GC^n continuity conditions for adjacent rational parametric surfaces

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Abstract

In this paper, the constraints on the homogeneous surface belonging to a certain rational surface are derived which are both necessary and sufficient to ensure that the rational surface is nth-order geometric continuous. This gives up the strong restriction that requires the homogeneous surface to be as smooth as the rational surface. Further the conditions for the rectangular rational Bézier patches are developed, and some simple and practical sufficient conditions are presented which might give a valid means for the construction of GC^n connecting surfaces.

Keywords: Rational surfaces; Bézier patches; Geometric continuity; Total differential vectors; Connecting functions; Parameter transformation

1. Introduction

In recent years geometric continuity between parametric surfaces has been receiving considerable attention in the field of computer aided geometric design (CAGD) (Barnhill, 1985). There was much research done on the geometric continuity condition and its applications. Most of the work concerns the sufficient conditions of tangent plane continuity or curvature continuity for Bézier surfaces based on certain simplifying assumptions (Boehm, 1988; Farin, 1982; Kahmann, 1983; Veron et al., 1976).

One effort to generalize this is the study of necessary and sufficient conditions for tangent plane continuity (Liu, 1986, Liu and Hoschek, 1989). Degen (1990)

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deduced explicit representations for $G^1$ and $G^2$ continuity between adjacent Bézier patches. However, for some special applications such as finite element analysis and ship hull design, higher orders of geometric continuity are needed. Therefore another generalization is the work on the geometric continuity of order $n$. Hahn (1989) described the characterization of geometric continuity in terms of diffeomorphism. Liang (1990) gave a theoretical foundation for the geometric continuity of arbitrary order.

In geometric modeling, rational surfaces, such as rational Bézier surfaces, are widely used. This is due in part to the fact that they possess many nice properties, one of which is the capability of describing exactly the conic surfaces which are commonly used in engineering. Nonetheless, very little previous work has been done on the rational surface of geometric continuity (Vincacu and Brunet, 1989; DeRose, 1990; Liu, 1990; Zheng et al., 1992). In general, the geometric continuity of rational surfaces is often ensured by requiring the associated homogeneous surfaces to possess the same continuity. However, it is only sufficient. DeRose and Liu, respectively, presented a system of necessary and sufficient conditions to ensure tangent plane continuity (DeRose, 1990; Liu, 1990). (Zheng et al., 1992) discussed curvature continuity between rational Bézier patches and its solutions. In this paper, we will derive the necessary and sufficient conditions of $n$th-order geometric continuity for the rational surface which are represented by its associated homogeneous surface and the so-called connecting functions. Then the conditions for rational Bézier patches and smooth connection are further analyzed. The main idea is using homogeneous coordinates for rational surfaces to make derivations simple, as for rational Bézier curves in (Degen, 1988; Hohmeyer and Barsky, 1989).

2. Preliminaries

This section introduces some terms used in the rest of the paper.

Let $E^m$ be the Euclidean space of dimension $m$, and $\Delta$ be a region in $E^2$. A $C^n$ surface of $E^m$ is defined as an $n$-times continuously differentiable mapping $r : \Delta \rightarrow E^m$, and is expressed by

$$r = r(u, v) \in C^n(\Delta).$$

(2.1)

In this paper, a surface $r$ of $E^3$ is always assumed to be regular, i.e.

$$\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \neq 0.$$

(2.2)

Assume that $r(u, v)$ and $\tilde{r}(\tilde{u}, \tilde{v})$ are two $C^n$ surfaces of $E^3$, and curve $CB$ is the intersection curve of $r$ and $\tilde{r}$ which is also $C^n$. We call $CB$ the common boundary curve of $r$ and $\tilde{r}$. $CB$ responds to edge $E_r(s) = (u(s), v(s))$ in domain $\Delta_1$ of $r$ and edge $E_{\tilde{r}}(s) = (\tilde{u}(s), \tilde{v}(s))$ in domain $\Delta_2$ of $\tilde{r}$ respectively.
**Definition 1.** Let \( r(u, v) \) and \( \tilde{r}(\tilde{u}, \tilde{v}) \) be two surfaces possessing a common boundary curve \( CB \). If there exists a \( C^n \)-diffeomorphism \( \varphi \):

\[
\begin{bmatrix}
\tilde{u} - \bar{u}(u, v) \\
\bar{u} = \tilde{\varphi}(u, v)
\end{bmatrix} \in C^n
\]  

such that

\[
\frac{\partial^i r}{\partial u^i \partial v^{3-i}} \bigg|_{CB} = \frac{\partial^i (\tilde{r} \circ \varphi)}{\partial \tilde{u}^i \partial \tilde{v}^{3-i}} \bigg|_{CB}, \quad i = 0, \ldots, s, \quad s = 0, \ldots, n
\]  

then \( r \) and \( \tilde{r} \) are said to meet along \( CB \) with geometric continuity of order \( n \) or \( GC^n \). Meanwhile the diffeomorphism \( \varphi \) is called the parameter transformation of \( GC^n \) between \( r \) and \( \tilde{r} \).

The definition has an intuitive geometric meaning (see Fig. 1). It expresses that if surfaces \( r \) and \( \tilde{r} \) meet along \( CB \) with \( GC^n \), then there exists a transformation (2.3) so that in the neighbourhood of every point \( P_0 \) along \( CB \), any point \( P \) in surface \( r \) (or \( \tilde{r} \)) and its corresponding point \( \tilde{P} \) in surface \( \tilde{r} \) (or \( r \)) under the transformation satisfy

\[
\lim_{P \to P_0} \| \tilde{P} - P \| = 0,
\]

where order \( n \) is the measure of contact between \( r \) and \( \tilde{r} \).

By differential geometry, it is possible to show that this definition accords with tangent plane continuity for \( n = 1 \) and curvature continuity for \( n = 2 \).
Definition 2. The first-order total differential vector of surface $\mathbf{r}$ is defined by $d\mathbf{r} = r_u \, du + r_v \, dv$. The total differential vectors of higher order are defined recursively by $d^0\mathbf{r} = \mathbf{r}$, and $d^k\mathbf{r} = d(d^{k-1}\mathbf{r})$.

