies (4) for incompressible flows. Nonetheless, given the equivalence of the

\[\text{1}\] The primary time interval \((t_0 - k, t_0)\) in the original paper [13] is shifted to \((t_0, t_0 + k)\).
DR and the flux sets with $n = \pm 1$, the proof of (4) is still an open problem, especially for the case of compressible flows. After all, the velocity field is not assumed to be solenoidal.

The integration of (3) over a fixed cell $\mathcal{C}$ and the application of (4) to the cell edges yield

$$\left\langle f(t_0 + k) \right\rangle_{\mathcal{C}} = \left\langle f(t_0) \right\rangle_{\mathcal{C}} - \frac{1}{\|\mathcal{C}\|} \sum_{\tilde{L}N \subset \partial \mathcal{C}} \int_{D_{\tilde{L}N}} f(x, t_0) \, dx,$$

where $\|\mathcal{C}\| = \int_{\mathcal{C}} dx$ and $\left\langle f(t) \right\rangle_{\mathcal{C}} = \frac{1}{\|\mathcal{C}\|} \int_{\mathcal{C}} f(x, t) \, dx$.

The identity (5) serves as the theoretical foundation of a family of highly accurate semi-Lagrangian methods [11] for solving the advection equation in incompressible flows. In these methods, the flow map is approximated by a numerical time integrator such as a Runge-Kutta method. The construction of a DR begins with obtaining sample points of the generating curves by appropriate numerical integration in time. Then these sampling points are connected by a spline as an approximation to the exact DR, which, together with algebraic quadrature formulas, reduce each DR integral in (5) to a linear combination of point values of $f$ at the initial time $t_0$. The aforementioned steps constitute a numerical algorithm to evolve (3) with the unknowns being cell-averaged scalars $\left\langle f(t) \right\rangle_{\mathcal{C}}$ in (5). Apart from their flexibility, these methods are free of the Eulerian stability constraint that the Courant number be less than or equal to one. For more details, the reader is referred to [11], where convergence rates from 2 to 8 with the Courant numbers ranging from 1 to 10000 are demonstrated.

This paper concerns explicitly constructing flux sets of all indices, i.e. the decomposition of the plane into pairwise disjoint flux sets. To this end, the original DRs are first generalized in Definition 4.3 via the equivalence classes induced by the winding numbers of an oriented closed curve, then in Theorem 4.9 they are shown, under certain conditions, to be equivalent to the flux sets for all integer indices. The adopted approach for proving the main conclusion is mostly algebraic and differs\(^2\) from that of the original paper [13]. Nonetheless, the two constructions are consistent since the generalized DRs reduce to the original ones when no particles have fluxing indices $|n| > 1$.

The rest of this paper is organized as follows. Section 2 contains fundamental definitions and theorems from geometry and topology. Section 3 concerns winding regions, the equivalence classes induced from the winding numbers of an oriented closed curve. In Section 4, the generalized DRs are defined as special cases of winding regions and, as the main contribution of this work, the normal ones are shown in Theorem 4.9 to be index-by-index equivalent to the flux sets. Section 5 concludes this paper with several future research prospects.

\(^2\)The different approach taken in this paper is necessary for generalizing the explicit construction of flux sets to $n \in \mathbb{Z}$. For example, the interior of a curvilinear polygon is not well defined when the boundary of the polygon has complex self-intersections.
2. Preliminaries

This section introduces notations and collects elementary definitions and theorems that are necessary for a (somewhat) self-contained exposition.

2.1. Planar curves

An open curve is a continuous map $\gamma : (0, 1) \to \mathbb{R}^2$; it is simple if $\gamma$ is injective. A closed curve is a continuous map $\gamma : [0, 1] \to \mathbb{R}^2$ satisfying $\gamma(0) = \gamma(1)$; it is simple closed or Jordan if the restriction of $\gamma$ to $(0, 1)$ is injective. Although curves are defined as continuous maps, they also refer to the images of the maps, e.g., the points not in the image of a closed curve $\gamma$ are denoted by $\mathbb{R}^2 \setminus \gamma$. When a simple open curve $\gamma$ satisfies $\lim_{s \to 0^+} \gamma(s) = \lim_{s \to 1^-} \gamma(s) = p \notin \gamma$, the corresponding Jordan curve can be obtained by extending the domain of $\gamma$ to $[0, 1]$.

An arc is a connected subset of a curve. A curve $\gamma$ is self-intersecting if it is not simple; a self-intersection point of $\gamma$ is a point on it whose fiber is more than a singleton set, i.e. $\gamma^{-1}(\gamma(s)) \neq \{s\}$. The degree of a self-intersection point is the cardinality of its fiber. A 1D self-intersection of a closed curve $\gamma$ is a maximal connected finite-length arc of self-intersection points with the same degree. These 1D self-intersections represent degenerate cases, and the inclusion of them in the analysis makes the proofs tediously technical and distracting. For ease of exposition, the author makes the following assumption for the rest of this paper (except in Proposition 4.6 where discussing 1D self-intersections is illuminating).

Assumption 2.1. A closed curve in this work is assumed to only have a countable number of self-intersection points, i.e. a closed curve is free of 1D self-intersections.

An orientation of a curve $\gamma$ is the assignment of a direction in which $\gamma$ is traversed. Let $\partial D$ denote the boundary of a compact set $D$. If $\partial D$ is a 1-manifold, i.e. a set of pairwise-disjoint non-self-intersecting closed curves, the induced orientation of $\partial D$ from $D$ is defined by the condition [9, p. 50] that the outward-pointing normal to $D$ at $\partial D$, followed by this induced orientation, yields the orientation of $\partial D$. Conversely, the orientation of $D$ can also be induced from $\partial D$ in a similar fashion. As usual, counterclockwise rotation is the positive orientation.

The study of topological invariants of planar curves mainly concerns smooth closed curves with a finite number of traversal double points [10, 2]. In comparison, closed curves in this work may contain self-tangency points, $C^1$ discontinuities, and self-intersection points with degrees more than 2.

2.2. Winding numbers

As a fundamental concept of algebraic topology, the winding number represents the total number of times that an oriented closed curve winds around a fixed point.
Definition 2.2 (Winding number). The winding number of an oriented closed curve $\gamma$ around a point $x \in \mathbb{R}^2 \setminus \gamma$ is the number

$$w_\gamma(x) := \frac{\theta(1) - \theta(0)}{2\pi},$$

where $\theta : [0, 1] \to \mathbb{R}$ is the continuous function defined as the angle of $\gamma(t)$ in the polar coordinate system whose pole is at $x$, i.e. $\gamma(t) = x + \rho(t)(\cos \theta(t), \sin \theta(t))$.

$w_\gamma(x)$ is clearly an integer for all points $x \notin \gamma$.

A closed curve $\gamma$ partitions $\mathbb{R}^2 \setminus \gamma$ into a family of disjoint connected open sets, one unbounded and the others bounded; they are called the cells of $\gamma$.

