Approximation of rational curves by polynomial curves

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Approximation techniques

♦ Offset approximation
♦ Degree reduction approximation
♦ Interpolation
♦ Fitting
♦ etc…

* Approximation of rational curves by polynomial curves
Where needed?

♦ Data exchanging

CAD system based on polynomials

CAD system based on rational polynomials

♦ Difficulty in calculation of derivatives or integrations
Approximation of …..

♦ Rational Bézier curves

♦ Circles or conic sections

♦ Rational curves
Previous work


♦ Floater, M.S. High order approximation of conic sections by quadratic splines, Computer Aided Geometric Design, 1995, 12, 617–637
Hybrid approximation (related work)


Hybrid approximation (target)

\[
R(t) = \frac{\sum_{i=0}^{m} B_i^m(t) \omega_i R_i}{\sum_{i=0}^{m} B_i^m(t) \omega_i}
\]

\[
P(t) = \sum_{i=0}^{n} B_i^n(t) P_i
\]
Hybrid approximation (Main idea)

\[ H^{r,p}(t) \equiv R(t) = \sum_{i=0}^{r+p} B_i^{r+p}(t) H_i^{r,p} + B_r^{r+p}(t) H_r^{r,p}(t) \]

where \[ R(t) = \frac{\sum_{i=0}^{m} B_i^m(t) \omega_i R_i}{\sum_{i=0}^{m} B_i^m(t) \omega_i} \]

Moving control point: \[ H_{r}^{r,p}(t) = \frac{\sum_{k=0}^{m} B_k^m(t) \omega_k M_k^{r,p}}{\sum_{k=0}^{m} B_k^m(t) \omega_k} \]

Exact representation not approximation!
Hybrid approximation (Main idea)

choose \( H_{r,p} \in C \left( \{ M_{k}^{r,p} \}_{k=0}^{m} \right) \)

Obtained Hybrid approximating polynomial curve

\[
\tilde{H}_{r,p}(t) = \sum_{i=0}^{r+p} B_{i}^{r,p}(t) H_{i}^{r,p}
\]
Hybrid approximation (Illustration)

Cubic rational Bézier curve

Quadratic Hybrid curve

Evaluating at $t=0.5$
Hybrid approximation (Error bound)

\[ |\tilde{\mathbf{H}}_{r,p}(t) - \mathbf{H}_{r,p}(t)| \leq \Delta \left( \binom{r + p}{r} \left( 1 - \frac{r}{r + p} \right)^p \left( \frac{r}{r + p} \right)^r \right) \]

Where \( \Delta \) denotes the half width of the bounding box.

If \( r + p \) is an even number, and choose \( r = p \), we get

\[ |\tilde{\mathbf{H}}_{r,p}(t) - \mathbf{H}_{r,p}(t)| \leq \Delta \binom{2r}{r} \frac{1}{2^{2r}} \]
Hybrid approximation (together with subdivision)
Hybrid approximation (Convergence)
Hybrid approximation (Convergence)

- Relation between Hermite approximation and Hybrid approximation
- Necessary and sufficient conditions of convergence with hybrid approximation to rational Bézier curve

Approximation to conic sections


Outline

Present a method for approximating conic sections using quintic polynomial curves. The constructed quintic polynomial curve has $G^3$ continuity with the conic section at the end points and $G^1$ continuity at the parametric mid-point.
Problem of interest

Given a conic section represented in the form of a rational quadratic Bézier curve as

\[ r(t) = \frac{B_0(t)p_0 + B_1(t)wp_1 + B_2(t)p_2}{B_0(t) + B_1(t)w + B_2(t)}, \quad t \in [0, 1], \]

try to find an approximating quintic polynomial curve that is $G^3$ continuous with the conic section at the end points and $G^1$ continuous at the parametric mid-point.
Conic section approximation

