Lecture I. Metrics, connections, curvatures and covariant differentiation

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1 Introduction to Differential Geometry

1.1 A topological manifold M^n, of dimension n, is a Hausdorff topological space, such that each point $x \in M^n$ has a neighborhood U homeomorphic to R^n. (Therefore, a manifold is locally compact and locally connected. A connected manifold is pathwise connected.)

1.2 A local chart on M^n is a pair (U, φ), where U is an open set of M^n, and φ a homeomorphism of U onto an open set of R^n.

A collection $(U_i, \varphi_i)_{i \in I}$ of local charts such that $\cup_{i \in I} U_i = M^n$ is called an **atlas**. The coodinates of $x \in M$, related to φ, are the coordinates of the point $\varphi(x)$ of R^n.

1.3 An atlas of class C^k (respectively C^∞, C^ω) on M^n is an atlas for which all changes of coodinates are C^k (respectively C^∞, C^ω). That is, if $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) are two local charts with $U_\alpha \cap U_\beta \neq \varnothing$, then the map $\varphi_\alpha \circ \varphi_\beta^{-1}$ of $\varphi_\beta(U_\alpha \cap U_\beta)$ onto $\varphi_\alpha(U_\alpha \cap U_\beta)$ is a diffeomorphism of class C^k (respectively C^∞, C^ω).

1.4 **A differentiable structure of class C^k** on topological manifold M^n is a family $\mathcal{U} = (U_\alpha, \varphi_\alpha)_{\alpha \in I}$ of coordinate neighborhoods such that

(1) the U_α cover M,

(2) for any α, β the neighborhoods $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) are C^k-compatible,

(3) any coordinate neighborhood (V, ψ) compatible with every $(U_\alpha, \varphi_\alpha) \in \mathcal{U}$ is itself in $\mathcal{U}$.

A differentiable manifold of class C^k (respectively C^∞, C^ω) is a topological manifold with a C^k (respectively C^∞, C^ω) differentiable structure.

1.5 **A mapping f of a differentiable C^k manifold N^p into another M^n, is called differentiable $C^r\,(r \leq k)$ at $x \in U \subset N^p$** if $\psi \circ f \circ \varphi^{-1}$ differentiable C^r at $\varphi(x)$. Here (U, φ) is a local chart of N^p and (V, ψ) is a local chart of M^n with $f(x) \in V \subset M^n$.

A C^r differentiable mapping f is an immersion if the rank of f is equal to p for every point $x \in N^p$. **It is an imbedding** if f is an injective immersion such that f is a homeomorphism of N^p onto $f(N^p)$ with the topology induced from M^n.

1.5.1 **Examples.**

(a) $F : R \to R^2$ is given by

$$F(t) = (2\cos(t - \frac{1}{2}\pi), \sin 2(t - \frac{1}{2}\pi)).$$

The image is a "figure eight". It is an immersion but not imbedding.

(b) $G : R \to R^2$ is given by

$$G(t) = (2\cos(g(t) - \frac{\pi}{2}), \sin 2(g(t) - \frac{\pi}{2})),$$

where $g(t)$ is a monotone increasing C^∞ function on $-\infty < t < \infty$ such that $g(0) = \pi, \lim_{t \to -\infty} g(t) = 0$ and $\lim_{t \to +\infty} g(t) = 2\pi$. For example, we may use $g(t) = \pi + 2\tan^{-1} t$. Then G is an injective immersion but not imbedding.

(c) $H : R \to R^2, H(t) = (t, t^3)$ is an imbedding.

1.6 **A tangent vector at $x \in M^n$ is a map $X : f \to X(f) \in R$** defined on the set of functions differentiable in a neighborhood of x, where X satisfies:

(a) If $\lambda, \mu \in R, X(\lambda f + \mu g) = \lambda X(f) + \mu X(g)$.

(b) $X(f) = 0$ if f is constant.

(c) $X(fg) = f(x) X(g) + g(x) X(f)$.