Proposition 2.3. If $x$ and $y$ are in the same cell of $\gamma$, then $w_\gamma(x) = w_\gamma(y)$.

Proof. See [7, p. 339].

In Definition 2.2, the oriented closed curve $\gamma$ is clearly bounded and thus can be covered by a disk. For any given $\delta > 0$, there exists a point $x$ away from the covering disk of $\gamma$ far enough such that $\theta(1) - \theta(0) = 0 < \delta$. Hence $w_\gamma(x) \to 0$ as $x \to \infty$ in (6). It then follows from Proposition 2.3 that the winding number of $\gamma$ around all points in the unbounded cell is zero.

2.3. Homotopy and Hopf theorem

A path in a topological space $\mathcal{X}$ is a continuous map $\zeta : [0, 1] \to \mathcal{X}$. A homotopy of paths in $\mathcal{X}$ is a family of paths $h_t : [0, 1] \to \mathcal{X}$, $t \in [0, 1]$, such that $h_t(0)$, $h_t(1)$ are independent of time and the associated map $H : [0, 1]^2 \to \mathcal{X}$ defined by $H(s, t) = h_t(s)$ is continuous. Two paths are homotopic if they are connected by a homotopy. A path in $\mathbb{R}^2$ can be attained from an open curve by adding its endpoints, written $\zeta = \lim_{s \to \alpha^+} \gamma(s) \cup \gamma \cup \lim_{s \to \beta^-} \gamma(s)$.

The relation of homotopy on paths with fixed endpoints in any space is an equivalence relation. The equivalence class of a path $\zeta$ under this equivalence relation is called the homotopy class of $\zeta$ and denoted by $[\zeta]$. For two paths $\zeta_1$ and $\zeta_2$ satisfying $\zeta_1(1) = \zeta_2(0)$, the composition of $\zeta_1$ and $\zeta_2$, denoted $\zeta_1 \cdot \zeta_2$, is defined as

$$(\zeta_1 \cdot \zeta_2)(s) := \begin{cases} \zeta_1(2s), & s \in [0, \frac{1}{2}], \\ \zeta_2(2s - 1), & s \in [\frac{1}{2}, 1]. \end{cases}$$

(7)

This composition preserves homotopy, i.e. $[\zeta_1 \cdot \zeta_2]$ is still a homotopy class [5, p. 26].

A loop or cycle is a path whose endpoints coincide; this common endpoint is the basepoint of the loop. Two loops are homotopic in $\mathcal{X}$ if they are homotopic in $\mathcal{X}$ as paths; in this case, the basepoint of the loop homotopy must be fixed. In comparison, a free homotopy between two loops $\gamma_1$ and $\gamma_2$ in $\mathcal{X}$ is a function $H_f : [0, 1]^2 \to \mathcal{X}$ such that $H_f(0, t) = \gamma_1(t)$ and $H_f(1, t) = \gamma_2(t)$ for all $t$, and $H_f(s, 0) = H_f(s, 1)$ for all $s$. Then $\gamma_1$ and $\gamma_2$ are said to be freely homotopic in $\mathcal{X}$. Clearly, any path homotopy between two loops can be interpreted as a free homotopy between them.
Any closed curve can be viewed as a loop, hence the aforementioned concepts directly apply to closed curves. As a special case of the celebrated Hopf degree theorem, the following statement relates two freely homotopic closed curves to their winding numbers around a fixed point.

**Theorem 2.4 (Hopf [6]).** Let a point $x \in \mathbb{R}^2$ be given. Two closed curves $\gamma_1$ and $\gamma_2$ are freely homotopic in $\mathbb{R}^2 \setminus \{x\}$ if and only if $w(\gamma_1, x) = w(\gamma_2, x)$.

In other words, one closed curve can be continuously deformed to the other without crossing $x$ in $\mathbb{R}^2$ if and only if the winding number of $\gamma_1$ around $x$ is equal to that of $\gamma_2$ around $x$. It is emphasized that the underlying space of the free homotopy is $\mathbb{R}^2 \setminus \{x\}$, not $\mathbb{R}^2$; see Figure 2 for illustrations of this crucial point.

### 2.4. Cycle decomposition

The following result is well known as the Seifert decomposition of an oriented closed curve.

**Proposition 2.5 (Cycle decomposition of an oriented closed curve [8]).**

An oriented closed curve $\gamma$ with a countable number of self-intersection points can be decomposed into oriented, pairwise non-crossing Jordan curves.

See Figure 3 for two simple examples. As discussed in Section 2.1, the orientation of the bounded complement of each individual Jordan curve can be induced from that of $\gamma$. 

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Figure 2: Examples of free homotopy and Hopf theorem. Two loops $\gamma_1$ (dashed lines) and $\gamma_2$ (thin solid lines) share as their common arc the simple curve $LN$ (the thick solid line). In $\mathbb{R}^2$, they are of the same homotopy class $H(s,t) = t\gamma_1(s) + (1-t)\gamma_2(s)$ with the fixed basepoint as $L = \gamma_1(0) = \gamma_1(1) = \gamma_2(0) = \gamma_2(1)$; they are also freely homotopic in $\mathbb{R}^2$. Furthermore, they are freely homotopic in $\mathbb{R}^2 \setminus \{x_1\}$ and $\mathbb{R}^2 \setminus \{x_3\}$ because of the Hopf theorem, $w(\gamma_1, x_1) = w(\gamma_2, x_1)$, and $w(\gamma_1, x_3) = w(\gamma_2, x_3)$. However, they are not freely homotopic in $\mathbb{R}^2 \setminus \{x_2\}$ because $w(\gamma_1, x_2) \neq w(\gamma_2, x_2)$. Simply put, there is no way $\gamma_1$ can be continuously deformed to $\gamma_2$ in $\mathbb{R}^2 \setminus \{x_2\}$, or, any continuous deformation of $\gamma_1$ to $\gamma_2$ in $\mathbb{R}^2$ must cross $x_2$.
Figure 3: Rearrange two crossing cycles to non-crossing ones. In both cases, the original cycles are the two circles. For case (a), the two rearranged cycles are represented by the boundaries of the two light gray regions. For case (b), one rearranged cycle is represented by the boundary of the dark gray area while the other by that of the light gray area that contains the dark gray area.

Figure 4: A negative crossing for a particle \( p \) of an oriented closed curve at a non-self-intersection point decreases its winding number by one. More precisely, \( w_{\gamma}(y) = w_{\gamma}(x) - 1 \), where \( x = p(t_0 - \tau) \), \( y = p(t_0 + \tau) \), and \( o = p(t_0) \). The dashed lines represents a local neighborhood of \( \gamma \) at \( o \), \( \gamma(s) \) a point on the curve, the arrow into \( \gamma(s) \) the orientation of \( \gamma \), and the hollow dots and doublelined arrows the polar coordinates \( (x, \lambda(y - x)) \) and \( (y, \lambda(y - x)) \).

