A FLUX IDENTITY OF SCALAR CONSERVATION LAWS FOR
LAGRANGIAN FLUX CALCULATION VIA DONATING REGIONS

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Abstract. For a simple planar curve in a nonautonomous flow, the fluxing index of a passively advected Lagrangian particle is the net number of its crossing the curve within a given time interval. Such indices naturally induce flux sets, equivalence classes of the particles at the initial time. Previously, we proposed donating regions as an explicit geometric construction of flux sets. In this work, we strengthen our earlier results by removing an assumption on the equivalence of flux sets and donating regions. More importantly, we propose and prove for scalar conservation laws a flux identity, which establishes the equivalence of the traditional Eulerian flux integral (a double integral in time as well as in space) to a Lagrangian flux integral at the initial time (a spatial integral independent of time). Thus the evolution of the scalar function is no longer needed in expressing its total flux through the curve over any time interval of finite length. To numerically exploit this identity, we also propose a simple algorithm of Lagrangian flux calculation, which is demonstrated by results of a variety of numerical tests to be highly accurate and highly efficient.

Key words. scalar conservation law; finite volume methods; donating regions; Hopf theorem; Lagrangian flux calculation; Reynolds transport theorem;

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1. Introduction. Given a fixed simple planar curve \( \tilde{LN} \) and a time-dependent vector field \( w : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2 \), the total flux of the flux density vector \( w(x, t) \) through \( \tilde{LN} \) within a time interval \([t_0, t_0 + k]\) can be expressed as

\[
\int_{t_0}^{t_0+k} \int_{\tilde{LN}} w(\gamma(s), t) \cdot \mathbf{n}_{\tilde{LN}} \, ds \, dt,
\]

where \( \gamma(s) \) and \( \mathbf{n}_{\tilde{LN}} \) are a parametrization and the normal vector of \( \tilde{LN} \), respectively. We also refer to the total flux in (1.1) as the Eulerian flux integral of \( w \) through the curve \( \tilde{LN} \).

The notion of fluxes is fundamental and ubiquitous in transport, mixing, and other physical processes, with the form of the flux density vector \( w(x, t) \) varying from applications to applications. In the context of hyperbolic conservation laws, we often write \( w(x, t) \) as the product of another vector \( u(x, t) \) and an integrable scalar function \( f(x, t) \), with \( u \) considered as a velocity field and \( f \) the scalar quantity being conserved. If the velocity \( u \) is continuous in time and Lipschitz continuous in space, it gives rise to a flow that conserves the integral of \( f \) over a moving region over time. As explained in Section 2.5, if \( f \) is further differentiable, we arrive at the scalar conservation law

\[
\frac{\partial f}{\partial t} + \nabla \cdot (fu) = 0.
\]

Flux integrals are also important in numerically solving hyperbolic conservation laws. In the finite volume (FV) methods, flux calculation is of paramount significance in ensuring stability and accuracy [9]. In using volume-of-fluid (VOF) methods to track free boundaries, the accuracy of the advection substep is solely determined...
by the algorithm of calculating fluxes through cell edges [24]. In traditional FV methods for solving (1.2), the curve $\tilde{LN}$ is identified with boundary edges of control volumes (or cells) and the flux identity vector $w$ with the vector $f\mathbf{u}$. Then Eulerian fluxes through cell edges within the computational domain are discretized as linear combinations of cell-averaged values of $f$; this spatial discretization process leads to a system of ordinary differential equations (ODEs). Lastly, a time-integration method is employed to solve the ODE system.

The aforementioned FV methods with method-of-lines (MOL) discretizations have been very successful in the past several decades [13, 14]. However, they are by no means perfect for all circumstances. For example, when the scalar $f$ is discontinuous in space (yet still integrable), it is difficult to derive the convergence rate from traditional FV analysis. Even if $f$ is indeed smooth, the time-step size is often subject to constraints of the well-known Courant-Friedrichs-Lewy (CFL) condition, which may be too restrictive when any component of the velocity has a large magnitude.

To analytically and numerically complement the FV-MOL approach, it is desirable to find a so-called donating region (DR) $D_{\tilde{LN}}(t_0, k)$ at the initial time $t_0$ such that the scalar conservation law (1.2) implies the following flux identity

$$\exists k > 0 \text{ s.t. } \int_{t_0}^{t_0+k} \int_{\tilde{LN}} f(s, t)\mathbf{u}(s, t) \cdot \mathbf{n}_{\tilde{LN}} \, ds \, dt = \int_{D_{\tilde{LN}}(t_0, k)} f(x, t_0) \, dx,$$

where $\tilde{LN}$ is a fixed simple curve, $\mathbf{u}$ is the velocity field, and $D_{\tilde{LN}}(t_0, k)$ is a point set at the initial time $t_0$ consisting of passively advected Lagrangian particles that will go across $LN$ once within the time interval $[t_0, t_0 + k]$ and contribute to the total flux. The right-hand side (RHS) of (1.3) is called a Lagrangian flux integral, as opposed to the Eulerian flux integral in the left-hand side (LHS) of (1.3). We list a number of prominent utilities of the flux identity (1.3) as follows.

(UFI-1) The time dependence of the Eulerian flux integral is removed so that the spatial-temporal integral reduces to a spatial integral at the initial time.

(UFI-2) The exact relation (1.3) is useful in determining local truncation errors of unsplit multidimensional FV algorithms, even if $f$ is discontinuous in space; see [24] for an example in analyzing a wide spectrum of VOF advection methods with the help of (1.3).

(UFI-3) Sometimes we are only interested in the total flux across a local curve, and it would be inefficient to solve for the evolution of the scalar over the entire computational domain. Instead, we construct the DR $D_{\tilde{LN}}$ from the given velocity $\mathbf{u}(x, t)$ and then calculate the Lagrangian flux integral by numerical quadrature over $D_{\tilde{LN}}$.

(UFI-4) The numerical quadrature in (UFI-3) can also be easily adapted to solve the scalar conservation law (1.2), with the advantage that the time-step size is not subject to the CFL condition. In addition, (1.2) can be solved for a single control volume instead of the entire computational domain. Hence this Lagrangian FV approach is more flexible than the traditional Eulerian FV-MOL approach; see [23] for more discussions.

(UFI-5) Both the DR construction and the numerical quadrature on it are amenable to parallel computing as they can be divided into atomic tasks that are highly independent; see Section 4 for the exposition and discussion on the proposed algorithm of Lagrangian flux calculation (LFC).

The aforementioned DR has been determined in our previous work [25, 26] as follows. First, for any particle labeled at the initial time $t_0$ and advected passively
in the flow, the number of its crossing \(LN\) within a time interval is an integer, which we call the fluxing index of that particle. Second, the fluxing index induces flux sets, equivalence classes of the particles at \(t_0\) in the plane. Third, an algebraically closed curve \(\gamma_p\) can be constructed from \(LN\), its preimage, and streaklines seeded at \(L\) and \(N\). The winding numbers of \(\gamma_p\) determine another family of equivalence classes at \(t_0\), which we designate as the desired DRs. Finally, we showed that DRs are index-by-index equivalent to fluxing sets. See Section 3.3 for precise statements of the above brief summary.

The flux identity (1.3) has been numerically verified in [23] for incompressible flows. As another intuitive justification, the divergence-free condition and (1.3) imply the zero constancy of the material derivative with respect to a passively advected Lagrangian particle \(p\), i.e., \(f(p(t)) \equiv f(p(t_0))\) holds within the whole time interval. However, the restriction of the velocity field to be divergence-free appears unnecessary, and we speculate that (1.3) holds so long as (1.2) holds, regardless of the velocity being solenoidal or not. It is also clear from the above discussion that, for a sufficiently large time increment \(k\), certain fluid parcels may cross the curve \(LN\) more than once and contribute to the total flux multiple times. These observations motivate the following flux identity that is more general than (1.3),

\[
\forall k > 0, \quad \int_{t_0}^{t_0+k} \int_{LN} f \cdot n_{LN} \, ds \, dt = \sum_{n \in \mathbb{Z}} n \int_{D_n^{LN}(t_0,k)} f(x, t_0) \, dx,
\]

where the integer \(n\) in the superscript denotes the fluxing index.

For incompressible flows, the flux identity (1.4) has been proved by Karrasch [7] via differential topology and the area formula. This approach is quite different from ours and may also work for compressible flows [5]. While Karrasch’s approach [7, 5] gives valuable insights on the theoretical side at a high level of abstraction, it is short of details on the topology and geometry of DRs. Yet these details, in our opinion, are essential in the design of efficient and accurate LFC algorithms.

With our approach to flux identities via DRs, we bring together some of our previous results and a diversity of elementary techniques to make several major contributions in this paper.

- The index-by-index equivalence of flux sets and DRs were proved with an assumption that the streaklines seeded at \(L\) and \(N\) neither intersect nor self-intersect [26]. In this paper we remove this assumption and give a substantially simplified proof for this equivalence.
- We show that the scalar conservation law (1.2) implies the flux identity (1.4) both for incompressible and compressible flows.
- To estimate Eulerian flux integrals from (1.4), we propose a LFC algorithm that are shown in Sections 4 and 5 to be simpler, more accurate, and more efficient than other methods such as that of Karrasch and colleagues [7, 5].

The rest of this paper is organized as follows. Section 2 contains fundamental definitions, lemmas, and theorems. We prove the flux identities (1.3) and (1.4) in Section 3 and propose the LFC algorithm in Section 4. The contents in Sections 2, 3, and 4 are largely self-contained. In Section 5, we preform numerical tests to confirm the validity of flux identities and demonstrate the extreme accuracy and efficiency of the proposed LFC algorithm. Finally, Section 6 concludes this paper.