1.7 **The tangent space $T_x(M)$ at** $x \in M^n$ is the set of tangent vectors at x. It has a natural vector space structure. In a coodinates system $\{x^i\}$ at x, the vectors $\frac{\partial}{\partial x^i}|_x$ defined by $\left(\frac{\partial}{\partial x^i}\right)_x f = \left[\frac{\partial\left(f \circ \varphi^{-1}\right)}{\partial x^i}\right]_{\varphi(x)}$, form a basis.

The tangent space $T(M) = \cup_{x \in M^n} T_x(M)$. **It has a natural vector fiber bundle structure**. if $T_x^*(M)$ denotes the dual space of $T_x(M)$, **the cotangental space** is $T^*(M) = \cup_{x \in M^n} T_x^*(M)$. Likewise, **the fiber bundle** $T_s^r(M)$ **of the tensor of type** (r, s) is $\cup_{x \in M^n} \overset{r}{\otimes} T_x(M) \overset{s}{\otimes} T_x^*(M)$.

1.8 Let $x \in M^n$ and Φ be a differentiable map of M^n into N^p. Set $y = \Phi(x)$. The map Φ induces a linear map $\Phi_* : T_x(M) \to T_y(N), (\Phi_* X)(f) = X(f \circ \Phi)$, where $X \in T_x(M)$ and f is a differentiable function in a neiborhood of y. We call Φ_* **the linear tangent mapping of** Φ.

By duality, we define the **linear cotangent mapping** $\Phi^* : T^*(N) \to T^*(M)$ as follows: $T_y^*(N) \to T_x^*(M), \forall \omega \in T_y^*(N), \Phi^*(\omega)$ defined as follows:

$$\langle \Phi^*(\omega), X \rangle = \langle \omega, \Phi_*(X) \rangle, \quad \text{for all } X \in T_x(M).$$

One verifies easily that $\Phi^*(df) = d(f \circ \Phi)$.

1.9 **A differentiable vector field** is a section of $T(M)$. **A section of vector fiber bundle** (E, π, M) is a differentiable map ξ of M into E, such that $\pi \circ \xi = id$. If $E = T(M)$, π is the mapping of E onto $M : T_x(M) \ni X \to x$.

The bracket $[X, Y]$ of two vector fields X and Y is the vector field defined by

$$[X, Y](f) = X(Y(f)) - Y(X(f)).$$

A differentiable tensor field of type (r, s) is a section of $T_s^r(M)$. Especially, $T_x^r(M) = \otimes^r T_x^*(M)$. We denote $T_x^0(M) = R, T_x^1(M) = T_x^*(M)$. The symmetric tensors in $T_x^r(M)$ form a subspace which we denote by $\Sigma_x^r(M)$ and the alternating tensors also form a subspace $\Lambda_x^r(M)$. Alternating mapping $\mathcal{A} : T_x^r(M) \to T_x^r(M)$ is defined by the formula:

$$(\mathcal{A}\Phi)(v_1, ..., v_r) = \frac{1}{r!} \Sigma_\sigma sgn\sigma \Phi(v_{\sigma(1),...,v_{\sigma(r)}}),$$

the summation being over all $\sigma \in \varphi(r)$, the group of all permutations of r letters. Then we have $\mathcal{A}(T_x^r(M)) = \Lambda_x^r(M)$. The mapping $\Lambda^r(M) \times \Lambda^s(M) \to \Lambda^{r+s}(M)$, defined by

$$(\varphi, \psi) \to \frac{(r+s)!}{r!s!} \mathcal{A}(\varphi \otimes \psi),$$

is called the exterior product (or wedge product) of φ and ψ and is denoted $\varphi \wedge \psi$.

1.10 An exterior differential p-form η is a section of $\Lambda^{p+1} T^* M$. In a local chart,

$$\eta = \Sigma_{j_1 < ... < j_p} \eta_{j_1 ... j_p} dx^{j_1} \wedge \cdots \wedge dx^{j_p},$$

and **the exterior differentiation** $d\eta$ **of** η is

$$d\eta = \Sigma_{j_1 < ... < j_p} d\eta_{j_1 ... j_p} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_p}.$$

It is clearly that $dd\eta = 0$.