3. Winding Regions

In this section, the change of its winding number of a moving particle is studied. Then the definition of winding numbers is extended to points on the closed curve so that the winding regions defined as the equivalence classes of winding numbers partition the plane.

**Definition 3.1 (Signed crossings [1]).** Let \( \gamma \) be a closed curve and \( p(t) \) a particle moving continuously in the plane. Suppose \( p \) crosses \( \gamma \) at time \( t_c \). A point \( \gamma(s_c) = p(t_c) \) is called a positive crossing if, in a sufficiently small open neighborhood of \( s_c \), \( \gamma \) is oriented from the left of the directed pathline of \( p \) to its right, and a negative crossing otherwise.

Although the following two propositions seem to be well known, the proofs are included here because they are essential for proving the main conclusion in Theorem 4.9.
Proposition 3.2. Let $\gamma$ be an oriented closed curve. A positive crossing of a moving particle $p$ increases its winding number $w_{\gamma}(p(t))$ by one and a negative crossing decreases $w_{\gamma}(p(t))$ by one.

Proof. We only prove the case of a negative crossing since the other case is symmetric. Referring to Figure 4, let $o = p(t_0) \in \gamma$ be the crossing point and denote $x = p(t_0 - \tau)$, $y = p(t_0 + \tau)$. As $\tau \to 0$, the direction of the ray $x \to y$ converges to that of $u(o, t_0)$, which splits the plane into two halfplanes: $\theta_y(s) = \theta_x(s) + \delta_L$ in the left one and $\theta_y(s) = \theta_x(s) - \delta_R$ in the right one. Choose the basepoint of $\gamma$ as $o = \gamma(0)$, $\gamma(1) = o$, then $\lim_{s \to 0} \delta_L = \lim_{s \to 1} \delta_R = \pi$ for a single traversal of $\gamma$. Finally $w_{\gamma}(y) = w_{\gamma}(x) - 1$ follows from Definition 2.2, $\gamma$ being closed, and $o$ not being a self-intersection point.

Multiple crossings are incurred as $p$ crosses $\gamma$ at a self-intersection point. Then the local neighborhood of each preimage in the fiber of the self-intersection point determines its own sign separately. Consider for example a closed curve homeomorphic to the figure “8” and a particle going from one ring to the other through the self-intersection point: the signs of the two crossings should be determined separately.

Proposition 3.3. If a particle crosses an oriented closed curve a finite number of times as it moves continuously from $x$ to infinity, the winding number $w_{\gamma}(x)$ is equal to the number of negative crossings minus the number of positive crossings.

Proof. Use Proposition 3.2 as the inductive step with the induction basis as the fact that the winding number of the unbounded cell is zero.

While the above proposition is well known, the viewpoint in the following lemma is original in that it relates winding numbers with the cycle decomposition of a closed curve.

Lemma 3.4. If the cycle decomposition of an oriented closed curve $\gamma$ consists of a finite number of Jordan curves, then for each $x \not\in \gamma$, its winding number $w_{\gamma}(x)$ is the number of counterclockwise Jordan curves containing $x$ minus that of clockwise Jordan curves containing $x$.

Proof. By Proposition 2.5, the cycle decomposition of $\gamma$ contains the arcs of $\gamma$ with the same orientation. For a particle that moves along a ray from $x$ to infinity, the Jordan curve theorem dictates that it cross each Jordan curve containing $x$ an odd number of times, alternating positive and negative crossings. The rest follows directly from Proposition 3.3.

The ultimate goal of this paper is to show that the equivalence classes induced by winding numbers are the same as those induced by fluxing indices. For this purpose, the winding numbers of $\gamma$ around points on $\gamma$ need to be defined. After all, for each point on the plane, the number of times it goes across a given simple curve $LN$ is a well-defined integer$^3$. Also, a definition of winding

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$^3$The requirement of “proper intersection” in Definition 1.1 leaves no room of ambiguity for the fluxing index of any point in the plane.
numbers of \( \gamma \) around points on \( \gamma \) is formally necessary because (i) so far we only have winding numbers for points in \( \mathbb{R}^2 \setminus \gamma \), and (ii) the equivalence classes of winding numbers have to partition \( \mathbb{R}^2 \), not \( \mathbb{R}^2 \setminus \gamma \). Such a definition usually varies from applications to applications; the following one is analogous to the condition of proper intersection in Definition 1.1.

**Definition 3.5 (Winding number of \( \gamma \) around a point \( x \in \gamma \)).** Consider an oriented closed curve \( \gamma \) and a point \( x \in \gamma \). Let \( B_r(x) \) denote the open disk centered at \( x \) with radius \( r \). For sufficiently small \( r > 0 \), \( B_r(x) \setminus \gamma \) contains the same set of winding numbers. \( w_\gamma(x) \) is defined as the winding number in this set that is closest to zero\(^4\).

Proposition 3.2 and Assumption 2.1 imply that the winding numbers of adjacent cells differ by most one. Hence, the set of winding numbers at the local neighborhood of a point \( x \in \gamma \) has consecutive integers, which ensures that Definition 3.5 is well defined. For example, the winding number set cannot be \( \{-1, 1\}\).

**Definition 3.6 (Winding regions).** The winding region (WR) \( W_\gamma \) associated with an oriented closed curve \( \gamma \) is

\[
\begin{align*}
W_\gamma^n &:= \{ x \in \mathbb{R}^2 : w_\gamma(x) = n \}, \\
W_\gamma &:= \bigcup_{n \in \mathbb{Z} \setminus \{0\}} W_\gamma^n,
\end{align*}
\]

where \( W_\gamma^n \) is called the WR of index \( n \). The degree of a WR is defined as

\[
Z_{W_\gamma} := \max_{W_\gamma^n \neq \emptyset} |m|.
\]

**Definition 3.7.** The WRs of index at least \( n \) and at most \( n \) are, respectively,

\[
W_\gamma^\geq n := \bigcup_{i \geq n} W_\gamma^i, \quad W_\gamma^\leq n := \bigcup_{i \leq n} W_\gamma^i.
\]

In particular, the positive and negative WRs associated with \( \gamma \) are, respectively,

\[
W_\gamma^+ := W_\gamma^{\geq 1}, \quad W_\gamma^- := W_\gamma^{\leq -1}.
\]

\(^4\)Picking the number closest to zero is consistent with (and a consequence from) the requirement of proper intersection in Definition 1.1. Referring to Figure 5(a) or [13, Fig. 4.1] as the simple scenario of the DR being a simply connected region, all Lagrangian particles initially on the preimage of the curve reach the fixed simple curve at the end of the time interval, but none of them incurs any proper intersections [13, Def. 4.1]. Consequently, their fluxing indices should not have a bigger magnitude than those of the particles initially inside and outside the DR.
It follows directly from the above definitions that
\[ W^n = W^{\geq n}_\gamma \setminus W^{\geq n+1}_\gamma, \] (12)
the indexed set \( \{ W^{\geq n}_\gamma : n \in \mathbb{Z}^+ \} \) is a descending filtration and \( \{ W^{\leq n}_\gamma : n \in \mathbb{Z}^- \} \) a filtration, i.e.
\[ \forall n \in \mathbb{N}^+, \quad W^{\geq n+1}_\gamma \subseteq W^{\geq n}_\gamma, \quad W^{\leq n-1}_\gamma \subseteq W^{\leq n}_\gamma. \] (13)

**Proposition 3.8.** For each \( n \in \mathbb{N}^+ \), both \( W^{\geq n}_\gamma \) and \( W^{\leq n}_\gamma \) are open in \( \mathbb{R}^2 \).