2. Preliminaries. In this section, we introduce notation and collect relevant definitions and results from a number of distinct mathematical branches to form the theoretical foundation for subsequent sections.
2.1. The flow map. The ordinary differential equation

\[ \frac{dx}{dt} = u(x, t) \]  

admits a unique solution for any given initial time \( t_0 \) and initial position \( p_0 \in \mathbb{R}^D \) if the time-dependent velocity field \( u(x, t) \) is continuous in time and Lipschitz continuous in space. This uniqueness gives rise to a flow map \( \phi : \mathbb{R}^D \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^D \) that takes the initial position \( p_0 \) of a Lagrangian particle \( p \), the initial time \( t_0 \) and the time increment \( \pm k \), and returns \( p(t_0 \pm k) \), the position of \( p \) at the final time \( t_0 \pm k \):

\[ \begin{cases} \phi_{t_0}^{+k}(p) := p(t_0 + k) = p(t_0) + \int_{t_0}^{t_0+k} u(p(t), t) \, dt, \\ \phi_{t_0}^{-k}(p) := p(t_0 - k) = p(t_0) + \int_{t_0}^{t_0-k} u(p(t), t) \, dt. \end{cases} \]  

The flow map also generalizes to point sets in a straightforward way,

\[ \phi_{t_0}^{\pm k}(M) = \{ \phi_{t_0}^{\pm k}(p) : p \in M \}. \]

We will also use the shorthand notation \( \hat{M} := \phi_{t_0}^{-k}(M) \) when \( t_0 \) and \( k \) are clear from the context.

2.2. Curves and winding numbers. An open curve is (the image of) a continuous map \( \gamma : (0, 1) \to \mathbb{R}^2 \), it is simple if \( \gamma \) is injective; otherwise it is self-intersecting. A closed curve is (the image of) a continuous map \( \gamma : [0, 1] \to \mathbb{R}^2 \) with \( \gamma(0) = \gamma(1) \); it is simple closed or Jordan if the restriction of \( \gamma \) to \([0, 1]\) is injective. An isolated intersection point \( p \) of two curves is a proper intersection or a crossing if at a sufficiently small local neighborhood of \( p \) one curve lies at both sides of the other curve; otherwise it is an improper intersection.

An orientation of a curve \( \gamma \) is the assignment of a direction in which \( \gamma \) is traversed. We follow the common convention that counterclockwise orientation is positive while clockwise orientation is negative. The well known Seifert decomposition [16] states that any oriented closed curve \( \gamma \) with a countable number of self-intersection points can be decomposed into a set of oriented, pairwise non-crossing Jordan curves.

Let \( \gamma \) be a closed curve and \( p(t) \) a particle moving continuously on the plane. Suppose \( p \) crosses \( \gamma \) at time \( t_c \). A proper intersection \( \gamma(s_c) = p(t_c) \) is called a positive crossing if, in a sufficiently small open neighborhood of \( s_c \), \( \gamma \) is oriented from the left of the directed path of \( p \) to its right; otherwise it is called a negative crossing [1]. As shown in [26, Sec. 3], the following definition is equivalent to the common notion of winding numbers in complex analysis.

**Definition 2.1 (Winding numbers).** Let \( \gamma \) be an oriented closed curve and \( x \notin \gamma \) a fixed point. The winding number of \( \gamma \) around \( x \), written \( w(\gamma, x) \), is the number of negative crossings of a Lagrangian particle \( p \) through \( \gamma \) minus its number of positive crossings, as \( p \) moves from its initial position \( x \) to \( \infty \).

Definition 2.1 has an immediate consequence as follows.

**Corollary 2.2.** Let \( \gamma \) be an oriented closed curve and \( x \) a fixed point in the unbounded complement of \( \gamma \). Then \( w_c(\gamma, x) = 0 \).

2.3. Pathlines and streaklines. One common characteristic curve of the flow map is the pathline, a curve generated by following a single particle in a time interval,

\[ \Phi_{t_0}^{\pm k}(p) = \{ \phi_{t_0}^{\pm \tau}(p) : \tau \in (0, k) \} . \]
A *backward streakline* is the loci of all particles that will pass continuously through a fixed seeding location $M$,

$$\Psi_{t_0-k}^{−k}(M) : \phi_{t_0+k}^{−\tau}(M) : \tau \in (0,k),$$

and a *forward streakline* is the loci of all particles that have passed $M$,

$$\Psi_{t_0-k}^{+k}(M) : \phi_{t_0-k}^{+\tau}(M) : \tau \in (0,k),$$

where the time increment $k > 0$.

Backward streaklines and forward streaklines are distinguished by the sign of the superscript of $\Psi$. A streakline in the above definition is a snapshot: all particles in (2.4) and (2.5) are at $t = t_0$. In comparison, a pathline is the history of a single Lagrangian particle. By (2.3), (2.4), and (2.5), pathlines and streaklines are indeed curves because they are continuous maps from an interval to the plane.

For more discussions on streaklines, the reader is referred to [20], [7, Sec. 3], and Sections 3.1 and 3.2.

### 2.4. Homotopy and the Hopf theorem.

A *path* in a topological space $X$ is a continuous map $\zeta : [0,1] \to X$. A *homotopy* of paths in $X$ is a family of paths $\gamma_t : [0,1] \to X$, $t \in [0,1]$, such that the endpoints $\gamma_t(0)$, $\gamma_t(1)$ are independent of time and the associated map $H : [0,1]^2 \to X$ defined by $H(s,t) = \gamma_t(s)$ is continuous. The relation of homotopy on paths with fixed endpoints in any space is an equivalence relation. The equivalence class of a path $\zeta$ under this equivalence relation is called the *homotopy class* of $\zeta$. For two paths $\zeta_1$ and $\zeta_2$ satisfying $\zeta_1(1) = \zeta_2(0)$, the *composition* of $\zeta_1$ and $\zeta_2$, denoted $\zeta_1 \cdot \zeta_2$, is defined as

$$\zeta_1 \cdot \zeta_2(s) := \begin{cases} 
\zeta_1(2s), & s \in [0, \frac{1}{2}] \\
\zeta_2(2s - 1), & s \in [\frac{1}{2}, 1]. 
\end{cases}$$

This composition preserves homotopy, i.e. $[\zeta_1 \cdot \zeta_2]$ remains a homotopy class [4, p. 26].

A *loop* or *cycle* is a path whose endpoints coincide; this common endpoint is the *basepoint* of the loop. Two loops are homotopic in $X$ if they are homotopic in $X$ as paths; in this case, the basepoint of the loop homotopy must be fixed. In comparison, a *free homotopy* between two loops $\gamma_1$ and $\gamma_2$ in $X$ is a function $H_f : [0,1]^2 \to X$ such that $H_f(0,t) = \gamma_1(t)$ and $H_f(1,t) = \gamma_2(t)$ for all $t$, and $H_f(s,0) = H_f(s,1)$ for all $s$. 

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**Fig. 2.1.** Examples of homotopic loops and Hopf theorem. Two loops $\gamma_1$ (solid lines) and $\gamma_2$ (dashed lines) have the simple curve $LN$ (the thick solid line) as their common arc. According to the Hopf theorem, $\gamma_1$ and $\gamma_2$ are homotopic in $\mathbb{R}^2$, in $\mathbb{R}^2 \setminus \{x_1\}$, and in $\mathbb{R}^2 \setminus \{x_2\}$. However, they are not homotopic in $\mathbb{R}^2 \setminus \{x_2\}$ because $w(\gamma_1,x_2) \neq w(\gamma_2,x_2)$. In other words, there is no way $\gamma_1$ can be continuously deformed to $\gamma_2$ in $\mathbb{R}^2 \setminus \{x_2\}$, or, any homotopic deformation of $\gamma_1$ to $\gamma_2$ in $\mathbb{R}^2$ must cross $x_2$. 
Then $\gamma_1$ and $\gamma_2$ are said to be freely homotopic in $\mathcal{X}$. Clearly, any path homotopy between two loops can be interpreted as a free homotopy between them.

**Theorem 2.3** (Hopf theorem [6]). Let a point $x \in \mathbb{R}^2$ be given. Two closed curves $\gamma_1$ and $\gamma_2$ are freely homotopic in $\mathbb{R}^2 \setminus \{x\}$ if and only if $w(\gamma_1, x) = w(\gamma_2, x)$.

Theorem 2.3 states that a closed curve can be continuously deformed to another without crossing $x$ in $\mathbb{R}^2$ if and only if $w(\gamma_1, x) = w(\gamma_2, x)$. In Figure 2.1 we emphasize that the underlying space of the free homotopy must be $\mathbb{R}^2 \setminus \{x\}$ instead of $\mathbb{R}^2$.

### 2.5. Jacobian of flow maps.

Although the lemmas in this subsection are well known, we still prove them here because the proofs are intimately related to our key ideas for proving the flux identity in Section 3.4.