Denote by $\Lambda^p(M)$ **the space of exterior differential p-forms**. For $\alpha \in \Lambda^p(M)$ and $\beta \in \Lambda^q(M)$,

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta.$$

It is easily to verify.

1.11 An **affine connection** is a map D (called the **affine covariant derivative**) of $TM \times \mathcal{X}(M)(M) \to TM$ ($\mathcal{X}(M)$ denotes the space of differentiable vector fields):

(a) $D(X_p, Y) = D_{X_p}Y \in T_pM$ when $X_p \in T_pM$.

(b) For any $p \in M$, any tangential vectors $X_p, Z_p \in T_pM$, any $Y \in \mathcal{X}(M)$ and any smooth functions $f, g \in C^\infty(M)$,

$$D_{f(p)X_p+g(p)Z_p}Y = f(p)\, D_{X_p}Y + g(p)\, D_{Z_p}Y$$

.

(c) If f, h are differentiable functions, $Y, Z \in \mathcal{X}(M)$

$$D_{X_p}(fY + hZ) = X_p(f)\, Y + f(p)\, D_{X_p}Y + X_p(h)\, Z + h(p)\, D_{X_p}Z.$$

If X and Y belong to $\mathcal{X}(M)$, X of class C^r and Y of class C^{r+1}, then D_XY is of class C^r.

In a local chart (U, φ), denote $\nabla_i Y = D_{\partial/\partial x^i}Y$. Conversely, if we are given, for all pair (i, j),

$$\nabla_i \frac{\partial}{\partial x^j} = \Gamma^k_{ij} \frac{\partial}{\partial x^k},$$

then a unique connection D is defined.

The functions Γ^k_{ij} are called the **Christoffel symbols** of the connection D with respect to the local coordinates system $x^1, ..., x^n$. 1.12 The **torsion** of the connection is the map $T : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$:

$$T(X, Y) = D_XY - D_YX - [X, Y].$$

$T^k\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \Gamma^k_{ij} - \Gamma^k_{ji}$ are the components of a tensor (hence, $T \equiv 0 \iff \Gamma^k_{ij} = \Gamma^k_{ji}$).

1.13 The **curvature** of the connection is the 2-form with values in $Hom(\mathcal{X}(M), \mathcal{X}(M))$ defined by

$$R(X, Y) = D_XD_Y - D_YD_X - D_{[X,Y]}.$$

One verifies that $R(X,Y)Z$ at $p \in M$ depends only upon the values of X, Y and Z at p.

In a local chart, denote by R^l_{kij} the l-th component of $R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k}$. R^l_{kij} are the components of a tensor, called the curvature tensor, and

$$R^l_{kij} Z^k = \nabla_i \nabla_j Z^l - \nabla_j \nabla_i Z^l.$$

It follows that

$$R^l_{kij} = \partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} + \Gamma^l_{im}\Gamma^m_{jk} - \Gamma^l_{jm}\Gamma^m_{ik}. \tag{1.1}$$

1.14 The definition of covariant derivative extends to differentiable tensor fields as follows:

(a) For functions, $D_X f = X(f)$.
(b) D_X preserves the type of the tensor.
(c) Let $\omega \in T^*M$ and $X, Y \in \mathcal{X}(M)$, $(D_X\omega)Y = X(\omega(Y)) - \omega(D_X Y)$.
(d) D_X commutes with the contraction.
(f) $D_X(u \otimes v) = (D_X u) \otimes v + u \otimes (D_X v)$, where u and v are tensor fields.

For (c), in local coordinates, $\nabla_i dx^j = -\Gamma^j_{ik} dx^k$.

2 Riemannian metrics, connections, curvatures and covariant differentiation

2.1 **Metrics and connections.**

A **Riemannian manifold** (M, g) consists of a (C^∞) manifold M and a Euclidean inner product g_x on all of the tangent spaces $T_x M$ of M. g is called a **Riemannian metric**.