A quintic polynomial curve represented in Hermite form

\[ Q(t) = (x(t), y(t)) = \begin{bmatrix} H_1(t) & H_2(t) & H_3(t) & H_4(t) & H_5(t) & H_6(t) \end{bmatrix} \begin{bmatrix} Q(0) \\ Q'(0) \\ Q''(0) \\ Q(1) \\ Q'(1) \\ Q''(1) \end{bmatrix}, \quad t \in [0, 1], \]

where

\[
\begin{align*}
H_1(t) &= 1 - 10t^3 + 15t^4 - 6t^5, \\
H_2(t) &= t - 6t^3 + 8t^4 - 3t^5, \\
H_3(t) &= \frac{1}{2}t^2 - \frac{3}{2}t^3 + \frac{3}{2}t^4 - \frac{1}{2}t^5, \\
H_4(t) &= 10t^3 - 15t^4 + 6t^5, \\
H_5(t) &= -4t^3 + 7t^4 - 3t^5, \\
H_6(t) &= \frac{1}{2}t^3 - t^4 + \frac{1}{2}t^5.
\end{align*}
\]
Conic section approximation ($G^2$ continuous)

A quintic Hermite curve can be $G^2$ continuous to $\mathbf{r}(t)$ at end points by setting

\[
\begin{align*}
Q(0) &= \mathbf{r}(0), & Q'(0) &= \alpha_0 \mathbf{r}'(0), & Q''(0) &= \alpha_0^2 \mathbf{r}''(0) + \beta_0 \mathbf{r}'(0), \\
Q(1) &= \mathbf{r}(1), & Q'(1) &= \alpha_1 \mathbf{r}'(1), & Q''(1) &= \alpha_1^2 \mathbf{r}''(1) + \beta_1 \mathbf{r}'(1),
\end{align*}
\]

where $\alpha_0, \beta_0, \alpha_1, \beta_1$ are arbitrary constants and the differential properties of $\mathbf{r}(t)$ at $t = 0$ and $t = 1$ are computed as

\[
\begin{align*}
\mathbf{r}(0) &= \mathbf{p}_0, & \mathbf{r}(1) &= \mathbf{p}_2, \\
\mathbf{r}'(0) &= 2w(\mathbf{p}_1 - \mathbf{p}_0), & \mathbf{r}'(1) &= 2w(\mathbf{p}_2 - \mathbf{p}_1), \\
\mathbf{r}''(0) &= (4w - 8w^2)(\mathbf{p}_1 - \mathbf{p}_0) + 2(\mathbf{p}_2 - \mathbf{p}_0), & \mathbf{r}''(1) &= (4w + 8w^2)(\mathbf{p}_2 - \mathbf{p}_1) - 2(\mathbf{p}_2 - \mathbf{p}_0).
\end{align*}
\]
G³ at end points and G¹ at t=1/2

\[-\alpha_0 \beta_0 + w \beta_1 + \left[ 10 - 3 \alpha_0^2 - 2(1 - w) \alpha_0^3 - 8w \alpha_1 - (1 + 2w - 4w^2) \alpha_1^2 \right] = 0.\]
\[w \beta_0 - \alpha_1 \beta_1 - \left[ 10 - 3 \alpha_1^2 - 2(1 - w) \alpha_1^3 - 8w \alpha_0 - (1 + 2w - 4w^2) \alpha_0^2 \right] = 0.\]
\[\frac{5}{16} w (\alpha_0 + \alpha_1) + \frac{1}{16} (w - 2w^2) (\alpha_0^2 + \alpha_1^2) + \frac{1}{32} w (\beta_0 - \beta_1) = \frac{w}{1+w}.\]
\[\frac{1}{16} (\alpha_0 - \alpha_1) (\alpha_0 + \alpha_1 - 2w) = 0\]

Conclusion: There exist only three solutions satisfying all of the four equations.
Approximation error (different criterions)

1. It is well known that for any point on the conic section \( r(t) \), its barycentric coordinates \( \tau_0, \tau_1, \tau_2 \), where \( \tau_0 + \tau_1 + \tau_2 = 1 \), with respect to the triangle \( \triangle p_0p_1p_2 \), satisfies

\[
f(r(t)) = \tau_1^2(t) - 4w^2\tau_0(t)\tau_2(t) = 0.
\]

Consequently, for any curve \( c(t) \) approximating the conic section \( r(t) \), we can use \( f(c(t)) \) to see how well the approximation is.

2. Hausdorff distance between \( c(t) \) and \( r(t) \).

Error bounds are given with respect to both.
Shape preservation

Approximation quality

Shape preserving: when \( Q(1/2) \) is the only point at which the tangent of \( Q(t) \) is parallel to line \( p_0p_2 \), we say that \( Q(t) \) preserves the basic shape of \( r(t) \).