**Lemma 2.4.** Let $u(x, t)$ be a given velocity field that is continuous in time and Lipschitz continuous in space. Let $\phi^{t_0}_{t_0} = \phi^{t_0}_{t_0}(x)$ be the associated flow map with fixed initial time $t_0$ and label a Lagrangian particle $x \in \mathcal{X}$ by its initial position $x$. Denote by $J(x, t)$ the Jacobian determinant of the flow map $\phi^{t_0}_{t_0}(x)$. Then

$$
\forall t = t_0 + k > t_0, \quad \frac{dJ}{dt}(x, t) = J(x, t)(\nabla \cdot u)(X(x, t), t).
$$

**Proof.** Denote by $u_i$ and $x_i$ the $i$th component of the velocity $u(x, t)$ and initial position $x$ of a Lagrangian particle. We have

$$
\phi^{t_0}_{t_0}(x_i) = x_i + ku_i(x, t_0) + O(k^2)
$$

$$
\Rightarrow \quad \frac{\partial}{\partial x_i}\phi^{t_0}_{t_0}(x_i) = \delta_{ij} + k\frac{\partial}{\partial x_j}u_i(x, t_0) + O(k^2),
$$

where the first line comes from a Taylor expansion of $X(x, t_0 + k)$ in time. The characteristic polynomial of a 2-by-2 matrix $A$ is

$$
p(\lambda) = \det(\lambda I - A) = \lambda^2 - \text{trace}(A)\lambda + \det(A) = \lambda^2 \det(I - \frac{1}{\lambda} A).
$$

Set $A$ as the matrix satisfying $A_{ij} = \frac{\partial}{\partial x_j}u_i(x, t_0) + O(k)$, and we have from (2.8)

$$
J(x, t) = \det(I + kA) = k^2p\left(-\frac{1}{k}\right) = 1 + k\text{trace}(A) + O(k^2) = 1 + k(\nabla \cdot u) + O(k^2),
$$

which yields

$$
\frac{dJ}{dt}(x, t_0) = (\nabla \cdot u)(x, t_0).
$$

Let $\mathcal{M}$ be a bounded open set such that $\partial \mathcal{M}$ consists of a countable number of pairwise disjoint Jordan curves. The total volume of $\phi^{t_0}_{t_0}(\mathcal{M})$ is then

$$
V(t) = \int_{\phi^{t_0}_{t_0}(\mathcal{M})} 1 dX = \int_{\mathcal{M}} J(x, t) dx.
$$

Differentiate (2.10) with respect to $t$, set $t = t_0$, apply (2.9), and we have

$$
\frac{dV}{dt}(t_0) = \int_{\mathcal{M}} (\nabla \cdot u)(x, t_0) dx,
$$

where $\mathcal{M}$ is a bounded open set such that $\partial \mathcal{M}$ consists of a countable number of pairwise disjoint Jordan curves. The total volume of $\phi^{t_0}_{t_0}(\mathcal{M})$ is then

$$
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$$
V(t) = \int_{\phi^{t_0}_{t_0}(\mathcal{M})} 1 dX = \int_{\mathcal{M}} J(x, t) dx.
$$

Differentiate (2.10) with respect to $t$, set $t = t_0$, apply (2.9), and we have

$$
\frac{dV}{dt}(t_0) = \int_{\mathcal{M}} (\nabla \cdot u)(x, t_0) dx,
$$
which must also holds if we switch the reference starting time to any \( t > t_0 \), i.e.

\[
\frac{dV(t)}{dt} = \int_{\phi_{t_0}^{(t-t_0)}(\mathcal{M})} (\nabla \cdot \mathbf{u})(X(x,t),t)dX = \int_{\mathcal{M}} (\nabla \cdot \mathbf{u})(X(x,t),t)J(x,t)dx.
\]

On the other hand, differentiating (2.10) with respect to \( t \) yields

\[
\frac{dV(t)}{dt} = \int_{\mathcal{M}} \frac{dJ}{dt}(x,t)dx.
\]

The comparison of the last two equations yields (2.7). \( \square \)

In particular, \( J = 1 \) for incompressible flows since \( \frac{dJ}{dt} = 0 \) and then \( \phi_{t_0}^0 \) is the identity operator. See [11, Chap. 1] for more details.

**Lemma 2.5.** Let \( \mathcal{M} \) denote a bounded open set of which its boundary consists of a countable number of pairwise disjoint Jordan curves. Let \( f(x,t) \) be an integrable scalar function satisfying

\[
\frac{d}{dt}\int_{\phi_{t_0}^{(t-t_0)}(\mathcal{M})} f(X,t)dX = 0.
\]

If \( f(x,t) \) is further differentiable, then (2.11) is equivalent to the scalar hyperbolic conservation law (1.2).

**Proof.** Change the integral variable in (2.11) so that the integral is over the fixed domain \( \mathcal{M} \). The chain rule and Lemma 2.4 yield

\[
\frac{d}{dt}\int_{\phi_{t_0}^{(t-t_0)}(\mathcal{M})} f(X,t)dX = \frac{d}{dt}\int_{\mathcal{M}} f(X(x,t),t)Jdx = \int_{\mathcal{M}} \left[ \frac{\partial f}{\partial t} + \frac{\partial f}{\partial X} \frac{dx}{dt} + f(\nabla \cdot \mathbf{u}) \right] Jdx
\]

\[
= \int_{\phi_{t_0}^{(t-t_0)}(\mathcal{M})} \left[ \frac{\partial f}{\partial t} + \nabla \cdot (f\mathbf{u}) \right] dX = 0.
\]

Equation (1.2) must hold because \( t \) and \( \mathcal{M} \) are both arbitrary. \( \square \)

Lemma 2.5 is more general than (1.2) in that the scalar function \( f \) needs not to be differentiable in order for its integral on a moving region to be conserved.

2.6. Reynolds transport theorem on the homotopy class of moving regions. A **moving Jordan loop** is a homotopy class of loops

\[
\check{\Gamma} = \{ \gamma(\tau) : \tau \in [0,k] \},
\]

where (i) each loop \( \gamma(\tau) : [0,1] \rightarrow \mathbb{R}^2 \) is an oriented piecewise smooth Jordan curve, (ii) the map associated with the homotopy is smooth in time, and (iii) all the Jordan curves have a consistent orientation.

A **moving region in the plane** is a family of regular subsets of \( \mathbb{R}^2 \),

\[
\{ \mathcal{M}(\tau) : \tau \in [0,k] \},
\]

such that \( \{ \partial \mathcal{M}(\tau) : \tau \in [0,k] \} \) is a set of moving Jordan loops. The following theorem is also known as the multidimensional Leibniz integral rule [15].

**Theorem 2.6 (Reynolds’ transport theorem [12]).** Let \( \{ \mathcal{M}(\tau) : \tau \in [0,k] \} \) denote a moving region in the plane and \( g(\cdot,\tau) : \mathcal{M}(\tau) \rightarrow \mathbb{R} \) a scalar function. We have

\[
\frac{d}{d\tau}\int_{\mathcal{M}(\tau)} g(x,\tau)dx = \int_{\partial \mathcal{M}(\tau)} \frac{\partial g(x,\tau)}{\partial \tau} dx + \int_{\partial \mathcal{M}(\tau)} g(s,\tau)V_n(s,\tau)ds,
\]

where \( V_n \) is the normal vector.
where \( V_n(s, \tau) \) is the normal speed of the moving Jordan loop \( \{ \partial M(\tau) : \tau \in (0, k) \} \).  

Proof. See [10] for a proof via differential forms and Stokes’ theorem. \( \square \)

In proving Lemmas 2.4 and 2.5, each pair of elements in the moving region must have a bijective correspondence through the flow map. In the Reynolds transport theorem, however, the bijective correspondence need not hold for the moving region under consideration; this is a key of our proving the flux identity in Section 3.4.

3. Analysis. After deriving two useful streakline identities in Section 3.1, we give examples of intersecting and self-intersecting streaklines in Section 3.2. Then we proceed in Section 3.3 to generalize the main result in [26] by removing its assumption that the streaklines of the edges of donating regions neither intersect nor self-intersect. Finally we prove the flux identity in Section 3.4.

3.1. Two streakline identities. By (2.4), the backward streakline \( \Psi^{-k}_{t_0+k}(M) \) contains the initial positions of all particles that will pass \( M \) within \( (t_0, t_0 + k) \). At \( t = t_0 + \tau \) with \( \tau \in (0, k) \), these particles can be classified into three types: those that will pass \( M \) within \( (t_0 + \tau, t_0 + k) \), the one that coincides with \( M \), and those that have passed \( M \) within \( (t_0, t_0 + \tau) \). Under the mapping of the diffeomorphism \( \phi^r_{t_0} \), these three sets of particles are \( \Psi^{-(k-r)}_{t_0+k}(M) \), \( M \), \( \Psi^{+\tau}_{t_0}(M) \), respectively. Hence the image of the backward streakline \( \Psi^{-k}_{t_0+k}(M) \) can be decomposed as

\[
\forall \tau \in (0, k), \quad \phi^r_{t_0} \left( \Psi^{-k}_{t_0+k}(M) \right) = \Psi^{-(k-r)}_{t_0+k}(M) \cup M \cup \Psi^{+\tau}_{t_0}(M).
\]  

For two time increments \( 0 < r < s \), similar arguments yield

\[
\phi^{s-r}_{t_0+r} \left( \Psi^{-(s-r)}_{t_0+s}(M) \right) = \Psi^{-s}_{t_0+s}(M) \setminus \Psi^{r}_{t_0+r}(M) \setminus \phi^{r}_{t_0+r} (M).
\]

The streakline inside the LHS parenthesis is at time \( t_0 + s - (s - r) = t_0 + r \) while those streaklines on the RHS at time \( t_0 + s - s = t_0 + r - r = t_0 \). Then the flow map \( \phi^{s-r}_{t_0+r} \) carries the former streakline to the RHS because they are exactly the same set of Lagrangian particles! Balasuriya [2] exploited this property of streaklines in locating a curve through which the flux quantifies the transport between two fluids.

3.2. Intersecting streaklines. By the uniqueness of the solution of the ODE (2.1), two different particles cannot be present at the same location and the same time. This observation is sometimes used to justify two widespread common beliefs on streaklines [22]: “a streakline does not intersect itself” and “two different streaklines do not intersect.” Unfortunately, none of these two beliefs is correct.

Starting from the same initial position \( x \), two particles with different initial times may reach the seeding location \( M \) at two different instants within the time interval \( [t_0, t_0 + k] \). Hence \( x \) could be a self-intersection of the backward streakline \( \Psi^{-k}_{t_0+k}(M) \). For two different seeding locations \( L \) and \( N \), although no particle can be simultaneously at \( L \) and \( N \) for any given time, a particle \( p \) may arrive at \( L \) and \( N \) at two different time instants, hence \( p \) might as well be an intersection point of \( \Psi^{-k}_{t_0+k}(L) \) and \( \Psi^{-k}_{t_0+k}(N) \). See Figure 3.1 for illustrations of the above arguments.