We shall assume that g_x varies smoothly. This means that for any two smooth vector fields X, Y, the inner product $g_x(X,Y)$ should be a smooth function of x. Thus we may think of g either as a positive-definite section of the boundle of symmetric covariant 2-tensors $T^*M \otimes T^*M$ or as positive-definite bilinear maps $g(x) : T_x M \times T_x M \to R$, for all $x \in M$.

Theorem 2.1 *On a paracompact C^∞ differentiable manifold, there exists a C^∞ Riemannian metric g.*

The proof of Theorem 2.1 depends on the partition of unity theorem on M.

A covering $\{A_\alpha\}$ of a manifold M by subsets is said to be **locally finite** if each $p \in M$ has a neighborhood U which intersects only a finite number of sets A_α. If $\{A_\alpha\}$ and $\{B_\beta\}$ are coverings of M, then $\{B_\beta\}$ is called a **refinement** of $\{A_\alpha\}$ if each $B_\beta \subset A_\alpha$ for some α.

Recall that the **support** of a function f on a manifold M is the set $\text{supp}(f) = \overline{\{x \in M |\ f(x) \neq 0\}}$, the closure of the set on which f does not vanish.

Definition. A C^∞ **partition of unity on** M is a collection of C^∞ functions $\{f_\gamma\}$ defined on M with the following properties:

(1) $f_\gamma \geq 0$ on M,
(2) $\{\text{supp}(f_\gamma)\}$ form a locally finite covering of M, and
(3) $\Sigma_\gamma f_\gamma(x) = 1$ for every $x \in M$.

Partition of unity Theorem. *Every open covering $\{A_\alpha\}$ has a partition of unity which is subordinate to it.*

Proof of Theorem 2.1. Let $(\Omega_i, \varphi_i)_{i\in I}$ be an atlas and $\{\alpha_i\}$ a C^∞ partition of unity subordinate to the covering $\{\Omega_i\}$. Such $\{\alpha_i\}$ exists since the manifold M^n is paracompact. Set $\epsilon = (\epsilon_{ij})$ be the Euclidean metric on R^n (in an orthonormal basis, $\epsilon_{ij} = \delta_i^j$). Then $g = \Sigma_{i\in I}\alpha_i\varphi_i^*(\epsilon)$ is a Riemannian metric on M^n, as one can easily verify. ■

The metric g defines an infinitesimal notion of length and angle. The length of a tangent vector X is defined by

$$|X| =: g(X,X)^{1/2}$$

and the angle between two nonzero tangent vectors X and Y is defined by

$$\angle(X,Y) = \cos^{-1}\left(\frac{\langle X,Y\rangle}{|X|\,|Y|}\right).$$

Let $\{x^i\}_{i=1}^n$ be local coordinates in a neighborhood U of some point of M. In U the vector fields $\{\partial/\partial x^i\}_{i=1}^n$ form a local basis for TM and the 1-forms $\{dx^i\}_{i=1}^n$ form a dual basis for T^*M, that is $dx^i(\partial/\partial x^j) = \delta_j^i$. The metric g may then be written in local coordinates as

$$g = g_{ij}dx^i \otimes dx^j,$$

where $g_{ij} = g(\partial/\partial x^i, \partial/\partial x^j)$.

Given a smooth immersion $\varphi : N^m \to M^n$ and a metric g on M^n, we can pull back g to a metric on N^m :

$$(\varphi^* g)(V, W) =: g(\varphi_* V, \varphi_* W),$$

where $\varphi_* : TN \to TM$ is the tangent map. If $\{y^\alpha\}$ and $\{x^i\}$ are local coordinates on N and M, respectively, then

$$(\varphi^* g)_{\alpha\beta} = g_{ij} \frac{\partial \varphi^i}{\partial y^\alpha} \frac{\partial \varphi^j}{\partial y^\beta},$$

where $(\varphi^* g)_{\alpha\beta} =: (\varphi^* g)\left(\partial/\partial y^\alpha, \partial/\partial y^\beta\right)$ and $\varphi^i =: x^i \circ \varphi$. More generally, given any covariant p-tensor α on M^n and a smooth map $\varphi : N^m \to M^n$, we define the pull back of α to N by

$$(\varphi^* \alpha)(X_1, ..., X_p) = \alpha(\varphi_* X_1, ..., \varphi_* X_p)$$

for all $X_1, ..., X_p \in T_y N$. If φ is a *diffeomorphism*, then the pull back of contravariant tensor is defined as the push forward by φ^{-1}.