Criterion in mathematical form:

The equation \( \frac{dQ(t)}{dt} \times (p_2 - p_0) = 0 \) has only one solution at \( t=1/2 \).
Example (bad case)

Fig. 4. The anomalies of $Q_0(t)$ (in thin lines) when $w > 3$: (a) “camel humps” and (b) loops.

Fig. 5. A close-up of $r(t)$ ($w = 100$, in thick line) and $Q_1(t)$ (in thin line) around $t = 1/2$. 
Example (circular arcs)

Fig. 6. A sample curvature graph for $r(t)$ ($w = 2.7$) and its approximation curves $Q_0(t)$ and $Q_1(t)$. 
Spline interpolation


Outline

Describe a scheme that produces a convexity-preserving parametric quartic spline interpolant to position, tangent and curvature data. The resulting interpolant is curvature continuous and the scheme is local and 6-th order accurate.
Problem description

curvature continuous:
the unit tangent vector $f^* = \frac{f'}{|f'|}$ and the signed curvature $f^{**} = \frac{f' \times f''}{|f'|^3}$ are continuous.

Let $f_i$, $d_i$ and $\kappa_i$ be prescribed positions, directions and curvature values.

$$f_i := f(t_i), \quad d_i := f^*(t_i) \quad \text{and} \quad \kappa_i := f^{**}(t_i).$$

The objective, then, is a geometric Hermite interpolant $p$ that satisfies

$$p(i) = f_i, \quad p^*(i) = d_i \quad \text{and} \quad p^{**}(i) = \kappa_i,$$

where the components of $p$ are polynomials on each interval.
The De Boor-Höllig-Sabin Scheme


$p$ can be constructed interval by interval, so it is sufficient that we restrict our attention to the interval $[0,1]$. Write $p$ on this interval in its Bézier form

$$p(t) = \sum_{j=0}^{3} b_j B_j^3(t), \quad (0 \leq t \leq 1)$$
P construction

\[ b_0 = f_0, b_1 = (1 - \rho_0)f_0 + \rho_0 \tilde{b}, b_2 = (1 - \rho_1)f_1 + \rho_0 \tilde{b} \text{ and } b_3 = f_1, \]

where

\[ \tilde{b} := \frac{(f_1 \times d_1)d_0 + (d_0 \times f_0)d_1}{d_0 \times d_1} \]

and \( \rho_0 \) and \( \rho_1 \) satisfy

\[ 0 < \rho_0, \rho_1 \leq 1, \quad \rho_0 = 1 - R_1 \rho_1^2 \quad \text{and} \quad \rho_1 = 1 - R_0 \rho_0^2, \]

where

\[ R_0 := \frac{3\kappa_0 |\tilde{b} - b_0|^2}{2d_0 \times (b_3 - \tilde{b})} \quad \text{and} \quad R_1 := \frac{3\kappa_1 |b_3 - \tilde{b}|^2}{2(\tilde{b} - b_0) \times d_1}. \]
Quartic Spline construction

Construction in Bézier form:

\[ p(t) = \sum_{j=0}^{4} b_j B_j(t), \quad (0 \leq t \leq 1) \]

\[ b_0 = f_0 \text{ and } b_4 = f_1, \]

\[ b_1 = (1 - \rho_0)b_0 + \rho_0 b \] and \[ b_3 = (1 - \rho_1)b_4 + \rho_1 b \]

\[ b_2 = \alpha_0 b_1 + \alpha_1 b_3 + (1 - \alpha_0 - \alpha_1)b \]

Fig. 1. Bézier control polygon of a quartic curve segment. To interpolate tangents and preserve convexity, it is sufficient that \( b_1 \) and \( b_3 \) be on the line segments \( \overline{b_0 b} \) and \( \overline{b_4 b} \) and that \( b_2 \) be inside the triangle \( \Delta b_1 b_2 b_4 \)
Quartic Spline construction

\[ \kappa_0 = \frac{3d_0 \times (b_2 - b_1)}{4|b_1 - b_0|^2} \quad \text{and} \quad \kappa_1 = \frac{3(b_3 - b_2) \times d_1}{4|b_4 - b_3|^2}. \]

\[ \kappa_0 = \frac{3\alpha_1(1 - \rho_1)d_0 \times (b_4 - \tilde{b})}{4\rho_0^2|\tilde{b} - b_0|^2} \quad \text{and} \quad \kappa_1 = \frac{3\alpha_0(1 - \rho_0)(\tilde{b} - b_0) \times d_1}{4\rho_1^2|b_4 - \tilde{b}|^2}. \]