Lemma 1. We have $d^0\mathbf{r} = \mathbf{r}$, and

$$d^s\mathbf{r} = \sum_{k=1}^s \sum_{r_1 + \cdots + r_k = s} \frac{A_{r_1 \cdots r_k}^{k \lambda}}{r_1! \cdots r_k!} \, d^{r_1}u \cdots d^{r_k}v \cdot \frac{\partial^s \mathbf{r}}{\partial u^{r_1} \partial v^{r_k}},$$

where $r_i$ are positive integers, and

$$A_{r_1 \cdots r_k}^{k \lambda} = \frac{s!}{h!(k-h)! r_1! \cdots r_k!}.$$

Proof. Formula (2.4) is obviously true for $s = 0, 1$. Assume it holds for $s$ ($s \geq 1$), then

$$d^{s+1}\mathbf{r} = d(d^s\mathbf{r})$$

$$= \sum_{k=2}^s \sum_{r_1 + \cdots + r_k = s+1} \sum_{r_2 = 1}^k \frac{A_{r_1 r_2 \cdots r_k}^{k \lambda}}{r_1! \cdots r_k!} \cdot d^{r_2}u \cdots d^{r_k}v \cdot \frac{\partial^{s+1} \mathbf{r}}{\partial u^{r_1} \partial v^{r_k}}$$

$$+ \sum_{k=2}^s \sum_{r_1 + \cdots + r_k = s} \sum_{r_2 = 1}^{k-1} \frac{A_{r_1 r_2 \cdots r_k}^{k \lambda}}{r_1! \cdots r_k!} \cdot \frac{\partial^s \mathbf{r}}{\partial u^{r_1} \partial v^{r_k}} \cdot \sum_{i=1}^{k-1} (d^{r_2}u \cdots d^{r_i}v \cdots d^{r_k}v)$$

$$= \sum_{k=2}^{s+1} \sum_{r_1 + \cdots + r_k = s+1} \frac{A_{r_1 r_2 \cdots r_k}^{k \lambda}}{r_1! \cdots r_k!} \cdot d^{r_2}u \cdots d^{r_k}v \cdot \frac{\partial^s \mathbf{r}}{\partial u^{r_1} \partial v^{r_k}}$$

$$+ \sum_{k=2}^{s+1} \sum_{r_1 + \cdots + r_k = s} \sum_{r_2 = 1}^{k-1} \frac{A_{r_1 r_2 \cdots r_k}^{k \lambda}}{r_1! \cdots r_k!} \cdot \frac{\partial^s \mathbf{r}}{\partial u^{r_1} \partial v^{r_k}} \cdot \frac{\partial^s \mathbf{r}}{\partial u^{r_2} \partial v^{r_k}}$$

$$= \sum_{k=2}^{s+1} \sum_{r_1 + \cdots + r_k = s+1} \frac{A_{r_1 r_2 \cdots r_k}^{k \lambda}}{r_1! \cdots r_k!} \cdot d^{r_2}u \cdots d^{r_k}v \cdot \frac{\partial^s \mathbf{r}}{\partial u^{r_1} \partial v^{r_k}}$$

$$+ \sum_{k=2}^{s+1} \sum_{r_1 + \cdots + r_k = s} \sum_{r_2 = 1}^{k-1} \frac{A_{r_1 r_2 \cdots r_k}^{k \lambda}}{r_1! \cdots r_k!} \cdot \frac{\partial^s \mathbf{r}}{\partial u^{r_1} \partial v^{r_k}} \cdot \frac{\partial^s \mathbf{r}}{\partial u^{r_2} \partial v^{r_k}}.$$
\[
\begin{align*}
&+ \sum_{k=1}^{s+1} \sum_{i=1}^{k} \sum_{r_1, \ldots, r_k = 0}^{s+1} \sum_{r_{s+1} = 0}^{k} r_i A^{k}_{r_1, \ldots, r_k} \cdot d^{r_1} u \cdot \cdots \cdot d^{r_k} v \cdot \frac{\partial^{s+1} r}{\partial u^{r_i} \partial v^{s+1-k}} \\
&= \sum_{k=1}^{s+1} \sum_{i=1}^{k} \sum_{r_1, \ldots, r_k = 0}^{s+1} \sum_{r_{s+1} = 0}^{k} r_i A^{k}_{r_1, \ldots, r_k} \cdot d^{r_1} u \cdot \cdots \cdot d^{r_k} v \cdot \frac{\partial^{s+1} r}{\partial u^{r_i} \partial v^{s+1-k}} \\
&= (s + 1) \cdot \sum_{k=1}^{s+1} \sum_{r_1, \ldots, r_k = 0}^{s+1} \sum_{r_{s+1} = 0}^{k} A^{k}_{r_1, \ldots, r_k} \cdot d^{r_1} u \cdot \cdots \cdot d^{r_k} v \cdot \frac{\partial^{s+1} r}{\partial u^{r_i} \partial v^{s+1-k}} \\
&= \sum_{k=1}^{s+1} \sum_{r_1, \ldots, r_k = 0}^{s+1} \sum_{r_{s+1} = 0}^{k} A^{(s+1)k}_{r_1, \ldots, r_k} \cdot d^{r_1} u \cdot \cdots \cdot d^{r_k} v \cdot \frac{\partial^{s+1} r}{\partial u^{r_i} \partial v^{s+1-k}}.
\end{align*}
\]

By induction, the formula (2.4) is obtained. \(\Box\)

By (2.4), the total differential of \(r\) to the \(s\)th-order is given by the partial derivatives of \(r\) and the total differentials of \(du, dv\), all of which up to the \(s\)th-order.

**Definition 3.** Rational parametric surface \(r(u, v)\) is defined as

\[
r(u, v) = R(u, v)/\omega(u, v), \quad (u, v) \in \Delta
\]

where \(R(u, v)\) is a surface of \(E^3\), \(\omega(u, v)\) is a function \(\Delta \rightarrow E^1\).

The rational surface \(r\) can be thought of as the composition of a surface \(Q(u, v)\) of \(E^4\) with a projection function \(p\), where

\[
p: (R, \omega) \rightarrow R/\omega,
\]

\[
Q(u, v) = \rho(u, v)(R(u, v), \omega(u, v)).
\]  

We refer to \(Q\) as the homogeneous surface associated with \(r\) and surface \(r\) as the projection of \(Q\). In general, we take \(\rho = 1\) and suppose that \(Q(u, v)\) is \(C^n\).