**Proof.** The statement clearly holds if \( W^{\geq n}_\gamma = \emptyset \) or \( W^{\geq n}_\gamma = \mathbb{R}^2 \). Otherwise consider a point \( x \in \partial W^{\geq n}_\gamma \subseteq \gamma \). Let \( B_r(x) \) be an open disk centered at \( x \) with sufficiently small radius \( r \) such that neither the set of winding numbers of points inside \( B_r(x) \setminus \gamma \) nor the set of self-intersection points of \( \gamma \) inside \( B_r(x) \) changes when \( r \) is further reduced.

- If \( x \) is not a self-intersection point of \( \gamma \), then \( B_r(x) \setminus \gamma \) contains two components. Suppose both components have its winding number equal to or greater than \( n \), then Definition 3.5 implies that \( w_{\gamma}(y) \geq n \) for all \( y \in B_r(x) \cap \gamma \), which contradicts the starting point of \( x \in \partial W^{\geq n}_\gamma \). Hence at least one component of \( B_r(x) \setminus \gamma \) has its winding number less than \( n \). Then Definition 3.5 implies \( w_{\gamma}(x) < n \).

- If \( x \) is a self-intersection point of \( \gamma \), then the arguments of the previous case imply \( w_{\gamma}(y) < n \) for all \( y \in \gamma \cap (B_r(x) \setminus x) \). Then Definition 3.5 implies that at least one component of \( B_r(x) \setminus \gamma \) must have its winding number less than \( n \). Hence \( w_{\gamma}(x) < n \).

In summary, \( w_{\gamma}(x) < n \) for all \( x \in \partial W^{\geq n}_\gamma \), i.e. \( x \in \partial W^{\geq n}_\gamma \) implies \( x \not\in W^{\geq n}_\gamma \). Therefore, \( W^{\geq n}_\gamma \) is open.

Finally, the case of \( W^{\leq n}_\gamma \) can be proved in a similar fashion, thanks to the symmetry of the descending filtration to filtration in (13).

The above topology of WRs will be useful in proving Theorem 4.9 as it is a natural consequence of characterizing fluxing indices in Definition 3.5. It is also emphasized that, by (12) and (13), Proposition 3.8 does not hold for \( W^n_\gamma \), i.e. \( W^n_\gamma \) is in general not open.

4. **Generalized Donating Regions**

The notion of a DR originated as a geometric construction of the flux sets of index \( \pm 1 \), \( F^\pm_{LN}(t_0, k) = F^\pm_{LN}(t_0, k) \cup F^{-1}_{LN}(t_0, k) \), i.e. the set of Lagrangian particles that will go across \( LN \) once in the sense of net effect. In this section, we study a generalization of DR based on winding regions. A crucial building block is the backward streakline.
Definition 4.1 (Streaklines [13]). A backward streakline is the loci of all particles that will pass continuously through a fixed seeding location $M$,

$$
\Psi_{t_0}^{-k}(M) := \{ \phi_{t_0 - k + \tau}^{-\tau}(M) : \tau \in (0, k) \}, \quad (14)
$$

and a forward streakline is the loci of all particles that have passed $M$,

$$
\Psi_{t_0}^{+k}(M) := \{ \phi_{t_0 + k - \tau}^{+\tau}(M) : \tau \in (0, k) \}, \quad (15)
$$

where the time increment $k > 0$.

Backward streaklines and forward streaklines are distinguished by the sign of the superscript of $\Psi$. A streakline in the above definition is a snapshot: all particles in (14) are at $t = t_0 - k$ and those in (15) are at $t = t_0 + k$.

In comparison, a pathline is the history of a single Lagrangian particle. The definitions (2), (14), and (15) show that both pathlines and streaklines are indeed curves because they are continuous maps from intervals to points. See [13, Sec. 3] for more discussions on connections of the characteristic curves of an ODE.

The following relation between the forward and backward streaklines is a key observation.

Proposition 4.2. The image of a backward streakline $\Psi_{t_0+k}^{-k}(M)$ can be decomposed as

$$
\forall \tau \in (0, k), \quad \phi_{t_0}^{+\tau}\left(\Psi_{t_0}^{-k+k}(M)\right) = \Psi_{t_0+k}^{-(k-\tau)}(M) \cup M \cup \Psi_{t_0}^{+\tau}(M). \quad (16)
$$

Furthermore, $\Psi_{t_0}^{+\tau}(M) \cap \Psi_{t_0+k}^{-(k-\tau)}(M) = \emptyset$ if $\Psi_{t_0+k}^{-k}(M)$ is simple.

Proof. By Definition 4.1, $\Psi_{t_0+k}^{-k}(M)$ contains the initial positions of all particles that will pass $M$ within $(t_0, t_0+k)$. At $t = t_0 + \tau$, these particles can be classified into three types: those that will pass $M$ within $(t_0 + \tau, t_0 + k)$, the one that coincides with $M$, and those that have passed $M$ within $(t_0, t_0 + \tau)$; they are respectively the three terms on the right-hand side of (16). $\Psi_{t_0+k}^{-k}(M) \cap \Psi_{t_0+k}^{-(k-\tau)}(M) = \emptyset$ follows from the curve being not self-intersecting and the flow map being a homeomorphism.

In particular, the limit $\tau \to k$ yields

$$
\phi_{t_0}^{+k}\left(\Psi_{t_0+k}^{-k}(M)\right) = \Psi_{t_0+k}^{+k}(M), \quad (17)
$$

which is a special case of (16) as the two streaklines are the loci of the same set of particles at two different time instants.

The generalization of the notion of DRs [13, Def. 4.5] is based on WRs.