Define:

\[ R_0 := \frac{4\kappa_0|\tilde{b} - b_0|^2}{3d_0 \times (b_4 - \tilde{b})} \quad \text{and} \quad R_1 := \frac{4\kappa_1|b_4 - \tilde{b}|^2}{3(\tilde{b} - b_0) \times d_1}, \]

Get:

\[ \alpha_0 = \frac{R_1\rho_1^2}{1 - \rho_0} \quad \text{and} \quad \alpha_1 = \frac{R_0\rho_0^2}{1 - \rho_1}. \]

\[ 0 < \rho_0, \rho_1 < 1 \quad \text{and} \quad \frac{R_1\rho_1^2}{1 - \rho_0} + \frac{R_0\rho_0^2}{1 - \rho_1} \leq 1. \]
Quartic Spline construction

Conclusion: the system of inequality always has infinitely many solutions.
Example (airfoil)

Table 1. Data points for Example 1

<table>
<thead>
<tr>
<th>i</th>
<th>( f_i )</th>
<th>( d_i )</th>
<th>( \kappa_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(1.00, 0.00322)</td>
<td>(-0.756, 0.655)</td>
<td>0.0677</td>
</tr>
<tr>
<td>1</td>
<td>(0.552, 0.337)</td>
<td>(-0.8577, 0.5141)</td>
<td>0.311</td>
</tr>
<tr>
<td>2</td>
<td>(0.220, 0.468)</td>
<td>(-1.00, 0.00)</td>
<td>4.32</td>
</tr>
<tr>
<td>3</td>
<td>(0.0617, 0.350)</td>
<td>(-0.433, -0.901)</td>
<td>3.16</td>
</tr>
<tr>
<td>4</td>
<td>(0.00, 0.00)</td>
<td>(0.00, -1.00)</td>
<td>0.391</td>
</tr>
<tr>
<td>5</td>
<td>(0.0676, -0.403)</td>
<td>(0.537, -0.844)</td>
<td>4.73</td>
</tr>
<tr>
<td>6</td>
<td>(0.316, -0.539)</td>
<td>(1.00, 0.00)</td>
<td>5.43</td>
</tr>
<tr>
<td>7</td>
<td>(0.514, -0.451)</td>
<td>(0.796, 0.605)</td>
<td>1.19</td>
</tr>
<tr>
<td>8</td>
<td>(0.728, -0.243)</td>
<td>(0.676, 0.737)</td>
<td>0.00</td>
</tr>
<tr>
<td>9</td>
<td>(1.00, 0.00)</td>
<td>(0.942, 0.336)</td>
<td>-5.23</td>
</tr>
</tbody>
</table>

Fig. 2. Dimensionless airfoil example. a. The data b. The fit
Floater’s recent approach


Outline:

the general problem by showing that a large class of rational parametric curves can be interpolated, in a Hermite sense, by a polynomial of degree \( m \) matching \( 2m - 2k + 4 \) data, where \( k \) is the total degree of the rational curve.
Target

Given \( r(t) = \frac{f(t)}{g(t)} \),

find a polynomial \( p \) of degree at most \( n+k-2 \) and scalar values \( \mu_1, \ldots, \mu_n \) satisfying the \( 2n \) interpolation conditions

\[
p(t_i) = r(t_i), \quad p'(t_i) = \mu_i r'(t_i), \quad i = 1, 2, \ldots, n.
\]

Two assumptions on the denominator \( g(t) \) needed

(1) \( g(t) \) has no roots in \([a, b]\),

(2) \( g(t) \) has no double roots (real or complex).
Basic idea

Let

\[ p(t) = r(t) + \lambda(t) \omega_n(t) r'(t), \quad \omega_n(t) = (t - t_1)(t - t_2) \cdots (t - t_n), \]

Key problem lying: how to select \( \lambda(t) \) to make \( P(t) \) be a polynomial?
Deduction:

\[
p(t) = \frac{g(t) - \lambda(t)\omega_n(t)g'(t)}{g_2(t)} f(t) + \frac{\lambda(t)\omega_n(t)}{g(t)} g'(t).
\]

\[X(t)\text{ undetermined}\]

\[\lambda(t) = g(t)X(t)\]

\[
p(t) = \frac{1 - X(t)\omega_n(t)g'(t)}{g(t)} f(t) + X(t)\omega_n(t)g'(t)
\]

polynomial

polynomial

\[\omega_n(t)g'(t)X(t) + g(t)Y(t) = 1,\]

\[Y(t)\]

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Deduction:

Find

\[ X(t) \quad Y(t) \]

satisfying

\[ \omega_n(t) g'(t) X(t) + g(t) Y(t) = 1 \]

Approximation result:

\[ p(t) = Y(t)f(t) + X(t)\omega_n(t)f'(t) \]
Euclid’s algorithm

Given two numbers not prime to one another, find their greatest common measure.