**Lemma 2.** If the rational surface \(r\) of \(E^3\) is regular at point \((u, v)\), then \(Q(u, v), Q_0(u, v)\) and \(Q_i(u, v)\) are linearly independent, i.e., there exists at least one \(i\) \((1 \leq i \leq 3)\) such that \(\langle Q, Q_i, Q_{-i}\rangle \neq 0\) at \((u, v)\), where the notation \(\langle \cdot, \cdot, \cdot \rangle\) means

\[
\langle R, S, T \rangle_i = (-1)^{i+1} \det \begin{bmatrix}
R_{i+1} & S_{i+1} & T_{i+1} \\
R_{i+2} & S_{i+2} & T_{i+2} \\
R_{i+3} & S_{i+3} & T_{i+3}
\end{bmatrix}, \quad i = 1, 2, 3, 4.
\]  

\(R, S, T\) are arbitrary vectors in \(E^4\), \(R_i\) denotes the \(i\)th component of \(R\). If \(i > 4\), \(R_i = R_{(i-1)4+1}\).
\[
+ \sum_{k=1}^{s+1} \sum_{r_1+\cdots+r_k=s+1} \sum_{h=0}^{k} r_i A^{(i)}_{r_1 \cdots r_k} \cdot d^{r_1} u \cdots d^{r_k} v \cdot \frac{\partial^k r}{\partial u^a \partial v^{k-h}}
\]

\[
= \sum_{k=1}^{s+1} \sum_{r_1+\cdots+r_k=s+1} \sum_{h=0}^{k} r_i A^{(i)}_{r_1 \cdots r_k} \cdot d^{r_1} u \cdots d^{r_k} v \cdot \frac{\partial^k r}{\partial u^a \partial v^{k-h}}
\]

\[
= (s+1) \sum_{k=1}^{s+1} \sum_{r_1+\cdots+r_k=s+1} \sum_{h=0}^{k} A^{(i)}_{r_1 \cdots r_k} \cdot d^{r_1} u \cdots d^{r_k} v \cdot \frac{\partial^k r}{\partial u^a \partial v^{k-h}}
\]

By induction, the formula (2.4) is obtained. \(\square\)

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The rational surface \(r\) can be thought of as the composition of a surface \(Q(u, v)\) of \(E^3\) with a projection function \(p\), where

\[
p : (R, \omega) \rightarrow R / \omega,
\]

\[
Q(u, v) = p(u, v)(R(u, v), \omega(u, v)).
\]

We refer to \(Q\) as the homogeneous surface associated with \(r\) and surface \(r\) as the projection of \(Q\). In general, we take \(p \equiv 1\) and suppose that \(Q(u, v)\) is \(C^n\).

**Lemma 2.** If the rational surface \(r\) of \(E^3\) is regular at point \((u, v)\), then \(Q(u, v)\), \(Q_i(u, v)\) and \(Q_j(u, v)\) are linearly independent, i.e., there exists at least one \(i\) \((1 \leq i \leq 3)\) such that \(\langle Q_i, Q_j \rangle \neq 0\) at \((u, v)\), where the notation \(\langle \cdot , \cdot, \cdot \rangle\), means

\[
\langle R, S, T \rangle_i = (-1)^{i+1} \det \begin{bmatrix}
R_{i+1} & S_{i+1} & T_{i+1} \\
R_{i+2} & S_{i+2} & T_{i+2} \\
R_{i+3} & S_{i+3} & T_{i+3}
\end{bmatrix}, \quad i = 1, 2, 3, 4.
\]

\(R, S, T\) are arbitrary vectors in \(E^4\). \(R_i\) denotes the \(i\)th component of \(R\). If \(i > 4\), \(R_i = R_{\text{mod}(i-1, 4)+1}\).
such that
\[ \bar{Q}(\bar{u}, \bar{v}) \bigg|_{CB} = c_0(s)Q(u, v) \bigg|_{CB}, \]
\[ = c_nQ \bigg|_{CB} + \sum_{m=0}^{m} \left( \prod_{j=1}^{m} \right) \sum_{k=1}^{k} \sum_{l=1}^{l} \sum_{m-k-h=0}^{m-k-h} \frac{\partial^m Q}{\partial u^m \partial v^{k-h}} \bigg|_{CB}, \]
\[ \quad \cdot p_{r_1} \cdots p_{r_m} \bar{a}_{r_m} \cdots \bar{q}_{r_1} \quad m = 1, \ldots, n. \quad (3.2) \]

**Proof.** With the given \( \bar{p}(s), \bar{q}_i(s) \), we construct a parameter transformation:
\[ \psi_i: \begin{align*} &\bar{u} = \bar{u}(s) + \sum_{i=1}^{m} \bar{p}_i t_i'/i!, \\ &\bar{v} = \bar{v}(s) + \sum_{i=1}^{n} \bar{q}_i t_i'/i!. \end{align*} \quad (3.3) \]

The transformation \( \psi_i \) is non-singular because of the condition
\[ \begin{vmatrix} \frac{\partial \bar{p}_i}{\partial s} & \frac{\partial \bar{q}_i}{\partial s} \\ \frac{\partial \bar{u}}{\partial s} & \frac{\partial \bar{v}}{\partial s} \end{vmatrix} \neq 0. \]

If \( \bar{r} \) meets \( r \) with \( GC^n \) along \( CB \), by Theorem 1, there exist a parameter transformation \( \varphi: \bar{u} = \bar{u}(u, v), \bar{v} = \bar{v}(u, v) \) and a function \( e(u, v) \) such that
\[ d^m(\bar{Q} \circ \varphi) \bigg|_{CB} = d^m(e \cdot Q) \bigg|_{CB}. \]
This gives us that
\[ \bar{Q}(\bar{u}, \bar{v}) \bigg|_{CB} = c_0(s)Q(u, v) \bigg|_{CB}, \quad c_0(s) = e(u, v) \bigg|_{CB}, \]
\[ d^m(\bar{Q} \circ \varphi) \bigg|_{CB} = \sum_{j=0}^{m} \left( \prod_{i=0}^{m} \right) d^m e \bigg|_{CB} d^i Q \bigg|_{CB}, \quad m = 1, \ldots, n. \]

Composing the transformation \( \tau = \varphi^{-1} \circ \psi_i : (s, t) \rightarrow (u, v) \), which is also non-singular, we have
\[ \frac{\partial^m}{\partial t^m} (\bar{Q} \circ \psi_i) \bigg|_{CB} = \sum_{j=0}^{m} \left( \prod_{i=0}^{m} \right) \frac{\partial^m e}{\partial t^m} \bigg|_{CB} \frac{\partial^i Q}{\partial t^i} \bigg|_{CB}. \quad (3.4) \]

Let \( c_i = \partial(e \circ \tau)/\partial t^i \). In terms of Lemma 1, we obtain
\[ \frac{\partial^m (\bar{Q} \circ \psi_i)}{\partial t^m} \bigg|_{CB} = \sum_{k=1}^{k} \sum_{l=1}^{l} \sum_{m-k-h=0}^{m-k-h} A_{r_1 \cdots r_m} \frac{\partial^m Q}{\partial u^m \partial v^{k-h}} \bigg|_{CB}, \quad m = 1, \ldots, n. \quad (3.5) \]
If \( p_1 = \frac{\partial u}{\partial t} \bigg|_{CB} \), \( q_1 = \frac{\partial v}{\partial t} \bigg|_{CB} \) are set, then
\[
\begin{bmatrix}
\frac{\partial u}{\partial s} \\
\frac{\partial v}{\partial s}
\end{bmatrix}
\neq 0.
\]

Therefore (3.2) will be arrived at by substituting (3.5) into (3.4).