As discussed in [21], streakline intersections and cusps are closely related to critical points of the advecting flow. Fortunately, Sard’s theorem states that these intersection points are of measure zero, and thus they do not affect the validity of integral equations such as the flux identity.
Fig. 3.1. Examples of streakline intersections. The velocity field is \( \mathbf{u} = (-y, x - t + 5)^T \), with \( t_0 = 0 \) and \( k = 10 \). The seeding locations of streaklines, \( L = (0, 0) \) and \( N = (1, 0) \), are represented by squares, with the backward streaklines \( \Psi_{t_0+k}^{-} (L) \) and \( \Psi_{t_0+k}^{-} (N) \) represented by red thick lines. Both streaklines contain cusps where tangent vectors of streaklines are undefined. A circle represents a self-intersection \( p \) or an intersection \( p \) of two different streaklines. A thin solid line represents a pathline \( \Phi_t^k (p) \) with \( k' = 10 \) for subplot (a) and \( k' = 7 \) for subplot (b). Note that in subplot (a) the pathline and streakline almost overlap between \( L \) and \( p \).

Fig. 3.2. Examples of fluxing indices. A thick solid line with pentagon endpoints represents the fixed curve \( \overline{LN} \) and a dotted line with round ends represents a pathline of the Lagrangian particle \( p \), with arrows indicating the direction of the flow. The shaded region is the bounded complement of the Jordan curve \( \hat{\gamma} \) that determines whether the intersection of \( \overline{LN} \) and a pathline is an outflux or influx. Note that those intersections in \( \hat{\gamma} \setminus \overline{LN} \) are irrelevant in determining the fluxing indices.

3.3. The index-by-index equivalence of flux sets and donating regions.

A fluxing particle to a fixed simple curve \( \overline{LN} \) over the time interval \((t_0, t_0 + k)\) is a particle \( p \) whose pathline \( \Phi_t^{+k} (p) \) properly intersects \( \overline{LN} \) at least once.

Consider the proper intersections of a fluxing particle to a fixed simple curve \( \overline{LN} \) within a time interval. Let \( \hat{\gamma} \) denote a Jordan curve satisfying \( \hat{\gamma} \supseteq \overline{LN} \). A proper intersection is called an outflux with respect to \( \hat{\gamma} \) if the fluxing particle crosses \( \hat{\gamma} \) from its bounded complement into its unbounded complement; otherwise it is called an influx with respect to \( \hat{\gamma} \). As illustrated in Figure 3.2, the fluxing index of a fluxing particle \( p \) to \( \overline{LN} \) with respect to \( \hat{\gamma} \supseteq \overline{LN} \) is the number of its outfluxes minus that of its influxes. We emphasize that the only reason of including \( \hat{\gamma} \) in this definition is to furnish a simple mechanism for choosing either one of the two sides of \( \overline{LN} \) in the determination of \( \mathbf{n}_{\overline{LN}} \), the outward normal vector of \( \overline{LN} \). Consequently, the fluxing
index of \( p \) is only dependent on the crossing of \( p \) through \( \tilde{LN} \) and it has nothing to do with any crossing of \( p \) through \( \gamma \setminus LN \).

The fluxing index furnishes an equivalence class on all points in the plane.

**Definition 3.1 (Flux sets).** The flux set of index \( n \) through a simple curve \( LN \) with respect to a Jordan curve \( \gamma \supseteq LN \) over the time interval \( (t_0, t_0 + k) \), written \( \mathcal{F}^n_{LN}(t_0, k) \), is the loci of all the fluxing particles of index \( n \) at time \( t_0 \).

Typically, the Jordan curve \( \gamma \) in Definition 3.1 and in Figure 3.2 is chosen according to the specific problem at hand. For example, consider the flux set \( \mathcal{F}^n_{LN}(t_0, k) \) through a cell face \( LN \) that simultaneously belongs to the boundaries of two neighboring control volumes. Clearly, if \( \gamma \) is chosen to be the boundary of one control volume and \( n = 1 \), then \( \gamma \) being the boundary of the other cell must yield \( n = -1 \).

The winding number of an oriented closed curve furnishes another family of equivalence classes of points in the plane.

**Definition 3.2 (Donating regions).** For a given velocity field \( u(x, t) \) that is continuous in time and Lipschitz continuous in space, the donating region of index \( n \) associated with a simple open curve \( LN \) fixed over the time interval \( (t_0, t_0 + k) \) is

\[
\begin{align*}
D^n_{LN}(t_0, k) := \{x \in \mathbb{R}^2 : w(\gamma_D, x) = n\},
\end{align*}
\]

\[
\gamma_D := L \cup LN \cup N \cup \Psi_{t_0+k}^{-k}(N) \cup \tilde{N} \cup \phi_{t_0+k}^{-k}(LN) \cup \tilde{L} \cup \Psi_{t_0+k}^{-k}(L),
\]

where \( \tilde{x} := \phi_{t_0+k}^{-k}(x) \), \( \gamma_D \), the generating curve of the DR, is oriented by the closed vertex sequence \( L \rightarrow N \rightarrow \tilde{N} \rightarrow \tilde{L} \rightarrow L \), and \( w(\gamma_D, x) \) is the winding number of \( \gamma_D \) around \( x \) as in Definition 2.1. For \( x \in \gamma_D \), the open disk centered at \( x \) with sufficiently small radius contains the same set of winding numbers; \( w(\gamma_D, x) \) is the number in this set that is closest to zero.

We also denote by \( D_{LN}(t_0, k) \) the union of donating regions with nonzero indices,

\[
(3.4) \quad D_{LN}(t_0, k) = \bigcup_{n \in \mathbb{Z} \setminus \{0\}} D^n_{LN}(t_0, k).
\]

**Definition 3.3.** For a nonempty DR, let \([n^-, n^+]\) denote the largest interval such that \( D^n_{LN} \neq \emptyset \) for each \( n \in [n^-, n^+] \). The unit normal vector of a simple open curve \( LN \) induced by a nonempty DR, written \( n^{DR}_{LN} \), is the normal vector that agrees with the outward unit normal of \( D^{n^+}_{LN} \) if \( n^+ > 0 \); otherwise \( n^{DR}_{LN} \) is the one that disagrees with the outward unit normal of \( D^{n^-}_{LN} \).

Definitions 3.2 and 3.3 are illustrated in Figure 3.3 by three examples.

A DR is **normal** if the two backward streaklines seeded at \( L \) and \( N \) neither intersect nor self-intersect. All DRs in Figure 3.3 are normal while the DR in Figure 3.4 is not. However, the DR in Figure 3.4 would be normal if \( k \) were chosen sufficiently small; see Figure 3.6 (a). In other words, the normality of a DR can be guaranteed by choosing the length of the time interval to be sufficiently small: this was proved in [26, Prop. 4.6]. Regardless of a DR being normal or not, the set of its generating curves at all intermediate time instants forms a homotopic class as follows.

**Lemma 3.4.** The set of generating curves of DRs for a simple open curve

\[
\Gamma_{LN}(t_0, k) := \{\gamma_{D_{LN}(t_0+t, k-\tau)} : \tau \in [0, k]\}
\]

is a homotopy class in \( \mathbb{R}^2 \).
Fig. 3.3. Examples of donating regions of a simple open curve $\overline{LN}$ and the induced outward normal vector of $LN$. A DR is represented by shaded regions, on the boundary of which a big arrow indicates the orientation of the generating curve. The orientation of dotted circles with small arrows indicates the sign of the index of DR. In subplot (c), the dark gray region and the light gray region represent $D_{LN}^{+1}$ and $D_{LN}^{-1}$, respectively. The double arrowed vectors indicate the outward normal vector of $LN$ induced by the DRs.

Fig. 3.4. A DR with intersecting streaklines. The velocity field is $u = (-y, x - t + 5)^T$, with $t_0 = 0$ and $k = 10$. The squares represent the seeding locations of streaklines $L = (0, 0)$ and $N = (1, 0)$, with the two streaklines the same as those in Figure 3.1. The curve $LN$ and its preimage are represented by thick lines. The light gray, dark gray, and white regions represent $D_{LN}^{+1}(t_0, k)$, $D_{LN}^{-1}(t_0, k)$, and $D_{LN}^0(t_0, k)$, respectively. See the associated movie file nonNormalDR.gif.

Proof. Construct a composite path via concatenation as

$$\zeta_r := \left( L \cup \Psi_{t_0+k}^{-((k-rk)}(L) \cup \overline{\overline{\overline{L}}} \right) \cdot \left( \overline{\overline{\overline{L}}} \cup \phi_{t_0+k}^{-(k-rk)}(\overline{\overline{\overline{LN}}} \cup \overline{\overline{\overline{N}}}) \right) \cdot \left( \overline{\overline{\overline{N}}} \cup \Psi_{t_0+k}^{-(k-rk)}(N) \cup N \right).$$

By (3.1), the set of backward streaklines $\{ \Psi_{t_0+k}^{-((k-rk)}(L) : r \in [0, 1] \}$ are the loci of a continuously diminishing set of particles that will pass $L$ during the continuously shortened time intervals $\{ [t_0 + rk, t_0 + k] : r \in [0, 1] \}$. The set of the preimages of $\overline{\overline{\overline{LN}}}$, $\{ \phi_{t_0+k}^{-(k-rk)}(\overline{\overline{\overline{LN}}}) : r \in [0, 1] \}$, is also continuous with respect to $r$. Hence the map $H(s, r) = \zeta_r(s)$ is continuous and $[\zeta_r]$ is a homotopy class. For the static path $\zeta_{LN} := L \cup LN \cup N$, the identity map $I(s, r) = \zeta_{LN}(s)$ is clearly continuous and $[\zeta_{LN}]$ is also a homotopy class. Hence their composition paths $\Gamma_{LN} = \zeta_r \cdot \zeta_{LN}$ form
a homotopy class of loops. □

We make two important observations. First, Lemma 3.4 holds regardless of the DR being normal or not; this is a key in proving Theorem 3.5. Second, Lemma 3.4 would not hold if the underlying space $\mathbb{R}^2$ were replaced by $\mathbb{R}^2 \setminus \{x\}$ for any $x \in \mathcal{D}_{LN}(t_0, k)$ in (3.4). Indeed, as $r \to 1$ in the above proof, the generating curve $\gamma_{\mathcal{D}_{LN}(t_0+k, 0)}$ is a trivial Jordan curve with the empty set as its bounded complement, hence at time $t_0 + k$ the winding number of $\gamma_{\mathcal{D}_{LN}(t_0+k, 0)}$ around any point in the plane is zero. On the other hand, for any $x \in \mathcal{D}_{LN}(t_0, k)$ at time $t_0$ we know from Definition 3.2 that the winding number of $\gamma_{\mathcal{D}_{LN}(t_0, k)}$ around $x$ is nonzero. Therefore, it would contradict the Hopf Theorem 2.3 if $\mathbb{R}^2$ in Lemma 3.4 were replaced with $\mathbb{R}^2 \setminus \{x\}$ for any $x \in \mathcal{D}_{LN}(t_0, k)$.