If \( b \mid a \) then \( \gcd(a, b) = b \).

This is indeed so because no number (\( b \), in particular) may have a divisor greater than the number itself.

If \( a = bt + r \), for integers \( t \) and \( r \), then \( \gcd(a, b) = \gcd(b, r) \).

Indeed, every common divisor of \( a \) and \( b \) also divides \( r \). Thus \( \gcd(a, b) \) divides \( r \). But, of course, \( \gcd(a, b) \mid b \). Therefore, \( \gcd(a, b) \) is a common divisor of \( b \) and \( r \) and hence \( \gcd(a, b) \gcd(b, r) \). The reverse is also true because every divisor of \( b \) and \( r \) also divides \( a \).
Euclid’s algorithm (Illustration)

- Divide a by b
- Let r be the complement

- Whether r=0?
  - Yes: Finish output r
  - No: a←b, b←r
Degree

\[ p(t) = Y(t)f(t) + X(t)\omega_n(t)f'(t) \]

Total degree of \( P(t) = N + M + n - 2 \), i.e., \( k + n - 2 \)
Approximation order

**Theorem 2.** There are constants \( h_0 > 0 \) and \( C > 0 \) depending only on \( r, a, b, \) and \( n \) such that

$$\max_{t_1 \leq t \leq t_n} |r(\phi(t)) - p(t)| \leq C h^{2n} \quad \text{for} \ h \leq h_0.$$ 

\[ \phi(t) = t + \lambda(t) \omega_n(t) \]
Special case (circle)

\[ \frac{(1-t^2, 2t)}{1 + t^2} \]

\[ r(t) = \frac{f(t)}{g(t)} = \left( \frac{1-t^2}{1 + t^2} , \frac{2t}{1 + t^2} \right) + (1,0) = \frac{(2,2t)}{1 + t^2} \]

Let \( t_1 < \cdots < t_n \) be arbitrary increasing values in \( \mathbb{R} \). If \( r \) is the circle above, a solution \( p \) is

\[ p(t) = Y(t)(2,2t) + X(t)\omega_n(t)(0,2), \]

where \( X \) and \( Y \) are the unique solutions of degrees at most 1 and \( n \) to

\[ 2t\omega_n(t)X(t) + (1 + t^2)Y(t) = 1, \]

and the degree of \( p \) is at most \( n + 1 \).
Circle \((n\ \text{odd and } t_i\ \text{symmetrically choosing})\)

\textbf{Theorem 5.} Suppose \(n = 2s + 1\) for some \(s \geq 0\) and that
\[
(t_1, \ldots, t_n) = (-u_s, \ldots, -u_1, 0, u_1, \ldots, u_s)
\]
for some values \(0 < u_1 < \cdots < u_s\). Then \(p\) in (17) has degree \(n\) and
\[
X(t) = \frac{1}{A_0(-1)}, \quad Y(t) = -\frac{A_0(t^2) - A_0(-1)}{A_0(-1)(1 + t^2)},
\]
where
\[
A_0(u) = 2u(u - u_1^2) \cdots (u - u_s^2).
\]
Figure

(a) \( n = 3, \, \mathbf{u} = (0.0), \, e = 7.8 \times 10^3 \)
(b) \( n = 3, \, \mathbf{u} = (0.5), \, e = 7.4 \times 10^4 \)
(c) \( n = 5, \, \mathbf{u} = (0.0, 0.0), \, e = 4.9 \times 10^4 \)
(d) \( n = 5, \, \mathbf{u} = (0.5, 0.5), \, e = 1.6 \times 10^5 \)
(e) \( n = 5, \, \mathbf{u} = (0.25, 0.5), \, e = 3.6 \times 10^6 \)
(f) \( n = 9, \, \mathbf{u} = (0.125, 0.25, 0.375, 0.5), \, e = 2.8 \times 10^{10} \)
The end

Thanks!