We now turn to prove the sufficiency. Define two non-singular transformations \( \psi_1 \) and \( \psi_2 \): \( \psi_1 \) is expressed by (3.3), and \( \psi_2 \) is expressed by
\[
\psi_2: \begin{cases}
u = u(s) + \sum_{i=1}^{n} p_i t^i i!, \\
v = v(s) + \sum_{i=1}^{n} q_i t^i i!.
\end{cases}
\tag{3.6}
\]

For \( c_i(s) \), it is easy to construct a function \( e(u, v) \) such that
\[
\left. \frac{\partial}{\partial t^i} (e \circ \psi_2) \right|_{CB} = c_i(s).
\tag{3.7}
\]

Thus from condition (3.2) and expression (3.5),
\[
\left. \frac{\partial^m (Q \circ \psi_1)}{\partial t^m} \right|_{CB} = \sum_{j=0}^{m} \binom{m}{j} \left. \frac{\partial^{m-j} (e \circ \psi_2)}{\partial t^{m-j}} \right|_{CB} \left. \frac{\partial^j (Q \circ \psi_2)}{\partial t^j} \right|_{CB}
\]
\[
= \left. \frac{\partial^n ((e Q) \circ \psi_2)}{\partial t^n} \right|_{CB}.
\]

Taking the partial derivatives of both sides of the above expression with respect to \( s \), yields
\[
\left. \frac{\partial^{m+j} (Q \circ \psi_1)}{\partial t^m \partial s^j} \right|_{CB} = \left. \frac{\partial^{m+j} ((e Q) \circ \psi_2)}{\partial t^m \partial s^j} \right|_{CB}.
\]

As \( m \) and \( j \) are arbitrary, we have
\[
\left. d^m (Q \circ \psi_1) \right|_{CB} = \left. d^m ((e Q) \circ \psi_2) \right|_{CB}.
\]

Therefore \( r \) and \( \tilde{r} \) are \( GC^n \) continuous along \( CB \), and \( \psi_1 \circ \psi_2^{-1} \) is the parameter transformation of \( GC^n \) between \( r \) and \( \tilde{r} \). \( \square \)

From the above proof, we can also obtain the following result.

**Theorem 3.** Suppose that the regular rational surfaces \( r(u, v) \) and \( \tilde{r}(\tilde{u}, \tilde{v}) \) share a common boundary curve \( CB: r(E_s(s)) = \tilde{r}(E_{\tilde{s}}(s)) \). If equations (3.2) hold for one system of functions \( \bar{p}(s), \bar{q}(s), i = 1, \ldots, n \), satisfying
\[
\begin{bmatrix}
\frac{\partial \bar{p}}{\partial s} \\
\frac{\partial \bar{q}}{\partial s}
\end{bmatrix}
\neq 0,
\]

then they hold for any other system of \( \bar{p}_i(s), \bar{q}_i(s), i = 1, \ldots, n \), satisfying

\[
\begin{vmatrix}
\frac{\partial \bar{p}_i}{\partial s} & \frac{\partial \bar{q}_i}{\partial s} \\
\frac{\partial \bar{p}_i}{\partial \tau} & \frac{\partial \bar{q}_i}{\partial \tau}
\end{vmatrix} \neq 0,
\]

with new scalar functions \( c_i(s), p_i(s) \) and \( q_i(s) \) obtained by the corresponding parameter transformation.

Generally, the scalar functions \( c_i(s), p_i(s) \) and \( q_i(s) \) are called the connecting functions of \( r \) and \( \bar{r} \). They are the solutions of differential equation (3.2). In order to determine them, we rewrite (3.2) as

\[
\bar{Q} |_{CB} = c_0(s)Q |_{CB},
\]

\[
e_{m}Q |_{CB} + c_0 p_m Q_n |_{CB} + c_0 q_m Q_v |_{CB} = F^m
\]

where

\[
F^m = \sum_{k=1}^{m} \sum_{k=2}^{m} \sum_{k=3}^{m} \cdot \cdot \cdot \sum_{k=m}^{m} A_{r_1 \ldots r_2}^{n_{k-1}} \cdot \bar{p}_{r_1} \cdot \bar{p}_{r_2} \cdot \bar{q}_{r_{k+1}} \cdot \ldots \cdot \bar{q}_{r_k} \cdot \frac{\partial^k \bar{Q}}{\partial u^k v^k} |_{CB}
\]

\[
- c_0(s) \sum_{k=2}^{m} \sum_{k=2}^{m} \sum_{k=3}^{m} \cdot \cdot \cdot \sum_{k=m}^{m} A_{r_1 \ldots r_2}^{n_{k-1}} \cdot p_{r_1} \cdot p_{r_2} \cdot q_{r_{k+1}} \cdot \ldots \cdot q_{r_k} \cdot \frac{\partial^k Q}{\partial u^k v^k} |_{CB}
\]

\[
- \sum_{j=1}^{m} \sum_{j=1}^{m} \sum_{j=1}^{m} \cdot \cdot \cdot \sum_{j=m}^{m} A_{r_1 \ldots r_2}^{n_{k-1}} \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \·
It can be verified that they are connected with $GC^3$ continuity along the common boundary $CB$: $E_i(s) = \{s, 1\}$ and $E_i(s) = \{s/2, s\}$. If we choose $\bar{p}_1 = 0.5$, $\bar{q}_1 = -1$ and $\bar{p}_i = \bar{q}_i = 0$ for $i > 1$, then the connecting functions are $c_0 = 1$, $c_2 = 0$, $p_1 = 1$, $c_i = p_i = q_i = 0$ for $j > 1$, $k > 0$ and $i \neq 0, 3$. After the parameter transformations:

$$\psi_1: \begin{cases} \bar{u} = \frac{1}{2}(s + t) \\
\bar{v} = s - t \end{cases} \quad \text{and} \quad \psi_2: \begin{cases} u = t + 1 \\
v = s \end{cases},$$

surfaces $r$ and $\tilde{r}$ become

$$r \circ \psi_2 = \frac{1}{1 + s} \{t + 1, s, 0\}$$

$$\tilde{r} \circ \psi_1 = \frac{1}{1 + s + (1 + s + t)^2 \{t + t^3, s + t^3(s - \sin t), 5st^2\}.}$$

We have

$$r \circ \psi_2 |_{r=0} = \tilde{r} \circ \psi_1 |_{r=0} = \frac{1}{1 + s} \{1, s, 0\},$$

$$\frac{\partial (r \circ \psi_2)}{\partial t} |_{r=0} = \frac{\partial (\tilde{r} \circ \psi_1)}{\partial t} |_{r=0}$$

$$= \frac{\partial^k (r \circ \psi_2)}{\partial t^k} |_{r=0} = \frac{\partial^k (\tilde{r} \circ \psi_1)}{\partial t^k} |_{r=0} = \{0, 0, 0\}, \quad k = 2, 3.$$
Fig. 2. Isoparametric lines of a surface consisting of patches \( r \) and \( \tilde{r} \).