The **Levi-Civita connection** (or **Riemannian covariant derivative**) $\nabla : TM \times \mathcal{X}(M) \to \mathcal{X}(M)$ is the unique connection on TM that is compatible with the metric and is torsion-free:

(a) $X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$,

(b) $\nabla_X Y - \nabla_Y X = [X, Y]$,

where

$$[X, Y] f =: X(Yf) - Y(Xf)$$

defines the Lie bracket acting on functions.

From this one can easily show that for any X, Y, Z

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X(g(Y, Z)) + Y(g(X, Z)) - Z(g(X, Y)) \\ &\quad + g([X, Y], Z) - g([X, Z], Y) - g([Y, Z], X). \end{aligned} \tag{2.1}$$

This gives the formula for $\nabla_X Y$. (a) is the product rule (compactibility with the metric) and (b) is a compactibility condition with the differentiable structure (torsion-free).

Local expression:

Let $\{x^i\}_{i=1}^n$ be local coordinates in a neighborhood U of some point of M. The **Christoffel symbols** are the components of the Levi-Civita connection and are defined in U by

$$\nabla_{\partial/\partial x^i}\frac{\partial}{\partial x^j} =: \Gamma_{ij}^k \frac{\partial}{\partial x^k}. \tag{2.2}$$

By (2.1) and $\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0$, we see that

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} \left(\frac{\partial}{\partial x^i} g_{jl} + \frac{\partial}{\partial x^j} g_{il} - \frac{\partial}{\partial x^l} g_{ij} \right).$$

Let $\{x^i\}$ and $\{y^\alpha\}$ be coordinate functions on a common open set. Using $g_{\alpha\beta} = g_{ij} \frac{\partial x^i}{\partial y^\alpha} \frac{\partial x^j}{\partial y^\beta}$, then we can show

$$\Gamma_{\alpha\beta}^\gamma \frac{\partial x^k}{\partial y^\gamma} = \Gamma_{ij}^k \frac{\partial x^i}{\partial y^\alpha} \frac{\partial x^j}{\partial y^\beta} + \frac{\partial^2 x^k}{\partial y^\alpha \partial y^\beta}.$$

Parallel Displacement, Geodesic

A vector field X along a path $\gamma : [a, b] \to M^n$ is **parallel** if

$$\nabla_{\dot{\gamma}} X = 0$$

along γ; the vector field $X(\gamma(t))$ is called the parallel translation of $X(\gamma(a))$. We say that a path γ is **geodesic** if the tangent vector field is parallel along γ :

$$\nabla_{\dot{\gamma}} \dot{\gamma} = 0.$$

We will show later the shortest path between two points is a geodesic and geodesics locally minimize length.

It is easy to show $|X|^2$ is constant along γ, if X is parallel along the path γ. So if γ is geodesic, $\left|\dot{\gamma}\right| = const.$.

2.2 Rieman Curvature

The **Riemann Curvature** $(3,1)$**-tensor** Rm is defined by

$$R(X, Y) Z =: \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{2.3}$$

One easily checks that for any function f

$$R(fX, Y) Z = R(X, fY) Z = R(X, Y)(fZ) = fR(X, Y) Z.$$

Rm is indeed a tensor. It is also nice to define

$$\nabla^2_{X,Y} Z =: \nabla_X \nabla_Y Z - \nabla_{\nabla_X Y} Z$$

so that

$$R(X,Y)Z = \nabla^2_{X,Y} Z - \nabla^2_{Y,X} Z,$$
$$\nabla^2_{fX,Y} Z = \nabla^2_{X,fY} Z = f \nabla^2_{X,Y} Z,$$

and

$$\nabla^2_{X,Y}(fZ) = f\nabla^2_{X,Y} Z + Y(f)\nabla_X Z + X(f)\nabla_Y Z \\ - ((\nabla_X Y) f) Z + X(Y(f)) Z$$

for any function f.