---

5Proposition 4.2 generalizes equation (3.7) in [13]; the latter contains a typo: it should be exactly the same as (17).
Definition 4.3 (Generalized donating regions). For a given velocity field \( \mathbf{u}(x,t) \) that is continuous in time and Lipschitz continuous in space, the generalized donating region (GDR) associated with a simple open curve \( \tilde{LN} \) over the time interval \( (t_0, t_0 + k) \) is the WR of a closed curve \( \gamma_D \),

\[
\left\{ \begin{align*}
\gamma_D &:= L \cup \tilde{L}N \cup N \cup \Psi_{t_0+k}^{-k}(N) \cup \tilde{N} \cup \phi_{t_0+k}^{-k} \tilde{N}L \cup \tilde{L} \cup \Psi_{t_0+k}^{-k}(L), \\
D_{\tilde{L}N}(t_0,k) &:= W_{\gamma_D},
\end{align*} \right.
\]

(18)

where \( \gamma_D \) is called the generating curve of the GDR and is oriented by the closed vertex sequence \( L \to N \to \tilde{N} \to \tilde{L} \to L \).

In the notation \( D^t_{\tilde{L}N}(t_0,k) \), \( t_0 \) is the time instance when the Lagrangian particles are identified with the positions inside the GDR, \( k \) is the time increment so that the GDR reduces to an empty set at \( t = t_0 + k \). See Figure 5 for three examples.

Since a GDR is always a WR, it inherits the various properties of a WR. For example, the GDRs with various indices and the degree of a GDR are as follows.

\[
\left\{ \begin{align*}
D^t_{\tilde{L}N}(t_0,k) &:= W^{t}_{\gamma_D}, & D^\geq_{\tilde{L}N}(t_0,k) &:= W^{\geq}_{\gamma_D}; \\
D^+_{\tilde{L}N}(t_0,k) &:= W^+_{\gamma_D}, & D^-_{\tilde{L}N}(t_0,k) &:= W^-_{\gamma_D}; \\
Z_{D_{\tilde{L}N}} &:= Z_{W_{\gamma_D}}.
\end{align*} \right.
\]

(19)

GDRs are more general than the original DRs defined as curvilinear polygons [13, Def. 4.5] since the interior of a curvilinear polygon might not be clear when its boundary contains multiple self-intersection points. Indeed, the concept of a polygon is historically imprecise [3] and invites ambiguity when the defining edges have complicated self-intersection.

The GDR of a Jordan curve can be deduced from Definition 4.3.
Figure 6: Examples of GDRs of simple closed curves. As $L \rightarrow N$, the shaded region in subplot (a) approaches an empty set. $C$ and $\phi_{t_0}^{-k}(C)$ clearly have different orientations induced from their boundaries. In subplot (b), the dotted circles with arrows indicate the induced orientations of sub-GDRs from the generating curve of the GDR.

**Proposition 4.4 (GDR of a Jordan curve).** Denote by $\partial C$ a Jordan curve and by $C$ its bounded complement. Then

\[
Z_{D_{\partial C}} = 1; \quad (20)
\]

\[
\begin{align*}
\partial C \text{ has counterclockwise orientation} & \Rightarrow \\
& \begin{cases} \\
D^+_{\partial C}(t_0, k) = C \setminus \phi_{t_0}^{-k}(C), \\
D^-_{\partial C}(t_0, k) = \phi_{t_0}^{-k}(C) \setminus C;
\end{cases} \\
\partial C \text{ has clockwise orientation} & \Rightarrow \\
& \begin{cases} \\
D^+_{\partial C}(t_0, k) = \phi_{t_0}^{-k}(C) \setminus C, \\
D^-_{\partial C}(t_0, k) = C \setminus \phi_{t_0}^{-k}(C).
\end{cases} \quad (21) \quad (22)
\]

**Proof.** As shown in Figure 6, the Jordan curve $C$ is viewed as a simple open curve $LN$ in the limiting case of $L \rightarrow N$. By Definition 4.3, the generating curve $\gamma_D$ of $D_{\partial C}$ is oriented by the vertex sequence $L \rightarrow N \rightarrow \overline{N} \rightarrow \overline{L} \rightarrow L$. The two segments $L \rightarrow N$ and $N \rightarrow \overline{L}$ imply that the cycles $\partial C$ and $\phi_{t_0}^{-k}(\partial C)$ are traversed with different orientations within $\gamma_D$.

As $L \rightarrow N$, the cycle decomposition of $\gamma_D$ consists of degenerate cycles from the backward streaklines seeded at $L$ and $N$. For example, a single degenerate cycle $N \rightarrow \overline{N} \rightarrow \overline{L} \rightarrow L$ exists for the case of Figure 6(b) while two degenerate
cycles exist in Figure 6(a). These degenerate cycles can be removed from \(\gamma_D\) without changing winding numbers of any points. Therefore \(\gamma_D\) is regarded as the union of \(\partial C\) and \(\phi_{-k}^{-t_0}(\partial C)\) for the rest of this proof. Consequently, a cell of \(\gamma_D\) must belong to one of the following four types:

(CT1) \( C \cap \phi_{-k}^{-t_0}(C) \),
(CT2) \( C \setminus \phi_{-k}^{-t_0}(C) \),
(CT3) \( \phi_{-k}^{-t_0}(C) \setminus C \),
(CT4) none of the above.

Clearly (CT4) is the unbounded cell and its winding number is zero. By the first paragraph of this proof, \(\partial C\) and \(\phi_{-k}^{-t_0}(\partial C)\) are traversed with different orientations within \(\gamma_D\); (CT1) is similar to the case shown in Figure 3(a). For any point inside a (CT1)-type cell to approach infinity, the Jordan curve theorem dictates that it must go across both \(\partial C\) and \(\phi_{-k}^{-t_0}(\partial C)\) an odd number of times, and that each pair of consecutive crossings must have different signs, cf. Definition 3.1. By Lemma 3.4, the winding number of such a point must be zero. (\(\gamma_D\) satisfies Assumption 2.1 after removing the degenerate cycles and hence Lemma 3.4 does apply here.) Similarly, for a point inside a cell of type (CT2) to approach infinity, the number of times it must go across \(\partial C\) and \(\phi_{-k}^{-t_0}(\partial C)\) is odd and even, respectively. By Lemma 3.4, its winding number is either +1 or −1 depending on the orientation of \(\partial C\). The same argument applies to (CT3).

Then (20) follows from (9).

Finally, (21) and (22) follow from (20), (11), and the arguments in the previous paragraph.

Definition 4.5. A GDR \(D_{\tilde{L}N}(t_0, k)\) is said to be normal if the backward streaklines \(\Psi_{-k}^{-t_0+k}(L)\) and \(\Psi_{-k}^{-t_0+k}(N)\) neither self-intersect nor intersect each other.

In addition, a GDR is canonical if its generating curve is Jordan; it is rotational if its generating curve \(\gamma_{\tilde{L}N}\) has exactly one self-intersection point that comes from \(L = \tilde{N}\), or \(N = \tilde{L}\), or \(L \in \Psi_{-t_0+k}(N)\), or \(N \in \Psi_{-t_0+k}(L)\). A GDR always satisfies \(\partial D \subseteq \gamma_D\); in particular, \(\partial D = \gamma_D\) holds for a canonical GDR. A rotational GDR can be decomposed into canonical GDRs [13, Lemma 4.10].