In [26], we showed that the DR of a simple curve $\tilde{L}N$ are index-by-index equivalent to the flux sets of $\tilde{L}N$, under the assumption of the DR being normal. A major contribution of this work is the removal of this assumption via a simpler proof.

**Theorem 3.5 (Index-by-index equivalence of flux sets and DRs).** Any DR of a fixed simple open curve $\tilde{L}N$ is index-by-index equivalent to the flux set of $\tilde{L}N$,

$$\forall n \in \mathbb{Z}, \forall k > 0, \quad \mathcal{D}_{LN}^n(t_0, k) = \mathcal{F}_{LN}^n(t_0, k),$$

if $u_{LN}^{DR}$, the unit normal vector of $\tilde{L}N$ induced by the DR, points from the bounded complement of $\tilde{\gamma}$ to the unbounded complement of $\tilde{\gamma}$, where $\tilde{\gamma}$ is the Jordan curve in Definition 3.1 for determining the sign of fluxing indices.

As for the other case of $u_{LN}^{DR}$, $\mathcal{D}_{LN}^n(t_0, k) = \mathcal{F}_{LN}^n(t_0, k)$.

**Proof.** Consider an arbitrary fixed point $x_0$. Define $v(x, t) = u(x, t) - u(x_0, t)$. It suffices to show that (3.6) holds under the flow of $v$ since the translation map of $-u(x_0, t)$ preserves all topological properties. Under the new flow map of $v$, $x_0$ is always static since $v(x_0, t) = 0$. As for the generating curve $\gamma_{\mathcal{D}_{LN}(t_0, k)}$, the curve $L \cup \tilde{L}N \cup N$ is moving with the uniform velocity $-u(x_0, t)$ while the velocity of the other part $\gamma_d := \gamma_{\mathcal{D}_{LN}(t_0, k)} \setminus (L \cup \tilde{L}N \cup N)$ is $v$.

Let $n_0$ be the integer such that $x_0 \in \mathcal{D}_{LN}^{n_0}(t_0, k)$. By Definition 3.2, the winding number of $\gamma_{\mathcal{D}_{LN}(t_0, k)}$ around $x_0$ is $n_0$. In contrast, it follows from Corollary 2.2 and the last sentence in Definition 3.2 that the winding number of $\gamma_{\mathcal{D}_{LN}(t_0+k, 0)}$ around $x_0$ is zero since $\gamma_{\mathcal{D}_{LN}(t_0+k, 0)} = \tilde{L}N$.

If $n_0 = 0$, then the Hopf theorem 2.3 implies that $\gamma_{\mathcal{D}_{LN}(t_0, k)}$ deforms homotopically to $\gamma_{\mathcal{D}_{LN}(t_0+k, 0)}$ in $\mathbb{R}^2 \setminus \{x_0\}$. Hence the net crossing of all parts of the moving loop $\gamma_d$, including $\tilde{L}N$, is zero.

If $n_0 \neq 0$, then Hopf theorem 2.3 implies that the deformation of $\gamma_{\mathcal{D}_{LN}(t_0, k)}$ to $\gamma_{\mathcal{D}_{LN}(t_0+k, 0)}$ cannot be freely homotopic in $\mathbb{R}^2 \setminus \{x_0\}$. Meanwhile, Lemma 3.4 states that the deformation of $\gamma_{\mathcal{D}_{LN}(t_0, k)}$ to $\gamma_{\mathcal{D}_{LN}(t_0+k, 0)}$ must be homotopic in $\mathbb{R}^2$. Therefore, the moving loop $\tilde{\gamma}_{\mathcal{D}_{LN}}$ in (3.5) must cross $x_0$; see Figure 2.1. Furthermore, the new flow map is a diffeomorphism, hence $\gamma_d$ never crosses $x_0$ and the only part of the generating curve that can cross $x_0$ is $\tilde{L}N$. In addition, the index of $\tilde{L}N$ crossing $x_0$ must equal $n_0$ so that the winding number of $\gamma_{\mathcal{D}_{LN}(t_0+k, 0)}$ around $x_0$ becomes zero at the final time $t_0 + k$. The proof is then completed by Definition 3.1. □

**3.4. The flux identity.** A DR is *canonical* if its generating curve is Jordan. The **degree of a DR** $\mathcal{D}_{LN}$ is the maximum integer $N$ such that $|n| > N$ implies $\mathcal{D}_{LN}^n = \emptyset$. 

for all \( n \in \mathbb{Z} \). Starting from the simplest form of DRs, we prove the flux identities (1.3) and (1.4) in three strategic steps:

(S1) prove the flux identity (1.3) for canonical DRs in Lemma 3.6,

(S2) utilize (S1) to prove the flux identity (1.4) for normal DRs of degree 1,

(S3) prove the flux identity (1.4) for arbitrary DRs.

**Lemma 3.6** (The flux identity for canonical DRs). Let \( f \) be a scalar function conserved by a nonautonomous flow \( u(x,t) \) such that (1.2) holds. If the DR \( D_{LN}(t_0,k) \)

for a fixed simple open curve \( LN \) is canonical, then

\[
\int_{t_0}^{t_0+k} \int_{D_{LN}(t_0,k)} f(x,t_0 + \tau) \, dx \, ds = \pm \int_{D_{LN}(t_0,k)} f(x,t_0) \, dx,
\]

where \( n_{LN} \) is the unit normal vector induced by \( D_{LN}(t_0,k) \) as in Definition 3.3, and \( \pm \) takes “+” and “−” for \( D_{LN}^+(t_0,k) \) and \( D_{LN}^-(t_0,k) \), respectively.

**Proof.** By Lemma 3.4, the DRs \( \{D_{LN}^- (t_0 + \tau, k - \tau) : \tau \in (0,k) \} \) form a moving region in the plane. Clearly it suffices to prove the case \( D_{LN}^+ (t_0,k) = D_{LN}^+ (t_0,k) \), for which we have

\[
\frac{d}{dt} \int_{D_{LN}^+(t_0+k)} f(x,t_0 + \tau) \, dx = \int_{D_{LN}^+(t_0+k)} \frac{\partial f(x,t_0+\tau)}{\partial \tau} \, dx + \int_{\partial D_{LN}^+(t_0+k)} f(s,t_0 + \tau) V_n(s,t_0 + \tau) \, ds,
\]

\[
= -\int_{D_{LN}^-(t_0+k)} V \cdot (f u) \, dx + \int_{\partial D_{LN}^-(t_0+k)} f(s,t_0 + \tau) V_n(s,t_0 + \tau) \, ds,
\]

\[
= \int_{D_{LN}^+} (s,t_0+\tau) V_n(s,t_0 + \tau) - u \cdot n_{LN} \, ds,
\]

where the first step follows from the Reynolds transport Theorem 2.6, the second step from the hyperbolic conservation law (1.2), and the third step from the divergence theorem. It is important to distinguish the normal speed of the moving Jordan loop \( \tilde{\Gamma}_{LN}(t_0,k) \) in (3.5) from those of the underlying flow. Because all Lagrangian particles in the initial DR \( D_{LN}^+ (t_0,k) \) are passively advected by the velocity field \( u \), we have \( V_n = u \cdot n \) for every point in \( \partial D_{LN}^+ (t_0+k-t, k-\tau) \setminus LN \). In comparison, \( V_n = 0 \) holds everywhere on the stationary curve \( LN \). Therefore, \( V_n - u \cdot n = -u \cdot n \) holds on \( LN \) and \( V_n - u \cdot n = 0 \) holds everywhere else on \( \partial D_{LN}^+(t_0, k, k) \). In the last step, Definition 3.3 implies that the unit normal of \( LN \) is exactly the same as the outward unit normal of \( D_{LN}^+(t_0+k, k) \). Finally, integrating the above equation in \([0,t_0+k]\) yields

\[
\int_{t_0}^{t_0+k} \left[ \frac{d}{dt} \int_{D_{LN}^+(t_0+k-t, k-\tau)} f(x,t_0+\tau) \, dx \right] d\tau \equiv \int_{D_{LN}^+(t_0+k,0)} f(x, t_0 + k) \, dx - \int_{D_{LN}^+(t_0,k)} f(x,t_0) \, dx = -\int_{t_0}^{t_0+k} \int_{LN} f u \cdot n_{LN} \, ds \, dt
\]

where the first step follows from the fundamental theorem of calculus and the second step from the fact that \( D_{LN}^+(t_0,k,0) = \emptyset \). \( \Box \)

It is shown in [25, Sec. 4] that a degree-1 DR with self-intersecting boundary can be decomposed into canonical DRs. This fact, together with Lemma 3.6, makes it straightforward to prove the following lemma.
Lemma 3.7 (The flux identity for normal DRs of degree one). Let $f$ be a scalar function conserved by a nonautonomous flow $u(x,t)$ such that (1.2) holds. If the DR $\mathcal{D}_{\tilde{L}N}(t_0,k)$ for a fixed simple open curve $\tilde{L}N$ is normal and has degree one, then

\begin{equation}
\int_{t_0}^{t_0+k} \int_{\tilde{L}N} f \cdot u \cdot n_{\tilde{L}N} \, ds \, dt = \sum_{n=\pm 1} n \int_{\mathcal{D}_{\tilde{L}N}^n(t_0,k)} f(x,t_0) \, dx,
\end{equation}

where $n_{\tilde{L}N}$ is the unit normal vector induced by $\mathcal{D}_{\tilde{L}N}(t_0,k)$ as in Definition 3.3.