In local coordinates $\{x^i\}$, we may write the components of the $(3,1)$-tensor Rm as

$$R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k} =: R^l_{kij}\frac{\partial}{\partial x^l}$$

and $R_{klij} =: g_{km} R^m_{lij}$. From (2.2) and (2.3), we may get the same formula as (1.1):

$$R^l_{kij} = \partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} + \Gamma^l_{im}\Gamma^m_{jk} - \Gamma^l_{jm}\Gamma^m_{ik}.$$

Note

$$R_{klij} = R\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}, \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) =: \left\langle R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^l}, \frac{\partial}{\partial x^k}\right\rangle$$

are the components of Rm as a $(4,0)$-tensor. Some basic symmetries of the Riemann curvature tensor are

$$R_{ijkl} = -R_{jikl} = -R_{ijlk} = R_{klij}.$$

Sectional curvature

If $P \subset T_x M^n$ is a 2-plane, then the **sectional curvature** of P is defined by

$$K(P) =: R(e_1, e_2, e_1, e_2)$$

where $\{e_1, e_2\}$ is an orthonormal basis of P; this definition is independent of the choice of such a basis. Equivalently, if $P = \mathit{span}\{X, Y\} \subset T_x M$, then

$$K(P) = \frac{R(X,Y,X,Y)}{|X|^2 |Y|^2 - \langle X, Y\rangle^2}.$$

Ricci Tensor

The **Ricci tensor** Rc is the trace of the Riemann curvature tensor:

$$Rc\,(Y,Z) =: trace\,(X \mapsto R\,(X,Y)\,Z)\,.$$

In terms of an orthonormal frame $\{e_i\}_{i=1}^n$, i.e. a frame with $g\,(e_i,e_j) = \delta_{ij}$, we have

$$Rc\,(Y,Z) = \Sigma_{i=1}^n \langle R\,(e_i,Y)\,Z, e_i\rangle\,.$$

Its components $R_{ij} =: Rc\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$ are given by

$$R_{ij} = \Sigma_k R_{ikj}^k = g^{kl} R_{likj}.$$

The **Ricci curvature** of a line $L \subset T_x M$ is defined by

$$Rc\,(L) =: Rc\,(e_1,e_1)\,,$$

where $e_1 \in T_x M$ is a unit vector spanning L.

Scalar curvature

The **scalar curvature** is the trace of the Ricci tensor

$$R =: \Sigma_{i=1}^n Rc\,(e_i,e_i)\,.$$

In local coordinates, $R = g^{ij} R_{ij}$.

A Riemannian manifold (M^n, g) has **constant sectional curvature** if the sectional curvature of every 2-plane is the same. That is, there exists a constant $k \in R$, such that for every $x \in M$ and 2-plane $P \subset T_x M, K\,(P) = k$. Similarly we say that a metric has **constant Ricci curvature** if the Ricci curvature of every line is the same. A metric has **constant scalar curvature** if the scalar curvature is constant at every point $x \in M$. In local coordinates, a Riemannian manifold (M^n, g) has constant sectional curvature k if and only if the curvature tensor Rm satisfies

$$R_{ijkl} = k\,(g_{ik} g_{jl} - g_{il} g_{jk})\,;$$

(M^n, g) has constant Ricci curvature k if and only if

$$R_{ij} = k g_{ij}. \tag{2.4}$$

One can show that $k = R/n$, where R is the scalar curvature. Then (2.4) becomes

$$R_{ij} = \frac{R}{n} g_{ij}. \tag{2.5}$$

Equation (2.5) is called Eistein equation. The metric g which satisfies (2.5) is called Einstein metric.