Since parametric continuity is a special case of geometric continuity which requires the transformation (2.3) being an identical mapping, we know \( p_i = 1, p_j = q_j = 0, i = 2, \ldots, n; j = 1, \ldots, n \) from the proof of Theorem 2. Thus we have

**Corollary.** Regular rational surfaces \( r(u, v) \) and \( \tilde{r}(\tilde{u}, \tilde{v}) \) are \( C^n \) parametric continuous along \( CB \). \( \tilde{r}(0, \tilde{v}) = r(1, v) \) (\( \tilde{v} = v \)) iff there exist scalar functions \( c_i(v) \), \( i = 0, \ldots, n \) such that

\[
\frac{\partial^i \tilde{Q}}{\partial \tilde{u}^i} \bigg|_{CB} = c_i Q \bigg|_{CB} + \sum_{j=1}^{i} \left[ \begin{array}{c} s \\ j \end{array} \right] c_{i-j}(v) \frac{\partial^j Q}{\partial u^j} \bigg|_{CB}, \quad s = 0, \ldots, n. \tag{3.10}
\]

Fig. 3. Cross sections of the same surface.
Now let us give the first few geometric continuity conditions.

\begin{align}
  s = 0: \quad & \bar{Q} \mid_{CB} = c_0(v) Q \mid_{CB}, \\
  s = 1: \quad & \bar{Q}_z \mid_{CB} = c_1 Q \mid_{CB} + c_0 p_1 Q_u \mid_{CB} + c_0 q_1 Q_v \mid_{CB}, \\
  s = 2: \quad & \bar{Q}_{uu} \mid_{CB} = c_2 Q \mid_{CB} + (2 c_1 p_1 + c_0 p_2) Q_u \mid_{CB} + (2 c_1 q_1 + c_0 q_2) Q_v \mid_{CB} \\
  & \quad + c_0 p_1^2 Q_{uu} \mid_{CB} + 2 c_0 p_1 q_1 Q_{uv} \mid_{CB} + c_0 q_1^2 Q_{vv} \mid_{CB}. 
\end{align}

It can be shown that the above formulas include many analogous conditions occurring in the computer aided geometric design literature as their special cases. In (Liu, 1990), the conditions are (3.11) and (3.12) with \( c_0(v) = 1 \) which is due to the fact that its position continuity is defined as “\( C^0 \) continuity”. In this paper, the position continuity is defined as “\( GC^n \) continuity” which belongs to the second definition presented in the “Remark” of (Liu, 1990). Therefore we have obtained more general results.

In fact, if we let \( c_0 = 1 \) and \( c_i = 0 \) for \( i \geq 1 \) then rational geometric continuity reduces to simple geometric continuity. On the other hand, if \( p_i = 1 \) and \( q_i = 0 \) for \( i \geq 2 \) and \( j \geq 1 \), then rational geometric continuity reduces to the rational parametric continuity. Finally, if \( c_i, p_i \) and \( q_i \) are all specified as above, the rational geometric continuity conditions reduce to simple parametric continuity conditions.

4. \( GC^n \) condition for rational polynomial patches

In CAGD applications, rational Bézier surfaces belong to the most widely used surfaces. Other rational polynomial surfaces can be converted into rational Bézier form. In this section we only discuss the geometric continuity for rational Bézier surfaces.

Suppose that the rational Bézier surfaces \( R \) of degree \( m \times l \) and \( \bar{R} \) of degree \( m \times l \) are expressed in homogeneous coordinates by

\[
  r: \quad Q(u, v) = \sum_{i=0}^{m} \sum_{j=0}^{l} Q_{ij} B_i^m(u) B_j^l(v), \quad 0 \leq u, v \leq 1
\]

\[
  \bar{r}: \quad \bar{Q}(\bar{u}, \bar{v}) = \sum_{i=0}^{m} \sum_{j=0}^{l} \bar{Q}_{ij} B_i^m(\bar{u}) B_j^l(\bar{v}), \quad 0 \leq \bar{u}, \bar{v} \leq 1
\]

where

\[
  Q_{ij} = \left( P_{ij}, \omega_{ij}, \omega_{ij} \right), \quad \bar{Q}_{ij} = \left( \bar{P}_{ij}, \omega_{ij}, \bar{\omega}_{ij} \right)
\]

\( P_{ij}, \bar{P}_{ij} \) are control points, \( \omega_{ij}, \bar{\omega}_{ij} \) are weights.

\[
  B_i^l(t) = \binom{l}{i} t^i (1-t)^{l-i} \text{ are Bernstein polynomials of degree } l.
\]

In general, if \( Q_{ij} \) are found, then \( P_{ij}, \omega_{ij} \) will be determined.
4.1. GC<sup>n</sup> necessary and sufficient condition

The GC<sup>n</sup> necessary and sufficient conditions for general rational surfaces have been given in Section 3. Here we concern ourselves with the rational Bézier surfaces and show that their connecting functions are rational polynomials.