The concrete cases of the topology of a GDR are enumerated in the proof of the following proposition. For example, case (i) is the GDR of a Jordan curve and case (iii) leads to a rotational GDR.

Proposition 4.6 (Normality of GDRs). For sufficiently short time intervals, GDRs are normal and have degree 1.

Proof. Consider self-intersection points of \(\gamma_D\). By (18), \(\gamma_D\) consists of eight parts: four points as the vertices and four open curves as the edges. By [13, Lemma 3.3], a streakline does not intersect itself if the time interval k is small enough. Because \(\tilde{L}N\) is simple and any flow map with fixed \(t_0\) and \(\tau\) is a homeomorphism, \(\phi_{-t_0+\tau}(\tilde{L}N)\) is also simple. Therefore a self-intersection point
of $\gamma_D$ must come from two distinct parts of $\gamma_D$. The $\binom{8}{2} = 28$ possibilities are grouped into the following ten cases,

(i) $L = N$ and $\widehat{L} = \widehat{N}$,

(ii) $L = \widehat{L}$ or $N = \widehat{N}$,

(iii) $L = \widehat{N}$ or $N = \widehat{L}$,

(iv) the two streaklines intersect each other,

(v) a streakline intersects $\widehat{LN}$ or its preimage (4 possibilities),

(vi) $\widehat{LN} \cap \phi^{-\tau}_{t_0+\tau} (\widehat{LN}) \neq \emptyset$,

(vii) a vertex is on a streakline (8 possibilities),

(viii) $L \in \widehat{LN}$ or $N \in \widehat{LN}$,

(ix) $\widehat{L} \in \phi^{-\tau}_{t_0+\tau} (\widehat{LN})$ or $\widehat{N} \in \phi^{-\tau}_{t_0+\tau} (\widehat{LN})$,

(x) $L \in \phi^{-\tau}_{t_0+\tau} (\widehat{LN})$ or $N \in \phi^{-\tau}_{t_0+\tau} (\widehat{LN})$ or $\widehat{L} \in \widehat{LN}$ or $\widehat{N} \in \widehat{LN}$,

where (i)–(iii) are the pairwise enumeration of the vertices, (iv)–(vi) that of the edges, and (vii)–(ix) that of a vertex and an edge. By [13, Corollary 3.5], (iv) can be avoided by reducing $k$. Hence $D_{\widehat{LN}}(t_0, \tau)$ is normal.

Case (i) is proved in Proposition 4.4.

For case (ii), either $L = \widehat{L}$ implies $L$ being a fixed point, i.e. $\Psi^{-\tau}_{t_0+\tau} = \emptyset$, or, $L = \widehat{L}$ can be avoided by reducing $k$.

For case (iii), either $L = \widehat{N}$ implies $N$ being a fixed point and $L = N$, or, $L = \widehat{N}$ can be avoided by reducing $k$.

Cases (v) and (vii) involve self-intersection points of a streakline and can also be avoided by setting $k$ to less than the smallest time increment of the intersections.

Cases (viii) and (ix) must occur together; they are combinations of case (i) and other cases.

For case (x), the four possibilities are symmetric and it suffices to only consider $L \in \phi^{-\tau}_{t_0+\tau} (\widehat{LN})$. If the streakline of $L$ coincides with part of $\phi^{-\tau}_{t_0+\tau} (\widehat{LN})$, the shaded region $D_{\widehat{LN}}$. 

Figure 7: A GDR with degree $-2$. The solid line represents the simple curve $\widehat{LN}$, the dashed line its preimage, the dotted lines the streaklines, the arrows the orientation of the generating curve $\gamma_D$, and the shaded region $D_{\widehat{LN}}$. 

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then they form a whisker\(^6\) of \(\gamma_D\) and thus can be removed from it. Otherwise a point \(p \in \phi_{t_0 + \tau}^{-1} (\tilde{L} \tilde{N})\) between \(L\) and \(\tilde{L}\) will not pass through \(L\) since it is not on the streakline of \(L\). Reducing \(k\) then removes this case as instantaneous.

Case (vi) is the only one left. Now suppose for each \(k > 0\) there exists \(\tau \in (0, k)\) such that there exists \(\tilde{D}_{\tilde{L}N}^\circ(t_0, \tau) \neq \emptyset\) with \(|n| > 1\). Then \(\gamma_D\) must contain a self-intersection point as shown in Figure 7. Maintaining the topology shown there as \(k \to 0\) would contradict the fact that \(\phi_{t_0 + \tau}^{-1} (\tilde{L} \tilde{N})\) is always simple.

In the original DR theorem [13, Thm. 4.11], the condition \(\tilde{L} \tilde{N} \subset \partial \tilde{D}_{\tilde{L}N}^\circ(t_0, \tau)\) is imposed. From the above proof, this condition clearly guarantees the degree of the GDR not exceeding 1 in the asymptotic range; see Figure 5(c) for an example. However, parts of \(\tilde{L} \tilde{N}\) and a streakline might form a whisker as in case (x), hence the condition \(\tilde{L} \tilde{N} \subset \partial \tilde{D}_{\tilde{L}N}^\circ(t_0, \tau)\) is a sufficient but not necessary condition for \(\tilde{D}_{\tilde{L}N}\) to have degree 1. Proposition 4.6 also connects DR with GDR: in the asymptotic range of the time interval being small enough, a DR is always a normal GDR with degree 1.

**Definition 4.7.** The unit normal vector of a simple open curve \(\tilde{L} \tilde{N}\) induced by a nonempty GDR \(\tilde{D}_{\tilde{L}N}\) is the normal vector that agrees with the outward unit normal of \(\tilde{D}_{\tilde{L}N}^+\), or, in the case of \(\tilde{D}_{\tilde{L}N}^+ = \emptyset\), the one that disagrees with the outward unit normal of \(\tilde{D}_{\tilde{L}N}^-\).

By velocity continuity, either \(\tilde{D}_{\tilde{L}N}^+\) or \(\tilde{D}_{\tilde{L}N}^-\) must contain a finite-length arc of \(\tilde{L} \tilde{N}\), which ensures that Definition 4.7 is well defined.

**Proposition 4.8.** The set of loops \(\{\gamma_{\tilde{D}_{\tilde{L}N}^\circ(t_0 + \tau, k-r)} : \tau \in [0, k]\}\) is a homotopy class in \(\mathbb{R}^2\).