Proof. As the main difficulty of the proof, the generating curve of a normal DR with degree 1 may be self-intersecting. Then the DR might not be canonical and its boundary is not orientable at the self-intersection points.

If any self-intersection point is fixed, i.e. their velocity being identically zero, then each DR can be decomposed into a number of disjoint simply connected regions. In this case, the evolution of each sub-DR is a moving regular open region and Lemma 3.6 yields (3.8) in a straightforward way.

If at least one self-intersection point of the generating curve of $\mathcal{D}_{\tilde{L}N}(t_0,k)$ is not a fixed point, we know from [25, Sec. 4] that, so long as $\mathcal{D}_{\tilde{L}N}(t_0,k)$ is normal, $\tilde{L}N$ can be decomposed into a number of disjoint arcs, each of which has a canonical DR within the given time interval $(t_0, t_0 + k)$. Applying Lemma 3.6 to each arc and summing up the integral completes the proof. $\Box$

Theorem 3.8 (The flux identity for arbitrary DRs). Let $f$ be a scalar function conserved by a nonautonomous flow $u(x,t)$ such that (1.2) holds. Then the DR $\mathcal{D}_{\tilde{L}N}(t_0,k)$ of a fixed simple open curve $\tilde{L}N$ as in Definition 3.2 satisfies

\begin{equation}
\forall k > 0, \quad \int_{t_0}^{t_0+k} \int_{\tilde{L}N} f \cdot u \cdot n_{\tilde{L}N} \, ds \, dt = \sum_{n=\pm 1} n \int_{\mathcal{D}_{\tilde{L}N}^n(t_0,k)} f(x,t_0) \, dx,
\end{equation}

where $n_{\tilde{L}N}$ is the unit normal vector induced by $\mathcal{D}_{\tilde{L}N}(t_0,k)$ as in Definition 3.3.

Proof. For a sufficiently small time increment, we know from [26, Prop. 4.6] that the corresponding DR is normal with degree 1. Hence for any given $k > 0$ there exist a finite number of positive real numbers $k_1, k_2, \ldots, k_m$, such that for each $i = 0, 1, \ldots, m$ the DR $\mathcal{D}_{\tilde{L}N}(t_0 + k_i, k_{i+1} - k_i)$ is normal with degree 1. Then we have

\[
\int_{t_0}^{t_0+k} \int_{\tilde{L}N} f \cdot u \cdot n_{\tilde{L}N} \, ds \, dt = \left( \sum_{i=0}^m \int_{\mathcal{D}_{\tilde{L}N}^n(t_0+k_i)} f(x,t_0+k_i) \, dx \right) \int_{\mathcal{D}_{\tilde{L}N}^n(t_0+k_i)} f(x,t_0+k_i) \, dx
\]

\[
= \sum_{i=0}^m \sum_{n=\pm 1} n \int_{\mathcal{D}_{\tilde{L}N}^n(t_0+k_i, k_{i+1} - k_i)} f(x,t_0+k_i) \, dx
\]

\[
= \sum_{i=0}^m \sum_{n=\pm 1} n \int_{\mathcal{D}_{\tilde{L}N}^n(t_0+k_i, k_{i+1} - k_i)} f(x,t_0+k_i) \, dx
\]

\[
= \sum_{n \in \mathbb{Z}} n \int_{\mathcal{D}_{\tilde{L}N}^n(t_0,k)} f(x,t_0+k_i) \, dx,
\]

where the first step breaks up the time interval into the sub-intervals, the second step follows from Lemma 3.7, and the third step follows from Lemma 2.5. The fourth step is justified as follows.

Denote by $\gamma_i$ as the generating curve of $\mathcal{D}_{\tilde{L}N}(t_0 + k_i, k_{i+1} - k_i)$. Then images of the generating curves $\phi_{t_0+k_i}^{-k_i}(\gamma_i)$ for $i = 0, 1, \ldots, m$ concatenate to $\gamma_D$, the generating
curve of $\mathcal{D}_{LN}(t_0, k)$. To see this, we calculate algebraically for a single image as

$$
\phi_{t_0+k_1}^{-k_i}(\gamma_i) = \phi_{t_0+k_1}^{-k_i} \left( LN \cup \Psi_{t_0+k_1}^{-k_i}(N) \cup \phi_{t_0+k_1}^{-k_i} \left( NL \cup \Psi_{t_0+k_1}^{-k_i}(L) \right) \right)
$$

where the first step follows from (3.3) and the second step follows from (3.2) and a group property of the flow map, i.e. $\phi_{t_0+k_1}^{-k_i} \circ \phi_{t_0+k_1}^{-k_i} = \phi_{t_0+k_1}^{-k_i+1}$. To avoid cluttering, we have ignored single points in the above derivation.

It follows that the concatenation $\gamma_0 \sqcup \phi_{t_0+k_1}^{-k_1}(\gamma_1) \sqcup \cdots \sqcup \phi_{t_0+k_m}^{-k_m}(\gamma_m)$ is

$$
\mathcal{D}_{L}^N \cup \Psi_{t_0+k_1}^{-k_1}(N) \cup \phi_{t_0+k_1}^{-k_1} (\mathcal{D}_{L} \cup \Psi_{t_0+k_1}^{-k_1}(L))
\sqcup \phi_{t_0+k_1}^{-k_1} (LN) \sqcup \Psi_{t_0+k_2}^{-k_2}(N) \sqcup \phi_{t_0+k_2}^{-k_2} (LN) \sqcup \Psi_{t_0+k_2}^{-k_2}(L) \sqcup \cdots
\sqcup \phi_{t_0+k_m}^{-k_m} (LN) \sqcup \Psi_{t_0+k_m}^{-k_m}(N) \sqcup \phi_{t_0+k_m}^{-k_m} (LN) \sqcup \Psi_{t_0+k_m}^{-k_m}(L)
$$

see Figures 3.5 and 3.6 for two illustrations. Here we have used the notation “⊆” to emphasize the convention that two curves concatenate to the empty set if they have the same image points but different orientations. The proof is then completed by Definition 3.2 and (3.4).

Equation (3.9) is the most general flux identity proposed in this work. On the other hand, it reduces to the motivating flux identity (1.3) for a sufficiently small time increment.

4. Algorithm. To utilize Theorem 3.8 for LFC, we have several possible approaches to design such an algorithm. In the first method, one divides the whole time interval and the given curve $LN$ into a number of sufficiently small sub-intervals to ensure that each sub-DR of a sub-interval is of degree 1; one then calculates the integral over each sub-DR and sum them up to obtain the RHS in (3.9). However, values of the scalar function at the starting time of each sub-interval must be known.
Fig. 3.6. Decomposing the non-normal DR in Figure 3.4 into a normal DR and the image of another normal DR, as shown in the two subplots.

and this undermines the leverage of the DR concept, as one of its advantages is to compute the Eulerian flux integral with no knowledge of the evolution of the scalar function except at the very initial time.

In the second method [7], one constructs the generating curve $\gamma_D$ of the DR for the whole time interval, divides $\gamma_D$ into oriented Jordan curves, calculates the winding number for each connected open region, computes the Lagrangian integral on these connected open regions, and finally sums up the sub-integrals weighed by corresponding winding numbers. The Lagrangian integral over irregular open region can be calculated via Green's theorem and one-dimensional (1D) quadrature formulas [18, 17]. As an advantage over the first method, values of the scalar function are only needed at the initial time. However, for a DR with complex topology, it might be very time-consuming to decompose the generating curve into oriented Jordan curves and to compute winding numbers for each connected open region. Furthermore, a straightforward generalization of the above method to higher dimensions may incur an algorithm complexity that is unacceptably expensive.

In the rest of this section we propose a new LFC algorithm by annihilating the needs of cycle decomposition and winding-number calculation. This algorithm is based on the following corollary.

**Corollary 4.1.** With the same conditions in Theorem 3.8, we have

\[
\int_{t_0}^{t_0+k} \int_{\mathcal{L}_N} f \mathbf{u} \cdot \mathbf{n}_{\mathcal{L}_N} ds dt = \oint_{\gamma_D} F(x, y, t_0) dy,
\]

where $F(x, y, t_0) = \int_{\zeta}^{x} f(v, y, t_0) dv$, $\zeta$ is an arbitrary fixed real number, and $\gamma_D$ is the generating curve of the DR $\mathcal{D}_{\mathcal{L}_N}(t_0, k)$ as in Definition 3.2.

**Proof.** Define

\[
\forall n \in \mathbb{Z}, \quad \mathcal{D}_{\mathcal{L}_N}^{\leq n} := \bigcup_{i \geq n; \zeta \in \mathbb{Z}} \mathcal{D}_{\mathcal{L}_N}^i, \quad \mathcal{D}_{\mathcal{L}_N}^{\geq n} := \bigcup_{i \leq n; \zeta \in \mathbb{Z}} \mathcal{D}_{\mathcal{L}_N}^i.
\]

It follows that the indexed set $\left\{ \mathcal{D}_{\mathcal{L}_N}^{\leq n} : n \in \mathbb{N}^+ \right\}$ is a descending filtration and the set
\[ \{D_{\tilde{L}N}^{\leq-n} : n \in \mathbb{N}^+ \} \] is a filtration, i.e.

\[ \forall n \in \mathbb{N}^+, \quad D_{\tilde{L}N}^{\geq n+1} \subseteq D_{\tilde{L}N}^{\geq n}; \quad D_{\tilde{L}N}^{\leq n-1} \subseteq D_{\tilde{L}N}^{\leq -n}. \]

Then we have

\[
\int_{t_0}^{t_0+k} \int_{\tilde{L}N} f \cdot n_{\tilde{L}N} \, ds \, dt = \sum_{n \in \mathbb{Z}} \int_{D_{\tilde{L}N}^{\leq n}} f(x, t_0) \, dx
\]

\[= \sum_{n \in \mathbb{N}^+} \int_{D_{\tilde{L}N}^{\geq n}} f(x, t_0) \, dx - \sum_{n \in \mathbb{N}^+} \int_{D_{\tilde{L}N}^{\leq -n}} f(x, t_0) \, dx
\]

\[= \int_{\gamma_D} F(x, t_0) \, dy + \sum_{n \in \mathbb{N}^+} \oint_{\partial D_{\tilde{L}N}^{\leq -n}} F(x, t_0) \, dy,
\]

where the first step follows from Theorem 3.8, the second step from (4.3), and the third step from applying the Green’s theorem on each \(D_{\tilde{L}N}^{\geq n}\) and \(D_{\tilde{L}N}^{\leq -n}\). Note that the orientation of \(\partial D_{\tilde{L}N}^{\leq -n}\) induced from \(\gamma_D\) is opposite to that of \(\partial D_{\tilde{L}N}^{\geq n}\), so the minus sign in the second step become the plus sign in the third step. The rest of the proof justifies the fourth step.