**Theorem 5.** A necessary and sufficient condition for GC<sup>n</sup> continuity between two adjacent rational Bézier patches rin degree m×l and r in degree m×l along CB: r(1, v) = r(0, v) (0 ≤ v ≤ v) (0 ≤ v ≤ v) is that

\[ G(v) = H(v)Q, \quad G(v)H(v) ≠ 0, \]

\[ D^{2s-1}H^{s-1} + \frac{\partial^s Q}{\partial u^s} = c_sQ + \sum_{k=1}^{s} \sum_{k_1+\ldots+k_s=k} A_{k_1\ldots k_s}k_{k_1\ldots k_s}G^{k-1}D^{k-1}p_r \ldots p_q \]

\[ + \sum_{d=1}^{s-1} \left( \sum_{k=1}^{s-d} \sum_{k_1+\ldots+k_s=k} A_{k_1\ldots k_s}k_{k_1\ldots k_s} \right) \frac{\partial^s Q}{\partial u^s} = c_sQ + \sum_{k=1}^{s} \sum_{k_1+\ldots+k_s=k} A_{k_1\ldots k_s}k_{k_1\ldots k_s}G^{k-1}D^{k-1}p_r \ldots p_q \]

\[ s = 1, \ldots, n \]

(4.1)

hold at every point of CB, where G, H, D, c, p, q, are all polynomials of v, D(v) ≠ 0, p, q ≠ 0, their degrees are as follows:

<table>
<thead>
<tr>
<th>G, H, D</th>
<th>c, p, q</th>
</tr>
</thead>
<tbody>
<tr>
<td>degrees ≤</td>
<td>l</td>
</tr>
<tr>
<td></td>
<td>3l-1</td>
</tr>
<tr>
<td></td>
<td>(2i-1)(3l-1)</td>
</tr>
<tr>
<td></td>
<td>(2i-1)(3l-1)</td>
</tr>
<tr>
<td></td>
<td>+ (i-1)l</td>
</tr>
<tr>
<td></td>
<td>+ (i-1)l+1</td>
</tr>
</tbody>
</table>

**Proof.** First, assume that r and r are GC<sup>n</sup> continuous. Then by Theorem 4, there exist c(0), c(v), p(v) and q(v), i = 1, ..., n, such that (3.9) holds.

When n = 0, Q|CB = c_0Q|CB implies

\[ \bar{R}|CB = c_0 R|CB \quad \text{and} \quad \bar{ω}|CB = c_0ω|CB. \]

If H = ω|CB, G = ω|CB are set, then GQ = HQ, and c_0 = H/G.

When n = 1,

\[ \bar{Q}_u|CB = c_1Q|CB + c_0p_1Q|CB + c_0q_1Q|CB. \]

In terms of Lemma 2, there exists i (1 ≤ i ≤ 3) such that \( \langle Q, Q_u, Q_i \rangle \|_{u=1, v-1} ≠ 0 \). Taking

\[ D(v) = \langle Q, Q_u, Q_i \rangle |_{CB}, \quad c_i = \langle \bar{Q}_u, Q_u, Q_i \rangle |_{CB}, \]

\[ p_i(v) = \langle Q, Q_u, Q_i \rangle |_{CB}, \quad q_i(v) = \langle Q, Q_u, Q_i \rangle |_{CB} \]

(4.2)
gives

\[ D\bar{Q}_u |CB = c_iQ|CB + p_1Q|CB + q_1Q|CB \]

\[ D\bar{Q}_u |CB = c_iQ|CB + p_1Q|CB + q_1Q|CB \]
and
\[ \xi_1 = Dc_1, \quad p_1 = Dc_0p_1, \quad q_1 = Dc_0q_1. \]

Obviously, the degrees of \( D, \xi_1, p_1, q_1 \) are not larger than \( 3l - 1, 3l - 1, 3l - 1 \) and \( 3l \) respectively. Now suppose that the necessity has been proved for \( n \leq s - 1 \). Then when \( n = s \),
\[ \frac{\partial^s \bar{Q}}{\partial t^s} \bigg|_{CB} = c_sQ \bigg|_{CB} + c_0p_sQ_u \bigg|_{CB} + c_0q_sQ_e \bigg|_{CB} + \delta \quad \text{(4.3)} \]
where
\[ \delta = \sum_{k=2}^{s} \sum_{r_1 + \cdots + r_s = s} \sum_{h=0}^{k} A_{r_1 \cdots r_s}^{kh} e_0 p_{r_1} \cdots p_{r_s} q_{r_s} \frac{\partial^k Q}{\partial u^h \partial t^{k-h}} \bigg|_{CB} \]
\[ + \sum_{d=1}^{s-1} \sum_{\xi \neq 0} \sum_{k=1}^{s} \sum_{r_1 + \cdots + r_s + d = k} A_{r_1 \cdots r_s}^{dh} p_{r_1} \cdots p_{r_s} q_{r_s} q_{r_s} \frac{\partial^k Q}{\partial u^d \partial t^{k-d}} \bigg|_{CB}. \]

Let
\[ \bar{\delta} = D^{2s-2} H^{s-1} \left( \frac{\partial^s \bar{Q}}{\partial t^s} \bigg|_{CB} - \delta \right) \]
\[ = D^{2s-2} H^{s-1} \left( c_sQ \bigg|_{CB} + c_0p_sQ_u \bigg|_{CB} + c_0q_sQ_e \bigg|_{CB} \right). \]

We have
\[ \bar{\delta} = D^{2s-2} H^{s-1} \left( \frac{\partial^s \bar{Q}}{\partial t^s} \bigg|_{CB} - \sum_{k=2}^{s} \sum_{r_1 + \cdots + r_s = s} \sum_{h=0}^{k} A_{r_1 \cdots r_s}^{kh} \cdot G^{k-1} D^{k-2} p_{r_1} \cdots p_{r_s} q_{r_s} \right) \]
\[ - \sum_{d=1}^{s-1} \sum_{\xi \neq 0} \sum_{k=1}^{s} \sum_{r_1 + \cdots + r_s + d = k} A_{r_1 \cdots r_s}^{dh} p_{r_1} \cdots p_{r_s} q_{r_s} q_{r_s} \frac{\partial^k Q}{\partial u^d \partial t^{k-d}} \bigg|_{CB} \cdot G^{k-1} D^{k-1} p_{r_1} \cdots p_{r_s} q_{r_s} q_{r_s} \frac{\partial^k Q}{\partial u^d \partial t^{k-d}} \bigg|_{CB}. \]

It is easy to show that \( \bar{\delta} \) is a curve of degree not larger than \( s(7l - 2) - 6l + 2 \). If we let
\[ c_s = \langle \bar{\delta}, Q_u, Q_e \rangle \bigg|_{CB}, \quad p_s(v) = \langle Q_s, \bar{\delta}, Q_s \rangle \bigg|_{CB}, \]
\[ q_s(v) = \langle Q_s, Q_s, \bar{\delta} \rangle \bigg|_{CB} \quad \text{(4.4)} \]
then
\[ c_s = c_s D^{2s-1} H^{s-1}, \quad p_s = c_0p_s D^{2s-1} H^{s-1}, \quad q_s = c_0q_s D^{2s-1} H^{s-1}, \quad \text{(4.5)} \]
and

\[ D \cdot \tilde{\mathcal{Q}} \mid \mathcal{C} \mathcal{B} = \xi_0 \mathcal{Q} \mid \mathcal{C} \mathcal{B} + p_i \mathcal{P}_i \mid \mathcal{C} \mathcal{B} + q_i \mathcal{Q}_i \mid \mathcal{C} \mathcal{B}. \]

The degrees of \( \xi_0, p_i, q_i \) are not larger than \( (2s - 1)(3l - 1) + (s - 1)l, (2s - 1)(3l - 1) + (s - 1)l \) and \( (2s - 1)(3l - 1) + (s - 1)l + 1 \) respectively, and the second equation of (4.1) holds for \( n = s \). Thus the necessity is fulfilled by induction.

