1. Let $p : \tilde{X} \to X$ be a simply connected covering space of $X$. Let $A \subset X$ be path connected and locally path connected, and let $\tilde{A} \subset \tilde{X}$ be a path component of $p^{-1}(A)$. In the course we proved that $p : \tilde{A} \to A$ is a covering space.

**Question:** Show that $p_*(\pi(\tilde{A}, \tilde{x}_0))$ is the kernel of the homomorphism $i_* : \pi(A, x_0) \to \pi(X, x_0)$, where $\tilde{x}_0 \in \tilde{A}$, $x_0 = p(\tilde{x}_0) \in A$, and $i : A \hookrightarrow X$ is the inclusion map.

**Solution:** Let $j : \tilde{A} \to \tilde{X}$ be the inclusion map. Then $i \circ p = p \circ j$. Hence $i_*p_* = p_*j_*$, so that $i_*p_*$ factors through $\pi(\tilde{X}, \tilde{x}_0)$, which is trivial. It remains to show that $\text{Ker} \ i^* \subset \text{Im} \ p_*$. Let $[f] \in \pi(A, x_0)$ so that $[i \circ f]$ is trivial in $\pi(X, x_0)$. We know that $f$ lifts uniquely to a path $\tilde{f}$ in $\tilde{A}$ with initial point $\tilde{x}_0$, and $i \circ f$ lifts uniquely to a path $\tilde{f}'$ in $\tilde{X}$ with initial point $\tilde{x}_0$. But $[i \circ f]$ is trivial, hence the lift $\tilde{f}'$ must be a loop in $\tilde{X}$. The uniqueness assertion also implies that $\tilde{f}' = j \circ \tilde{f}$. Therefore $\tilde{f}$ is a loop in $\tilde{A}$ and $[f] = p_*[\tilde{f}]$. □

2. Show that if a path connected, locally path connected space $X$ has finite fundamental group, then every continuous map $f : X \to S^1$ is homotopic to a constant map.

**Solution:** Consider the universal covering space $p : \mathbb{R} \to S^1$. Since $f_* (\pi(X))$ is a finite subgroup of $\pi(S^1) \cong \mathbb{Z}$, it must be trivial. Hence $f_* (\pi(X)) = p_* (\pi(\mathbb{R}))$, and therefore $f$ lifts to a map $g : X \to \mathbb{R}$ such that $f = p \circ g$. Let $F : \mathbb{R} \times I \to \mathbb{R}$ be a deformation retract of $\mathbb{R}$ to a point. Then $X \times I \to S^1$, $(x, t) \mapsto p \circ F(g(x), t)$ is a homotopy from $f$ to a constant map. □

3. (a) Show that $H_0(X, A) = 0$ if and only if $A$ meets each path component of $X$.

(b) Show that $H_1(X, A) = 0$ if and only if $H_1(A) \to H_1(X)$ is surjective and each path component of $X$ contains at most one path component of $A$.

**Solution:** Recall the long exact sequence

$$\cdots \to H_1(A) \to H_1(X) \to H_1(X, A) \to H_0(A) \to H_0(X) \to H_0(X, A) \to 0.$$

(a) $H_0(X, A) = 0$ if and only if $H_0(A) \to H_0(X)$ is surjective, if and only if $H_0(A \cap X_\gamma) \to H_0(X_\gamma)$ is surjective for each path component $X_\gamma$ of $X$, if and only if $A \cap X_\gamma \neq \emptyset$ for each $X_\gamma$. 

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(b) $H_1(X, A) = 0$ if and only if $H_1(A) \to H_1(X)$ is surjective and $H_0(A) \to H_0(X)$ is injective. That $H_0(A) \to H_0(X)$ is injective if and only if $H_0(A \cap X_0) \to H_0(X_0)$ is injective for each $X_0$, if and only if $A \cap X_0$ is path connected whenever it is non-empty, i.e. $X_0$ contains at most one path component of $A$. \hfill \Box

4. Let $X_n \subset \mathbb{R}^2$ be the union of $n$ circles of diameter 1 centered at $(1,0), (2,0), \ldots, (n,0)$. Compute $\pi(X_n)$ and $H_1(X_n)$.

**Solution:** It is clear that $X_1 \cong \mathbb{Z}$. We prove by induction that $\pi(X_n)$ is a free product on $n$ generators. Indeed, assume that $\pi(X_{n-1}) \cong \mathbb{Z} \ast \cdots \ast \mathbb{Z}$ ($n-1$ copies of $\mathbb{Z}$). Let $Y_n$ be the unit circle centered at $(n,0)$. Let $U, V$ be open subsets which have deformation retracts to $X_{n-1}$ and $Y_n$ such that $U \cap V$ is contractible. Then by the van Kampen Theorem, $\pi(X_n)$ is the free product of $\pi(X_{n-1})$ and $\pi(Y_n) \cong \mathbb{Z}$. Hence the induction follows.