As a degenerate case, a 1D self-intersection of a closed curve is a maximal connected arc of self-intersection points with the same degree, i.e. the number of parameters mapped to the same point by the curve. In particular, a whisker is a special type of 1D self-intersection where each self-intersection point has an even degree. These whisker arcs do not have any contribution to the line integral as the even degree causes the line integral to cancel. Hence we can assume that \(\gamma_D\) is an oriented closed curve with a finite number of self-intersection points. Then it follows from [26, Prop 2.5] that \(\gamma_D\) can be decomposed into a number of oriented, pairwise non-crossing Jordan curves via cycle decomposition of an oriented closed curve, which is also known as the Seifert decomposition [16]. Due to (3.3), (4.1), and (4.3), this set of oriented Jordan curves is exactly the oriented Jordan curves in \(\bigcup_{n \in \mathbb{N}^+} \partial D_{\tilde{L}N}^{\geq n}\) and \(\bigcup_{n \in \mathbb{N}^+} \partial D_{\tilde{L}N}^{\leq -n}\). This justifies the fourth step and completes the proof.

The main steps of Lagrangian flux calculation is summarized in Algorithm 1.

Algorithm 1 mainly consists of two stages. In the first stage, i.e. steps 1-5, we calculate characteristic points of the preimage of \(\tilde{L}N\) and the backward streaklines seeded at \(L\) and \(N\), and then concatenate these points as a spline approximated donating region (SADR) of the exact DR \(D_{\tilde{L}N}^{\leq n}(t_0, k)\). More precisely, the closed spline \(\bar{\gamma}_D \approx \gamma_D\) is the discrete representation of the boundary of the DR; see [23, Sec. 4] for more details. In the second stage, i.e. step 6 in Algorithm 1, the flux integral \(I_E\) is estimated by (4.5) where the nodes \((x_{sij}, y_{sij})\) and the weights \(w_{sij}\) of numerical quadrature depend only on the SADR and the quadrature rule. For the Gauss-Legendre rule, values of \((x_{sij}, y_{sij})\) and \(w_{sij}\) are given in [23, Eqn. (42)]; see [23, Sec. 5] for more details.

An important detail of Algorithm 1 is the choice of the time step size \(\Delta t\) in constructing SADR. Denote by \(|\tilde{L}N|\) the arc length of \(\tilde{L}N\). We set

\[ \Delta t_p := \frac{k}{m}, \quad \text{where } m := \frac{|\tilde{L}N|}{h}, \]  

for time integration in obtaining breakpoints of the preimage of \(\tilde{L}N\). For a breakpoint
Algorithm 1: Lagrangian Flux Calculation.

**Input:** a velocity field \( u(x, t) \) and a scalar field \( f(x, t_0) \) that satisfy (1.2), a time interval \([t_0, t_0 + k]\), a fixed simple curve \( LN \) and its normal vector \( n_{LN} \), a length scale \( h \), a \( \kappa \)th-order ODE solver.

**Output:** a real number \( I_\kappa \) as an estimate of the flux integral

\[
I_E := \int_{t_0}^{t_0 + k} \int_{LN} f(s, t) u(s, t) \cdot n_{LN}(s, t) \, ds \, dt.
\]

1 divide \( LN \) into curve segments with roughly the same arc length.
2 calculate the preimage \( \phi^{-k}_{t_0 + k}(LN) \) by [23, Algorithm 1] with the \( \kappa \)th-order ODE solver.
3 construct streaklines \( \Psi^{-k}_{t_0 + k}(L) \) and \( \Psi^{-k}_{t_0 + k}(N) \) by [23, Algorithm 2] with the \( \kappa \)th-order ODE solver.
4 if the distance between any two adjacent breakpoints in step 1 or 2 is greater than \( h \), insert more breakpoints so that \( h \) is no less than the maximum distance between any two adjacent breakpoints of the same type.
5 concatenate the breakpoints of \( LN, \Psi^{-k}_{t_0 + k}(N), \phi^{-k}_{t_0 + k}(LN) \), and \( \Psi^{-k}_{t_0 + k}(L) \) to form a spline \( \gamma_D \) of order \( \kappa \) as an approximation to \( \gamma_D \).
6 compute

\[
\oint_{\gamma_D} f(x, y, t_0) \, dx \, dy \approx I_\kappa(h) = \sum_{s=1}^{4n} \sum_{i=1}^{n} \sum_{j=1}^{m} w_{sij} f(x_{sij}, y_{sij}, t_0),
\]

where the quadrature nodes \((x_{sij}, y_{sij})\) and weights \(w_{sij}\) are the same as those given in [23, Eqn. (42)].

on a streakline that arrives at the seeding location at time \( t_0 + \tau \), we set

\[
\forall \tau \in (0, k), \quad \Delta t_x(t_0 + \tau) := \min\left(\frac{k}{m}, \tau\right).
\]

In other words, each breakpoint with \( \tau \leq \frac{k}{m} \) is obtained by a single time step of the \( \kappa \)th-order ODE solver.

We also define the relative LFC error of Algorithm 1 as

\[
E_\kappa(h) := \left| \frac{I_\kappa(h) - I_E}{I_E} \right|
\]

and the corresponding LFC convergence rate as

\[
O_\kappa(h) = \log \frac{E_\kappa(rh)}{E_\kappa(h)},
\]

where the refinement factor \( r \) is usually taken to be 2. With (4.6), (4.7), and the analysis in [23], it can be shown that the convergence rate of Algorithm 1 is \( \kappa \).

We summarize distinguishing features of Algorithm 1 as follows.
(Adv-1) Simplicity: neither self-intersections of the DR boundary nor the winding numbers need to be computed.

(Adv-2) High accuracy: results of $\kappa = 4, 6$ are much more accurate than those of $\kappa = 2$; see Section 5.

(Adv-3) Well conditioning: (i) a algorithm that computes the intersection of lines or splines can be arbitrarily ill-conditioned, which is avoided in Algorithm 1, (ii) the spline fitting in step 5 of Algorithm 1 concerns solving a well-conditioned linear system, thanks to enforcing a uniform upper bound on the distances between adjacent breakpoints\(^1\), and (iii) the final formula (4.5) only involves function evaluation, addition, and multiplication.

(Adv-4) Amenability to parallel computing: each step of Algorithm 1 can be divided into almost independent tasks so that the synchronization from local processors into the global result require a minimum amount of time.

5. Numerical Tests. In this section we perform a variety of numerical tests for incompressible and compressible flows, for normal and non-normal DRs, and for DRs with degree 1 or greater. Results of these tests confirm the validity of Theorem 3.8 and Corollary 4.1, and demonstrate the high accuracy and efficiency of Algorithm 1.

The convergence rates $\kappa = 2, 4, 6$ are tested, with the matching splines being linear, cubic, and quintic, respectively, and with the matching ODE solvers being the modified Euler method [8, p. 155], the classical fourth-order Runge-Kutta method [3, p. 138], and Verner’s method of order 6(5) [19], respectively. These ODE solvers are chosen for the sole reason of easy implementation; other time integrators with the same order-of-accuracy and larger stability regions would yield qualitatively the same results.

As another important detail, the order of the quadrature rule as in the RHS of (4.5) is chosen to be $\kappa + 2$ so that asymptotically the error caused by the quadrature formula is not the leading term in the LFC error $E_\kappa(h)$. This choice helps us focusing on the main purposes of this subsection.

5.1. A canonical DR in exponentially divergent flow. The velocity field of this compressible flow and the passively advected scalar function are given as

\[
\mathbf{u}(x, y, t) = \begin{bmatrix} \exp(x) - 1 - t \\ \exp(y) + t \end{bmatrix},
\]

(5.1)

\[
f(x, y, t) = \exp(-(x + y + t)).
\]

(5.2)

It is easily checked that $\mathbf{u}$ and $f$ satisfy the scalar conservation law (1.2).

For the line segment that connects $L = (1, 0)$ and $N = (0, 0)$, we first approximate its DR within the time integral $[t_0, t_0 + k] = [0, 1]$ by a SADR constructed in steps 1-5 of the LFC Algorithm 1; see Figure 5.1. We then calculate $I_\kappa$ by (4.5) as an estimate of the flux integral $I_F$. As shown in Table 5.1, all of the desired convergence rates $\kappa = 2, 4, 6$ are achieved. In particular, the LFC errors for cubic and quintic SADRs are smaller than those of linear SADRs by orders of magnitudes.

\[^1\text{Strictly speaking, one should also enforce a uniform lower bounded on the distances between adjacent breakpoints by, for example, replacing a pair of breakpoints too close to each other with their midpoint. However, the lower bound should be chosen carefully so that it neither deteriorates the convergence rates nor introduce a large variation of distances between adjacent breakpoints.}\]
A SADR in the exponentially divergent flow (5.1) constructed by the steps 1-5 of Algorithm 1 with \([t_0, t_0 + k] = [0, 1]\), \(\kappa = 2\), \(h = \frac{1}{16}\), and \(LN\) being the thick line segment connecting \((1, 0)\) and \((0, 0)\). Breakpoints on the streaklines and the preimage of \(LN\) are represented by “***” and “♦,” respectively. The shaded region represents \(D_{LN}^{+1}(t_0, k)\).