2.3 **Covariant differentiation**

Acting on $(r,0)$-tensors, we define covariant differentiation by

$$\nabla_X : C^\infty\left(\otimes^r TM\right) \to C^\infty\left(\otimes^r TM\right),$$

where

$$\nabla_X\left(Z_1 \otimes \cdots \otimes Z_r\right) =: \Sigma_{i=1}^r Z_1 \otimes \cdots \otimes \nabla_X Z_i \otimes \cdots \otimes Z_r.$$

Let $\otimes_s^r M = (\otimes^r TM) \otimes (\otimes_s T^*M)$. The covariant derivative of an (r,s)-tensor α is defined by

$$\begin{aligned}
&\left(\nabla_X \alpha\right)\left(\omega^1, ..., \omega^r, Y_1, ..., Y_s\right) \\
&=: X\left(\alpha\left(\omega^1, ..., \omega^r, Y_1, ..., Y_s\right)\right) \\
&- \Sigma_{i=1}^r \alpha\left(\omega^1, ..., \nabla_X \omega^i, ..., \omega^r, Y_1, ..., Y_s\right) \\
&- \Sigma_{j=1}^s \alpha\left(\omega^1, ..., \omega^r, Y_1, ..., \nabla_X Y_j, ...Y_s\right)
\end{aligned}$$

where

$$\left(\nabla_X \omega\right)(Y) = X\left(\omega(Y)\right) - \omega\left(\nabla_X Y\right).$$

The covariant derivative may be considered as

$$\nabla : C^\infty\left(\otimes_s^r M\right) \to C^\infty\left(\otimes_{s+1}^r M\right),$$

where

$$\begin{aligned}
&\left(\nabla \alpha\right)\left(\omega^1, ..., \omega^r, Y_1, ..., Y_s; X\right) \\
&=: \left(\nabla_X \alpha\right)\left(\omega^1, ..., \omega^r, Y_1, ..., Y_s\right)
\end{aligned}$$

By the definition of the Levi-Civita connection (a), we know that

$$\nabla g = 0.$$

That is the Riemannian metric g is parallel w.r.t. Levi-Civita connection. In local coordinates $\{x^i\}$, let

$$\alpha = \alpha_{j_1 \ldots j_s}^{i_1 \ldots i_r} \frac{\partial}{\partial x^{i_1}} \otimes ... \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes ... \otimes dx^{j_s}.$$

Then

$$\nabla_X \alpha = X^k \alpha^{i_1 \ldots i_r}_{j_1 \ldots j_s, k} \frac{\partial}{\partial x^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_s}$$

where

$$\begin{aligned} \alpha^{i_1 \ldots i_r}_{j_1 \ldots j_s, k} = & \frac{\partial \alpha^{i_1 \ldots i_r}_{j_1 \ldots j_s}}{\partial x^k} + \Sigma^r_{m=1} \Gamma^{i_m}_{hk} \alpha^{i_1 \ldots i_{m-1} h i_{m+1} \ldots i_r}_{j_1 \ldots j_s} \\ & - \Sigma^s_{m=1} \Gamma^h_{j_m k} \alpha^{i_1 \ldots i_r}_{j_1 \ldots j_{m-1} h j_{m+1} \ldots j_s}. \end{aligned}$$

$$\nabla \alpha = \alpha^{i_1 \ldots i_r}_{j_1 \ldots j_s, k} dx^k \otimes \frac{\partial}{\partial x^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_s}.$$

Particular, we have

$$\nabla_i \beta_j =: \beta_{j,i} = \frac{\partial \beta_j}{\partial x^i} - \Gamma^k_{ij} \beta_k$$

and

$$\nabla_i \nabla_j f = f_{j,i} = \frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial f}{\partial x^k}.$$