Since $H_1(X_n)$ is the abelianization of $\pi(X_n)$, we see immediately that $H_1(X_n) \cong \mathbb{Z}^n$. Alternatively, using homology of graph one can compute easily that $\chi(X_n) = 1-n$ thus rank $H_1(X_n) = n$, which implies that $H_1(X_n) \cong \mathbb{Z}^n$. \hfill \Box

5. Find the local homology groups $H_n(X, X - \{x\})$ for various points $x$ in the Mobius band and in the annulus $S^1 \times I$. Then show that these two spaces are not homeomorphic. [Consider their boundaries.]

**Solution:** We can find a contractible neighborhood $N$ of $x$ so that $N - \{x\}$ has deformation retract to $B$, the boundary of $N - \{x\}$. Thus from the long exact sequence

$$\cdots \to \tilde{H}_n(N - \{x\}) \to \tilde{H}_n(N) \to H_n(N, N - \{x\}) \to \tilde{H}_{n-1}(N - \{x\}) \to \cdots$$

we deduce that $H_n(N, N - \{x\}) \cong \tilde{H}_{n-1}(B)$.

If $x$ is an interior point, then $N$ can be taken to be a closed disk and $B$ is homeomorphic to $S^1$. In this case $H_n(X, X - \{x\}) \cong \tilde{H}_{n-1}(S^1) \cong \mathbb{Z}$ for $n = 2$ and 0 otherwise. If $x$ is a boundary point, then again $N$ is a closed disk but $x$ is a boundary point of $N$. In this case $B$ is homeomorphic to an open interval, so that $H_n(X, X - \{x\}) \cong \tilde{H}_{n-1}(B) = 0$ for all $n$.

Thus points $x$ with $H_2(X, X - \{x\}) = 0$ are exactly the boundary points. The boundary of the Mobius band is $S^1$, but the boundary of the annulus is a disjoint union of two $S^1$ which is disconnected. Thus the Mobius band and the annulus cannot be homeomorphic. \hfill \Box

6. Let $X$ be an oriented compact surface of genus 2, or equivalently the connected sum of two tori $a_1 \times b_1$ and $a_2 \times b_2$, where $a_i, b_i, i = 1, 2$ are all homeomorphic to $S^1$. There is a simple closed curve $\gamma$ that is homotopic to $a_1 b_1 a_1^{-1} b_1^{-1}$. Let $Y$ be the union of $X$ with a 2-dimensional disk $D$, where the boundary of $D$ is identified with $\gamma$.

Let $U$ and $V$ be open neighborhoods of $X$ and $D$, such that $X, D$ and $\gamma$ are deformation retracts of $U, V$ and $U \cap V$ respectively.

(a) Apply van Kampen’s Theorem for $U, V$ to compute $\pi(Y)$.

(b) Apply Mayer-Vietoris exact sequence for $U, V$ to compute $H_n(Y)$. 

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Solution: (a) We have $\pi(U) \cong (\mathbb{Z}a_1 * \mathbb{Z}b_1 * \mathbb{Z}a_2 * \mathbb{Z}b_2)/N$, where $N$ is the smallest normal subgroup containing $[a_1,b_1][a_2,b_2]$, $\pi(V)$ is trivial, and $\pi(U \cap V) \cong \mathbb{Z}[a_1,b_1]$. The van Kampen Theorem implies that $\pi(Y)$ is the quotient of $\pi(U)$ by the smallest normal subgroup containing $\pi(U \cap V)$. Thus $[a_1,b_1] = [a_2,b_2] = 1$ in $\pi(Y)$ and $\pi(Y) \cong \mathbb{Z}^2 * \mathbb{Z}^2$.

(b) Since $V$ is contractible and $U \cap V$ has deformation retract to $S^1$, from the Mayer-Vietoris exact sequence for reduced homology we have an exact sequence

$$0 \to H_2(X) \to H_2(Y) \to H_1(S^1) \to H_1(X) \to H_1(Y) \to 0.$$  

The map $H_1(S^1) \to H_1(X)$ sends the generator $[\gamma]$ of $H_1(S^1) \cong \mathbb{Z}$ to $a_1 + b_1 - a_1 - b_1 = 0$, hence $H_1(Y) \cong H_1(X) \cong \mathbb{Z}^4$, and we have a split exact sequence

$$0 \to H_2(X) \to H_2(Y) \to \mathbb{Z} \to 0,$$

which implies that $H_2(Y) \cong \mathbb{Z}^2$. Of course $H_0(Y) \cong \mathbb{Z}$ because everything is connected.

Note that $H_1(Y)$ is the abelianization of $\pi(Y)$, as it should be. $\square$