On the other hand, take \( c_j(v) = H(v) / G(v) \), and assume that \( D(v) \) has \( j \) zero points \( v_1, \ldots, v_j \) in \([0, 1]\). From Eq. (4.5), we know that \( v_1, \ldots, v_j \) are also the roots of \( \xi_j, p_j, q_j \) with multiplicity \( 2s - 1 \). These zero factors can be eliminated by

\[ \tilde{D} = D / V^j, \quad \tilde{\xi}_j = \xi_j / V^j \tilde{\xi}^{2j - 1}, \quad \tilde{p}_j = p_j / V^j \tilde{p}^{2j - 1}, \quad \tilde{q}_j = q_j / V^j \tilde{q}^{2j - 1} \]

where \( V^j = (v - v_1) \cdots (v - v_j) \).

It is clear that \( \tilde{D} \neq 0, \forall v \in [0, 1] \), and (4.1) still holds if \( D, \xi_j, p_j, q_j \) are replaced by \( \tilde{D}, \tilde{\xi}_j, \tilde{p}_j, \tilde{q}_j \) respectively. Thus the sufficiency follows immediately if we take

\[ c_j = \tilde{\xi}_j / (\tilde{D} \tilde{\xi}^{2j - 1} H^{2j - 1}), \quad p_j = \tilde{G} \tilde{p}_j / (\tilde{D} \tilde{p}^{2j - 1} H^{2j - 1}), \quad q_j = \tilde{G} \tilde{q}_j / (\tilde{D} \tilde{q}^{2j - 1} H^{2j - 1}). \]

Note that there is no loss of generality by assuming the degree of \( r \) with respect to \( v \) to be equal to that of \( \tilde{r} \) with respect to \( \tilde{v} \) in Theorem 5 because of the elevation formula.

We now deal with two special cases.

**Case 1.** If the position continuity of \( r \) and \( \tilde{r} \) along \( \mathcal{C} \mathcal{B} \) is defined as "\( C^0 \) continuity" of (Liu, 1990), i.e. \( \mathcal{Q} \mid \mathcal{C} \mathcal{B} = \mathcal{Q} \mid \mathcal{C} \mathcal{B} \), then \( c_0 = 1 \). Without loss of generality, taking \( H = G = 1 \), conditions (4.1) become

\[ \mathcal{Q} \mid \mathcal{C} \mathcal{B} = \mathcal{Q} \mid \mathcal{C} \mathcal{B}, \]

\[ D^{2j - 1} \frac{\partial^j \mathcal{Q}}{\partial u^j} \mid \mathcal{C} \mathcal{B} = \xi_0 \mathcal{Q} \mid \mathcal{C} \mathcal{B} + \sum_{k = 1}^{s} \sum_{r_1 = 0}^{d} \cdots \sum_{r_s = 0}^{d} \sum_{p = 0}^{k} \mathcal{A}^{r_1 \cdots r_s} \frac{\partial^k \mathcal{Q}}{\partial u^k} \mid \mathcal{C} \mathcal{B} \]

\[ + \sum_{d = 1}^{s} \sum_{r = 0}^{d} \xi^{d - r} \sum_{p = 1}^{d} \sum_{q = 1}^{d} \sum_{r = 0}^{k} \mathcal{A}^{r_1 \cdots r_s} \frac{\partial^k \mathcal{Q}}{\partial u^k} \mid \mathcal{C} \mathcal{B}, \quad s = 1, \ldots, n. \]

(4.6)

After an analogous discussion, we obtain

**Corollary 1.** If regular rational Bézier surfaces \( r \) of degree \( m \times l \) and \( \tilde{r} \) of degree \( m \times l \) satisfy "\( C^0 \) continuity" along their common boundary curve \( \mathcal{C} \mathcal{B} \), then they meet with \( GC^* \) iff there exist some polynomials \( D(v), \xi_j(v), p_j(v), q_j(v) \), whose
degrees are respectively not larger than \(3l-1\), \((2i-1)(3l-1)\), \((2i-1)(3l-1)+1\), \(D(v) \neq 0\), \(p_i(v) \neq 0\) such that (4.6) hold.

Case 2. If surfaces \(r\) and \(\bar{r}\) are integral Bézier surfaces, then it is a natural choice that \(H = G = 1\). For any \(s \geq 1\), \(\delta \omega / \delta u^s \delta v^r = 0\), and (4.1) imply \(0 = \zeta_s \omega + \zeta_s = 0\). Thus conditions (4.1) become

\[
\bar{r} \bigg|_{CB} = r \bigg|_{CB},
\]

\[
D^{2s-1} \frac{\delta \bar{r}}{\delta u^s} \bigg|_{CB} = \sum_{k=1}^{s} \sum_{r_1+\cdots+r_k=s} A^{i_1\cdots i_k} \cdot D^{k-1} \prod_{i=1}^{k} \frac{\delta r}{\delta u^{i_j} \delta v^{r_j}} \bigg|_{CB}.
\]

(4.7)

Similarly, the degrees of \(D\), \(p_i\), \(q_j\) must be properly adjusted.

Corollary 2. Bézier surfaces \(r\) of degree \(m \times l\) and \(\bar{r}\) of degree \(\bar{m} \times l\) are \(GC^n\) continuous along \(CB\): \(r(0, v) = r(1, v)\) \((0 \leq v < 1)\) if conditions (4.7) hold, where \(D(v) \neq 0\), \(p_i(v) \neq 0\), \(q_j(v)\), \(i = 1, \ldots, n\), are all polynomials of \(v\) with degrees not larger than \(2l-1\), \((2i-1)(2l-1)\) and \((2i-1)(2l-1)+1\).

4.2. Some practical sufficient conditions

One application of geometric continuity is the construction of smoothly connecting surfaces by using the continuity conditions. In general, as shown in (Zheng et al., 1992), conditions (4.1) provide the designer numerous coefficients while they include a system of constraint equations which these coefficients have to satisfy. In practice, too many free coefficients will confuse the designer and the complicated constraints will give rise to difficulties for CAGD application. For this reason, we reduce the degrees of the connecting functions and let the position continuity be “\(C^0\) continuity”. Thus some \(GC^n\) sufficient conditions are obtained.