The **square of the covariant derivative operator**:

$$\nabla^2 : C^\infty \left(\otimes^r_s M \right) \to C^\infty \left(\otimes^r_{s+2} M \right)$$

is given by

$$\begin{aligned} & \left(\nabla^2 \alpha \right) \left(\omega^1, \ldots, \omega^r, Y_1, \ldots, Y_s; Y, X \right) \\ & = \left(\nabla_X \left(\nabla \alpha \right) \right) \left(\omega^1, \ldots, \omega^r, Y_1, \ldots, Y_s; Y \right) \\ & = \nabla_X \left(\left(\nabla \alpha \right) \left(\omega^1, \ldots, \omega^r, Y_1, \ldots, Y_s; Y \right) \right) \\ & - \Sigma^r_{i=1} \left(\nabla \alpha \right) \left(\omega^1, \ldots, \nabla_X \omega^i, \ldots, \omega^r, Y_1, \ldots, Y_s; Y \right) \\ & - \Sigma^s_{j=1} \left(\nabla \alpha \right) \left(\omega^1, \ldots, \omega^r, Y_1, \ldots, \nabla_X Y_j, \ldots Y_s; Y \right) \\ & - \left(\nabla \alpha \right) \left(\omega^1, \ldots, \omega^r, Y_1, \ldots, Y_s; \nabla_X Y \right) \\ & = \nabla_X \left(\left(\nabla_Y \alpha \right) \left(\omega^1, \ldots, \omega^r, Y_1, \ldots, Y_s \right) \right) \\ & - \Sigma^r_{i=1} \left(\nabla_Y \alpha \right) \left(\omega^1, \ldots, \nabla_X \omega^i, \ldots, \omega^r, Y_1, \ldots, Y_s \right) \\ & - \Sigma^s_{j=1} \left(\nabla_Y \alpha \right) \left(\omega^1, \ldots, \omega^r, Y_1, \ldots, \nabla_X Y_j, \ldots Y_s \right) \\ & - \left(\nabla_{\nabla_X Y} \alpha \right) \left(\omega^1, \ldots, \omega^r, Y_1, \ldots, Y_s \right) \\ & = \left(\nabla_X \left(\nabla_Y \alpha \right) \right) \left(\omega^1, \ldots, \omega^r, Y_1, \ldots, Y_s \right) \\ & - \left(\nabla_{\nabla_X Y} \alpha \right) \left(\omega^1, \ldots, \omega^r, Y_1, \ldots, Y_s \right), \end{aligned}$$

that is

$$\left(\nabla^2\alpha\right)(...;Y,X) = \left(\nabla_X\left(\nabla_Y\alpha\right)\right)(...) - \left(\nabla_{\nabla_X Y}\alpha\right)(...).$$

If we use the notation

$$\nabla^2_{X,Y}\alpha\,(...) =: \nabla^2\alpha\,(...;Y,X)\,,$$

then we have

$$\nabla^2_{X,Y}\alpha = \nabla_X\left(\nabla_Y\alpha\right) - \nabla_{\nabla_X Y}\alpha.$$

Commuting covariant derivative

For any tensor α, we may define

$$R\,(X,Y)\,\alpha = \nabla_X\nabla_Y\alpha - \nabla_Y\nabla_X\alpha - \nabla_{[X,Y]}\alpha.$$

Then we have the following **commutation formulas** (**Ricci identities**):

$$\nabla^2_{X,Y}\alpha - \nabla^2_{Y,X}\alpha = R\,(X,Y)\,\alpha.$$

In local coordinates, the Ricci identities is

$$\begin{aligned}&\alpha^{i_1\ldots i_r}_{j_1\ldots j_s,lk} - \alpha^{i_1\ldots i_r}_{j_1\ldots j_s,kl}\\ &= \Sigma^s_{m=1}\alpha^{i_1\ldots i_r}_{j_1\ldots j_{m-1}hj_{m+1}\ldots j_s}R^h_{j_mlk} - \Sigma^r_{m=1}\alpha^{i_1\ldots i_{m-1}hi_{m+1}\ldots i_r}_{j_1\ldots j_s}R^{i_m}_{hlk}.\end{aligned}$$