**Proof.** Define \(\zeta_{\tilde{L}N} := L \cup \tilde{L} \cup N\) and

\[
\zeta_r := \left( L \cup \psi_{t_0+k}^{-1}(r) (L) \cup \tilde{L} \right) \cdot \left( L \cup \phi_{t_0+k}^{-1}(r) (\tilde{L} \tilde{N}) \cup \tilde{N} \right) \cdot \left( \tilde{N} \cup \psi_{t_0+k}^{-1}(r) (N) \cup N \right).
\]

By Proposition 4.2, the set of backward streaklines \(\{\psi_{t_0+k}^{-1}(r) (L) : r \in [0, 1]\}\) are the loci of a continuously diminishing set of particles that will pass \(L\) during the continuously shortened time intervals \(\{[t_0 + rk, t_0 + k] : r \in [0, 1]\}\). The set of the preimages of \(\tilde{L} \tilde{N}\), \(\{\phi_{t_0+k}^{-1}(r) (\tilde{L} \tilde{N}) : r \in [0, 1]\}\), is also continuous with respect to \(r\). Hence the map \(H(s, r) = \zeta_r(s)\) is continuous and \([\zeta_r] \) is a homotopy class. The identity map \(I(s, r) = \zeta_{\tilde{L}N}(s)\) is clearly continuous and \([\zeta_{\tilde{L}N}] \) is also a homotopy class. By Definition 4.3, their composition paths \(\zeta_r \cdot \zeta_{\tilde{L}N}\) are the generating curves of a GDR and form a homotopy class of loops.

---

\(^{6}\)As the image of a single interval under the map of the curve \(\gamma\), a whisker is a special type of 1D self-intersection where each self-intersection point has degree 2. This term is borrowed from Grunbaum and Shephard [4], who originally used it for linear polygons.
Figure 8: Constructive substeps for the i-th interval \([t_i, t_i + \tau]\) in proving Theorem 4.9. The thick solid line represents \(LN\), the hollow dots the particle \(p\), the dashed curve in subplot (a) \(\partial D_{LN}^{z_i}(t_i, k + t_0 - t_i) \setminus LN\), the thin solid curves \(\partial D_{LN}^{z_i}(t_i + 1, k + t_0 - t_{i+1}) \setminus LN\), and the dotted curves with arrows in subplots (a) and (b) the velocity field and the pathline \(\Phi_{\tau_i}(p)\), respectively. In the first substep, the boundary of the GDR \(\partial D_{LN}^{z_i}(t_i, k + t_0 - t_i)\) is deformed to \(\partial D_{LN}^{z_i}(t_i + 1, k + t_0 - t_{i+1})\) without crossing the fixed \(p(t_i)\). By Proposition 4.8 and Theorem 2.4, the winding number of the generating curve around \(p(t_i)\) remains the same during its deformation. In the second substep, the particle \(p\) is advected along its pathline to \(p(t_i + 1)\) with the GDR fixed. By the uniqueness of the ODE solution and the GDR being normal, a crossing of the generating curve, if there is any, must be through \(LN\).

By the discussion in Section 2.3, Proposition 4.8 would not hold if the underlying space \(\mathbb{R}^2\) were replaced by \(\mathbb{R}^2 \setminus \{x\}\) where \(x \in D_{LN}(t_0, k)\).

**Theorem 4.9 (GDRs in two dimensions).** A normal GDR of a simple open curve \(LN\) is index-by-index equivalent to the flux set of \(LN\),

\[
\forall n \in \mathbb{Z}, \quad D_{LN}^{n}(t_0, k) = F_{LN}^{n}(t_0, k),
\]

(23)

if \(n_{LN}\), the unit normal vector of \(LN\) induced by \(D_{LN}(t_0, k)\) as in Definition 4.7, points from the bounded complement of \(\hat{\gamma}\) to the unbounded complement of \(\hat{\gamma}\), where \(\hat{\gamma}\) is the Jordan curve in Definition 1.3 for determining the sign of fluxing indices.

As for the other case of \(n_{LN}\), \(D_{LN}^{n}(t_0, k) = F_{LN}^{-n}(t_0, k), \forall n \in \mathbb{Z}\).

**Proof.** The strategy of the proof is as follows. Let

\[
z^+ = \max_{D_{LN}^{m} \neq \emptyset} m; \quad z^- = \min_{D_{LN}^{m} \neq \emptyset} m,
\]

(24)

we first prove

\[
\forall n^+ = 1, 2, \ldots, z^+, \quad D_{LN}^{\geq n^+} = F_{LN}^{\geq n^+}.
\]

(25)

In particular, \(D_{LN}^{\geq z^+} = D_{LN}^{z^+}\) and \(F_{LN}^{\geq (z^+ + 1)} = \emptyset\). Then (12), (13), and (25) yield (23) for each \(n^+\). Consider a Lagrangian particle \(p(t_0) \in D_{LN}^{\geq z^+}\). To prove (25),
the entire time interval is split into a countable number of discrete steps,

\[ [t_0, t_0 + k] = [t_0, t_0 + \tau_0] \cup \cdots \cup [t_i, t_i + \tau_i] \cup \cdots \cup [t_0 + k - \tau_m, t_0 + k],\]

where each step consists of two substeps; see Figure 8. In the first substep, we deform the generating curve of the GDR without crossing the fixed \( p \), hence keeping \( w(\gamma_D, p) \) unchanged. In the second substep, we advect \( p \) along its pathline and argue that any crossing of \( \gamma_D \) must be through \( LN \). By setting \( \max_i \tau_i \to 0 \), the above discrete steps approximate the continuous deformation of the GDR arbitrarily well. During the entire time interval, the initially nonzero winding number of \( \gamma_D \) around \( p \) changes to 0 at the final time; this must be caused by the fluxes of \( p \) through \( LN \) in the second substeps. This sequence of deformation-advection substeps is the main idea of the proof.

The rest of the proof contains the details of the aforementioned sketch. Consider a point \( x \in D_{\tilde{L}N}^{\geq n+} \) for some \( n^+ = 1, 2, \ldots, z^+ \). Definition 3.6 implies \( w_0 := w(\gamma_{D(t_0,k)}, x) \geq n^+ \). Label a Lagrangian particle as \( p(t_0) = x \). Proposition 3.8 states that \( D_{\tilde{L}N}^{\geq n} \) is open. Hence in the first deformation substep, there exists a time increment \( \tau_0^{\max} > 0 \) such that, for all \( \tau_0 \in (0, \tau_0^{\max}) \), \( \partial D_{\tilde{L}N}^{\geq n} \) does not sweep across \( p(t_0) \) as it deforms to \( \phi_0^{\tau_0} \left( \partial D_{\tilde{L}N}^{\geq n} \right) \), which is a subset of \( \phi_0^{\tau_0} (\gamma_{D(t_0,k)}) \). By Proposition 4.8, \( \gamma_{D(t_0,k)} \) and \( \gamma_{D(t_0+\tau_0,k-\tau_0)} \) are homotopic in \( R^2 \), hence they are freely homotopic in \( R^2 \). Furthermore, \( \gamma_{D(t_0,k)} \) and \( \gamma_{D(t_0+\tau_0,k-\tau_0)} \) are freely homotopic in \( R^2 \setminus \{p(t_0)\} \) because \( \tau_0 \) is chosen so that the deformation of \( \partial D_{\tilde{L}N}^{\geq n} \) to \( \phi_0^{\tau_0} \left( \partial D_{\tilde{L}N}^{\geq n} \right) \) does not sweep across \( p(t_0) \); see Figure 2. It then follows from Theorem 2.4 that \( w(\gamma_{D(t_0+\tau_0,k-\tau_0)}, p(t_0)) = w_0 \). The above arguments are illustrated in Figure 8 (a).