Theorem 6. Regular rational Bézier surfaces \(r\) of degree \(m \times l\) and \(\bar{r}\) of degree \(\bar{m} \times l\) are \(GC^n\) along \(CB\): \(r(0, v) = r(1, v)\) \((0 \leq v < 1)\) if conditions (4.6) hold, where \(D(v) \neq 0\), \(\zeta_i\), \(p_i\) \((p_i \neq 0)\), \(q_j\) are all polynomials of \(v\) respectively with degrees not larger than \(\sigma\), \((2l-1)\sigma\), \((2i-1)\sigma\), \((2i-1)\sigma + 1\). The number \(\sigma\) is an integer ranging from \(0\) to \(3l-1\). If \(\sigma = 3l-1\), the condition is also necessary.

As an example, consider the case of \(\sigma = 0\). Then \(D\), \(\zeta_i\), \(p_i\) are constants, and \(q_j\) are linear polynomials. Assume that \(D(v) = 1\), \(\zeta_i = \gamma_i\), \(p_i = \alpha_i\), \(q_j = \beta_i^0(1-v) + \beta_i^1 v\).

Then conditions (4.6) become

\[
\bar{Q} \bigg|_{CB} = Q \bigg|_{CB},
\]

\[
\frac{\partial \bar{Q}}{\partial u^s} \bigg|_{CB} = \gamma_i \bar{Q}_i \bigg|_{CB} + \sum_{s=1}^{s} \sum_{d=1}^{s} \sum_{d=1}^{d} \sum_{r_1+\cdots+r_k=s} A^{i_1\cdots i_k} \cdot D^{k-1} \prod_{i=1}^{k} \frac{\delta Q}{\delta u^{i_j} \delta v^{r_j} \delta (1-v)} \bigg|_{CB},
\]

\[
\times \left( \alpha_{i_1} \cdots \alpha_{i_k} \left( \beta_{i_1}^0 (1-v) + \beta_{i_1}^1 v \right) \cdots \left( \beta_{i_k}^0 (1-v) + \beta_{i_k}^1 v \right) \right)
\]

\[
\frac{\partial \bar{Q}}{\delta u^s \delta v^r \delta (1-v)} \bigg|_{CB},
\]

\(s = 1, \ldots, n\).
After some calculations, it is possible to obtain the first \((n+1)\)-columns of the control points of surface \( \tilde{r} \) which meets \( r \) with GC\(^n\) along CB:

\[
\tilde{Q}_{i0} = Q_{i0}, \quad i = 0, \ldots, l,
\]

\[
\tilde{Q}_{ij} = -\sum_{j=0}^{s-1} \binom{s}{j} (-1)^{s-j} \tilde{Q}_{ji} + \frac{(m-s)!}{m!} \times \left\{ \begin{array}{c}
\gamma_s Q_{mi} + \sum_{d=1}^{s} \binom{s}{d} \sum_{k=1}^{d} \sum_{l-k-h=0}^{d} \sum_{j=0}^{l-k} \sum_{i_{k+1}=0}^{l-k} \sum_{i_0=0}^{l-k} \sum_{i_{k+1}+\cdots+i_k+i_l=j} \mu_{rl}^{d,k,h} \frac{m!}{l!} \frac{l!}{(m-h)! (l-k+h)!} \\
\times \left[ \frac{\binom{l-k+h}{j}}{\binom{l}{j}} \gamma_{l-j} \alpha_{r_1} \cdots \alpha_{r_k} \beta_{r_{k+1}} \cdots \beta_{r_l} \nabla_1^{k-h} \nabla_2^{l-k} Q_{m-i_l} \right] \end{array} \right. 
\]

\[s = 1, \ldots, n\]  \hspace{1cm} (4.8)

where

\[
\nabla_i Q_{ij} = Q_{i+1,j} - Q_{ij}, \quad \nabla_i^T Q_{ij} = \nabla_i (\nabla_i^{T-1} Q_{ij}),
\]

\[
\nabla_2 Q_{ij} = Q_{ij+1} - Q_{ij}, \quad \nabla_2^T Q_{ij} = \nabla_2 (\nabla_2^{T-1} Q_{ij}).
\]

and \( \gamma_s, \alpha_i, \beta_i^0, \beta_i^1 (\alpha_i \neq 0) \) are free coefficients which are sometimes called shape parameters in the literature.

Further, suppose that \( r \) and \( \tilde{r} \) are bicubic, that is \( m = l = \tilde{m} = 3 \). If we choose the first two columns of the control points of \( \tilde{r} \) as follows, then \( \tilde{r} \) meets \( r \) with GC\(^1\) continuity.

\[
\tilde{Q}_{i0} = Q_{i0}, \quad i = 0, \ldots, 3,
\]

\[
\tilde{Q}_{10} = \gamma_1 Q_{30} + \alpha_1 (Q_{30} - Q_{20}) + \beta^0_1 (Q_{31} - Q_{20}),
\]

\[
\tilde{Q}_{11} = \gamma_1 Q_{31} + 3 \alpha_1 (Q_{31} - Q_{21}) + \frac{3}{2} \beta^0_1 (Q_{32} - Q_{21}) + \frac{1}{2} \beta^1_1 (Q_{31} - Q_{21}),
\]

\[
\tilde{Q}_{12} = \gamma_1 Q_{32} + 3 \alpha_1 (Q_{32} - Q_{22}) + \frac{3}{2} \beta^0_1 (Q_{33} - Q_{22}) + \frac{3}{2} \beta^1_1 (Q_{32} - Q_{22}),
\]

\[
Q_{13} = \gamma_1 Q_{33} + \alpha_1 (Q_{33} - Q_{23}) + \beta^1_1 (Q_{33} - Q_{23}).
\]

5. Conclusion

This paper has presented a set of necessary and sufficient conditions of GC\(^n\) continuity for two adjacent rational surfaces along a general intersection curve, which do not require the homogeneous surfaces to be smooth. These conditions
are represented by the associated homogeneous surfaces and a set of connecting functions. Thus they can be further treated like integral surfaces. Specifically, for rational Bézier surfaces, it can be shown that these conditions can be converted into a series of constraints represented by the control points and weights as in (Zheng et al., 1992). This is convenient for constructing the connecting surfaces and examining by a program if two given rational Bézier patches are $G^m$.

This paper has also shown that geometric continuity is, in essence, the existence of a reparameterization. If two given surface patches are $G^m$, then transformations (3.3) and (3.6) present a method to reparameterize the two surfaces so that they are $C^m$. This is an important feature in some CAGD applications.

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