**Table 5.1**

Errors and convergence rates of LFC Algorithm 1 for the exponentially divergent test in Figure 5.1. \(E_\kappa(h)\) and \(O_\kappa\) are defined in (4.8) and (4.9), respectively.

<table>
<thead>
<tr>
<th>(\kappa)</th>
<th>(E_\kappa\left(\frac{1}{16}\right))</th>
<th>(O_\kappa)</th>
<th>(E_\kappa\left(\frac{1}{32}\right))</th>
<th>(O_\kappa)</th>
<th>(E_\kappa\left(\frac{1}{64}\right))</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>7.73e-04</td>
<td>2.04</td>
<td>1.87e-04</td>
<td>2.02</td>
<td>4.61e-05</td>
</tr>
<tr>
<td>4</td>
<td>9.84e-07</td>
<td>4.03</td>
<td>6.02e-08</td>
<td>4.04</td>
<td>3.67e-09</td>
</tr>
<tr>
<td>6</td>
<td>1.44e-09</td>
<td>6.48</td>
<td>1.62e-11</td>
<td>6.22</td>
<td>2.17e-13</td>
</tr>
</tbody>
</table>

**5.2. A non-normal DR of degree 1 in an incompressible flow.** The velocity field of this test

\[
\mathbf{u}(x, y, t) = \left[\begin{array}{c} -y \\ x - t + 5 \end{array}\right]
\]

has already been used in Figure 3.4 to generate a SADR within the time interval \([t_0, t_0 + k] = [0, 10]\) for the line segment connecting \(N = (0, 0)\) and \(L = (1, 0)\). As shown in Figure 3.4, the streakline \(\Psi_{t_0+k}^{-k}(N)\) not only is self-intersecting but also intersects the other streakline \(\Psi_{t_0+k}^{-k}(L)\). Hence the resulting DR is indeed non-normal.

The initial scalar function is chosen as

\[
f(x, y) = x^2 + y^2,
\]

and it clearly satisfies the scalar conservation law (1.2) with the velocity in (5.3).

Since the DR is associated with a long time interval, values of the length-scale \(h\) for Algorithm 1 are set to be much smaller than those in the previous test. The corresponding LFC errors and convergence rates are listed in Table 5.2, where the second-order and fourth-order convergence rates are verified.
Table 5.2
Errors and convergence rates of LFC Algorithm 1 for the non-normal DR test of Section 5.2; see Figure 3.4 for the corresponding SADR. $E_\kappa(h)$ and $O_\kappa$ are defined in (4.8) and (4.9), respectively.

<table>
<thead>
<tr>
<th>$\kappa$</th>
<th>$E_\kappa(\frac{1}{64})$</th>
<th>$O_\kappa$</th>
<th>$E_\kappa(\frac{1}{128})$</th>
<th>$O_\kappa$</th>
<th>$E_\kappa(\frac{1}{256})$</th>
<th>$O_\kappa$</th>
<th>$E_\kappa(\frac{1}{512})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>7.54e-03</td>
<td>2.01</td>
<td>1.88e-03</td>
<td>2.03</td>
<td>4.60e-04</td>
<td>2.02</td>
<td>1.13e-04</td>
</tr>
<tr>
<td>4</td>
<td>9.44e-06</td>
<td>4.01</td>
<td>5.88e-07</td>
<td>4.17</td>
<td>3.28e-08</td>
<td>3.92</td>
<td>2.17e-09</td>
</tr>
<tr>
<td>6</td>
<td>8.18e-07</td>
<td>4.32</td>
<td>4.10e-08</td>
<td>-0.75</td>
<td>6.90e-08</td>
<td>5.74</td>
<td>1.29e-09</td>
</tr>
</tbody>
</table>

Table 5.3
Errors and convergence rates of LFC Algorithm 1 for tests in Section 5.2 where the DRs have degrees greater than one. $E_\kappa(h)$ and $O_\kappa$ are defined in (4.8) and (4.9), respectively.

<table>
<thead>
<tr>
<th>$\kappa$</th>
<th>$E_\kappa(\frac{1}{64})$</th>
<th>$O_\kappa$</th>
<th>$E_\kappa(\frac{1}{128})$</th>
<th>$O_\kappa$</th>
<th>$E_\kappa(\frac{1}{256})$</th>
<th>$O_\kappa$</th>
<th>$E_\kappa(\frac{1}{512})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>DRs with degree 2, $[t_0, t_0+k] = [0, 1.25]$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>3.56e-03</td>
<td>1.73</td>
<td>1.07e-03</td>
<td>1.88</td>
<td>2.91e-04</td>
<td>1.95</td>
<td>7.56e-05</td>
</tr>
<tr>
<td>4</td>
<td>6.12e-06</td>
<td>3.87</td>
<td>4.18e-07</td>
<td>3.94</td>
<td>2.72e-08</td>
<td>3.97</td>
<td>1.74e-09</td>
</tr>
<tr>
<td>6</td>
<td>2.62e-10</td>
<td>6.03</td>
<td>4.00e-12</td>
<td>6.03</td>
<td>6.13e-14</td>
<td>4.11</td>
<td>3.56e-15</td>
</tr>
<tr>
<td>DRs with degree 3, $[t_0, t_0+k] = [0, 2.25]$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>7.94e-03</td>
<td>1.36</td>
<td>3.10e-03</td>
<td>1.76</td>
<td>9.13e-04</td>
<td>1.90</td>
<td>2.45e-04</td>
</tr>
<tr>
<td>4</td>
<td>5.70e-05</td>
<td>3.74</td>
<td>4.27e-06</td>
<td>3.89</td>
<td>2.88e-07</td>
<td>3.95</td>
<td>1.87e-08</td>
</tr>
<tr>
<td>6</td>
<td>9.66e-09</td>
<td>6.09</td>
<td>1.42e-10</td>
<td>6.03</td>
<td>2.16e-12</td>
<td>6.10</td>
<td>3.15e-14</td>
</tr>
</tbody>
</table>

In the case of $\kappa = 6$, results in Table 5.2 shows a temporary deterioration of convergence rates on the first three grids; this is probably caused by the $C^1$ discontinuities (i.e. the cusps) of the streaklines in Figure 3.4 and our no special treatments for these cusps. However, the convergence rate becomes very close to six once we refine the grids further. This is not unexpected since the LFC error as defined in (4.8) is essentially a 1-norm and therefore the cusps as a set of measure zero should not affect the asymptotic convergence rate. Nonetheless, the accuracy of the cubic LFC is close to that of the quintic LFC, which suggests that, for a DR with potential cusps on its edges, $\kappa = 4$ is probably a more efficient choice than $\kappa = 6$.

5.3. Normal DRs of Degree greater than 1 in compressible flows. In this test, the velocity field and the passively advected scalar function are

$$u(x, y, t) = \begin{bmatrix} x + 2\pi y \\ -2\pi x + y \end{bmatrix},$$

$$f(x, y, t) = (x^2 + y^2)e^{-4t}.$$  

It is easily checked that $\nabla \cdot u \neq 0$ and that $u$ and $f$ satisfy the scalar conservation law (1.2).

For time intervals $[t_0, t_0+k] = [0, 1.25]$, $[0, 2.25]$ and the line segment that connects $L = (1, 0)$ and $N = (0, 0)$, we plot two corresponding linear SADRs in Figure 5.2. Due to the singular velocity at $N$, the streakline $\Psi_{t_0+k}(N)$ reduces to the singleton.
point set \( \{N\} \). In contrast, the large magnitude of the velocity field away from \( N \) leads to the long spiraling streakline \( \Psi_{-k}^{-k}(L) \) and the consequent DRs with degrees greater than 1; see Figure 5.2.

We input four small length scales into Algorithm 1 to resolve fine details of the long streakline seeded at \( L \). In Table 5.3, all the desired convergence rates \( \kappa = 2, 4, 6 \) are achieved. In particular, the LFC error for the degree-2 DR on the finest grid is very close to machine precision. Apart from explaining the slightly reduced convergence rate “4.11,” this extremely small error strongly supports the discussion in Section 4 that our LFC Algorithm 1 has the property of excellent conditioning.

In summary, the test results in Tables 5.1, 5.2, and 5.3 confirms the validity of Theorem 3.8 and Corollary 4.1. In addition, we observe that our LFC algorithm based on cubic or quintic SADRs is much more accurate than one based on linear SADRs.

Finally, the test results in all tables for all length scales are generated within 40 minutes by using the first author’s laptop with a single Intel® Core™ i7-5500 CPU running at 2.40GHz. Considering the accuracy of results reported in this section, this short CPU time demonstrates the advantages of flexibility and efficiency of the proposed LFC algorithm over the traditional FV-MOL approach, as discussed in Section 1. It would be extremely difficult, if not impossible, for any FV-MOL method to achieve the same efficiency and accuracy of estimating the total flux across a curve by solving (1.2) over the whole computational domain!

6. Conclusion. By extending our previous work on donating regions \([25, 23, 26]\), we prove a flux identity for scalar conservation laws and propose an LFC algorithm for calculating flux integrals through a simple curve within a time interval \([t_0, t_0 + k]\) of any \( k \in (0, \infty) \). Our LFC algorithm requires no evolution information of the scalar function during the interval \([t_0, t_0 + k]\) except at the very initial time \( t_0 \). Thanks to the generality of the flux identity, our LFC algorithm is also free of spline-intersection computation and winding-number calculation. Other notable advantages of it include its simplicity, flexibility, excellent conditioning, high efficiency, and extreme accuracy.
In future research, we will extend the flux identity and augment the LFC algorithm to higher-dimensional Euclidean spaces and to flows on manifolds.

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