In the second advection substep, \( \gamma_{D(t_0+\tau_0,k-\tau_0)} \) is fixed while \( p \) is being advected from \( t_0 + \tau_0 \) to \( t_1 = t_0 + \tau_0 + \tau_1 \) along its pathline: \( p(t_1) = \phi_0^{\tau_0+\tau_1}(p) \); see Figure 8(b). If the GDR \( D_{\tilde{L}N}(t_0, k) \) is normal, then any GDR \( D_{\tilde{L}N}(t_0, k') \) with \( k' \in [0, k] \) is normal, otherwise it would contradict Proposition 4.2. The particle may or may not cross \( \gamma_{D(t_0+\tau_0,k-\tau_0)} \) within this advection substep. If it does, it cannot cross any streaklines because of the normality of the GDR. In addition, \( p \) cannot cross the preimage of \( LN \) because of the uniqueness of the ODE solutions. Therefore, the crossing of \( p \) through the generating curve, if there is any, must be a flux through \( LN \).

The above two substeps are repeated for \( i = 1, 2, \ldots, m \) until \( t_m + \tau_m = t_0 + k \). Setting \( \max_i \tau_i \to 0 \) yields a sequence of substeps that approximates the continuous process arbitrarily well.

At the final time, \( w_k := w(\gamma_{D(t_0+k,k-k)}, p(t_0 + k)) = 0 \) because \( \gamma_{D(t_0+k,k-k)} \) reduces to a degenerate Jordan curve with its bounded complement as an empty set. The change from \( w_0 \geq n^+ > 0 \) to \( w_k = 0 \) is not caused by deforming the

\[ \text{The openness of } D_{\tilde{L}N}^{\geq n} \text{ guarantees the existence, hence we work on } D_{\tilde{L}N}^{\geq n} \text{ instead of } D_{\tilde{L}N}^{n-} \text{ as the latter might not be open. This also explains the importance of Definition 3.5.} \]
generating curve in any of the first substeps because \( \tau_i \)'s are explicitly chosen to avoid so. Hence the change of the winding number must occur in the second substeps of advecting \( p \) along its pathline. Note that \( p(t_j) \in LN \) might be true at the end of an intermediate step; in this case \( p \) crosses \( \bar{LN} \) within \([t_{j-1}, t_{j+1}]\). Each crossing of \( p \) through \( \bar{LN} \) could be either a positive crossing or negative crossing, and the total number of crossings might be much bigger than \( w_0 \). However, the overall effect of these crossings must be such that the fluxing index of \( p \) as in Definition 1.2 equal \( w_0 \), cf. Proposition 3.3 and the choice of the sign-determining Jordan curve \( \gamma \). To sum up, we have shown that

\[
\forall n^+ = 1, 2, \ldots, z^+, \quad p(t_0) \in \mathcal{D}^n_{\bar{LN}} \Rightarrow p(t_0) \in \mathcal{F}^n_{\bar{LN}}. \quad (26)
\]

Consider \( n^+ = z^+ \). Suppose the fluxing index of such a particle \( p(t_0) \in \mathcal{D}^{z^+}_{\bar{LN}} \) is greater than \( z^+ \). Then the above arguments in this paragraph and Proposition 3.3 imply \( w_0 > z^+ \), which contradicts (24). Hence the maximum fluxing index must also be \( z^+ \). Then (12), (13), and the identity \( \mathcal{D}^z_{\bar{LN}} = \mathcal{D}^{2z^+}_{\bar{LN}} \) allow us to strengthen (26) as

\[
\forall n^+ = 1, 2, \ldots, z^+, \quad p(t_0) \in \mathcal{D}^n_{\bar{LN}} \Rightarrow p(t_0) \in \mathcal{F}^n_{\bar{LN}}. \quad (27)
\]

Now consider \( p(t_0) \in \mathcal{F}^{n^+}_{\bar{LN}} \). An out-flux of \( p \) through \( \bar{LN} \) must be a negative crossing of the generating curve and an in-flux a positive crossing, due to Definition 1.2, Definition 3.1, the definition of \( \mathcal{D}^+ \) in (19), Definition 4.7, and the specified choice of the sign-determining Jordan curve \( \gamma \). By invoking the deformation-advection substeps and applying Proposition 3.2, we have \( w_0 = n^+ \). Hence \( p(t_0) \in \mathcal{F}^{n^+}_{\bar{LN}} \Rightarrow p(t_0) \in \mathcal{D}^{n^+}_{\bar{LN}} \); (23) then follows from (27) for each \( n^+ \).

Thanks to the symmetry of the descending filtration to the filtration in (13), this construction can be repeated in a similar fashion to prove (23) for each \( n^- \). The differences are that all the sub-GDRs are now oriented clockwise and that the number of positive crossings are bigger than that of negative crossings.

Finally, the last sentence in the statement of the theorem can be shown by simply switching the choice of the sign-determining Jordan curve \( \gamma \) to the other case for \( \mathcal{F}^{n}_{\bar{LN}} \)'s, which changes out-fluxes to in-fluxes, although the sub-GDRs \( \mathcal{D}^+ \) and \( \mathcal{D}^- \) remain the same. More precisely, the fluxing index of a particle \( p(t_0) \in \mathcal{D}^{n^+}_{\bar{LN}} \) is changed to \(-n^+ \). This shows the last sentence of the statement and completes the proof of Theorem 4.9.

The above theorem is more general than the original DR theorem [13, Thm. 4.11]: the condition \( LN \subset \partial \mathcal{D}^{\infty}_{\bar{LN}}(t_0, \tau) \) is removed and, more importantly, the equivalence of normal GDRs and flux sets now holds not only for indices \( \pm 1 \), but also for an arbitrary integer. By Proposition 4.6, the assumption of the GDR being normal can be satisfied by choosing \( k \) sufficiently small.
5. Conclusion

Based on the equivalence relations induced by winding numbers of closed curves, the author proposes a generalization of the original DRs in [13, Def. 4.5]. As the main conclusion, Theorem 4.9 states that normal GDRs of a fixed simple open curve are index-by-index equivalent to the corresponding flux sets of the same curve.

The immediate next step along this line of research is to prove the flux identity (4) and to generalize it to the case of GDRs. For the latter, the author conjectures that the right-hand side of (4) be scaled by the indices of the sub-GDRs. Another future research prospect is the extension of the main results to three and higher dimensions. Finally, the author speculates that the condition of the GDR being normal in Theorem 4.9 may be relaxed for certain types of intersections of streaklines. Future research coupling GDRs to the classification of critical points might shed light on this issue.

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Reference


