Section 3. Curvature decomposition, conformal metric and Cartan structure equations

Weimin Sheng

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1. Decomposition of the curvature tensor.

The Riemann curvature $(0,4)$-tensor is a sectin of the bundle $\wedge^2 T^*M \otimes_S \wedge^2 T^*M$, where $\wedge^2 T^*M$ denotes the vector bundle of 2-forms and $\otimes_S$ denotes the symmetric tensor product. By the first Bianchi identity, Rm is a section of the subbundle $\ker(b)$, the kernel of the linear map:

$$b : \wedge^2 T^*M \otimes_S \wedge^2 T^*M \to T^*M \otimes_S \wedge^3 T^*M$$

defined by

$$\begin{aligned} & b(\Omega)(X,Y,Z,W) \\ & = \frac{1}{3}\left(\Omega(X,Y,Z,W) + \Omega(X,Z,W,Y) + \Omega(X,W,Y,Z)\right). \end{aligned}$$

We shall call $CM =: \ker(b)$ the bundle of curvature tensors. For every $x \in M^n$, the fiber C_xM has the structure of an $O(T_x^*M)$-module, given by

$$\times : O(T_x^*M) \times C_xM \to C_xM,$$

where

$$A \times (\alpha \otimes \beta \otimes \gamma \otimes \delta) := A\alpha \otimes A\beta \otimes A\gamma \otimes A\delta$$

for $A \in O(T_x^*M)$ and $\alpha, \beta, \gamma, \delta \in T_x^*M$. As an $O(T_x^*M)$ representation space, C_xM has a natural decomposition into its irreducible components. This yields a corresponding decomposition of the Riemann curvature tensor. To describe this, it will be convenient to consider the Kulkarni-Normizu product

$$\odot : S^2M \times S^2M \to CM$$

defined by

$$(\alpha \odot \beta)_{ijkl} := \alpha_{ik}\beta_{jl} + \alpha_{jl}\beta_{ik} - \alpha_{il}\beta_{jk} - \alpha_{jk}\beta_{il}.$$

Here $S^2M = T^*M \otimes_S T^*M$ is the bundle of symmetric 2-tensors. The irreducible decomposition of C_xM as an $O(T_x^*M)$-module is given by

$$CM = \mathbb{R}g \odot g \oplus \left(S_0^2M \odot g\right) \oplus WM,$$

where S_0^2M is the bundle of symmetric, trace-free 2-tensors and

$$WM := Ker\,(b) \cap Ker\,(c)$$

is the bundle of Weyl curvature tensors. Here

$$c : \Lambda^2 M^n \otimes_S \Lambda^2 M \to S^2M$$

is the contraction map defined by

$$c\,(\Omega)\,(X, Y) := \sum_{i=1}^{n} \Omega\,(e_i, X, e_i, Y)\,.$$

Note also that $(g \odot g)_{ijkl} = 2\,(g_{ik}g_{jl} - g_{il}g_{jk})$.

The irreducible decomposition of CM yields the following irreducible decomposition of the Riemann curvature tensor:

$$Rm = fg \odot g + (h \odot g) + W,$$

where $f \in C^\infty(M)\,, h \in C^\infty(S_0^2M)$ and $W \in C^\infty(WM)$. Take the contraction c of this equation implies

$$R_{jk} = 2\,(n-1)\,fg_{jk} + (n-2)\,h_{jk}.$$

Taking two contractions, we find that

$$R = 2n\,(n-1)\,f.$$

Therefore we have for $n \geq 3$

$$Rm = -\frac{R}{2\,(n-1)\,(n-2)} g \odot g + \frac{1}{n-2} Rc \odot g + Weyl \tag{3.1}$$

$$= \frac{R}{2\,(n-1)\,n} g \odot g + \frac{1}{n-2} \overset{\circ}{Rc} \odot g + Weyl. \tag{3.2}$$

where $\overset{\circ}{Rc} := Rc - \frac{R}{n}g$ is the traceless Ricci tensor and Weyl is the **Weyl tensor** which is defined by (3.1). The Weyl tensor has the same algebraic symmetries as the Riemann curvature tensor and in addition the Weyl tensor is totally trace-free, all of its traces are zero. Furthermore, the Weyl tensor is conformally invariant:

$$Weyl\left(e^{2f}g\right) = e^{2f}Weyl\left(g\right)$$

for any smooth function f on M.

In local coordinates, (3.1) says that for $n \geq 3$

$$\begin{aligned} R_{ijkl} = &-\frac{R}{\left(n-1\right)\left(n-2\right)}\left(g_{ik}g_{jl} - g_{il}g_{jk}\right) \\ &+\frac{1}{n-2}\left(R_{ik}g_{jl} + R_{jl}g_{ik} - R_{il}g_{jk} - R_{jk}g_{il}\right) + W_{ijkl}. \end{aligned} \tag{3.3}$$

In particular, if $n \leq 3$ then the Weyl tensor vanishes. If $n = 2$, we have

$$R_{ijkl} = \frac{1}{2}R\left(g_{ik}g_{jl} - g_{il}g_{jk}\right),$$

and $R_{ij} = \frac{1}{2}Rg_{ij}$. When $n = 3$,

$$R_{ijkl} = R_{ik}g_{jl} + R_{jl}g_{ik} - R_{il}g_{jk} - R_{jk}g_{il} - \frac{1}{2}R\left(g_{ik}g_{jl} - g_{il}g_{jk}\right). \tag{3.4}$$

2. Conformal metric

Definition 3.1. *Let (M^n, g) be a Riemannian manifold and f be a smooth function on M^n. Then $h = e^{2f}g$ is called a* ***conformal metric*** *of g.*

Proposition 3.1. *If $\tilde{g} = e^{2f}g$, then*

$$\tilde{R}^i_{jkl} = R^i_{jkl} - a^i_k g_{jl} - a_{jl}\delta^i_k + a^i_l g_{jk} + a_{jk}\delta^i_l, \tag{3.5}$$

where

$$a_{ij} := \nabla_i\nabla_j f - \nabla_i f\nabla_j f + \frac{1}{2}\left|\nabla f\right|^2 g_{ij}.$$

That is, as $(0,4)-$tensors,

$$e^{-2f}\tilde{Rm} = Rm - a \odot g. \tag{3.6}$$

From (3.5) or (3.6) we may get

$$\tilde{R}_{ij} = R_{ij} - (n-2)\, a_{ij} - \left(\Delta f + \frac{n-2}{2} \left| \nabla f \right|^2 \right) g_{ij} \tag{3.7}$$

and

$$e^{2f} \tilde{R} = R - 2\,(n-1) \left(\Delta f + \frac{n-2}{2} \left| \nabla f \right|^2 \right). \tag{3.8}$$

The Yamabe problem is to find a conformal metric $\tilde{g} \in [g]$ (the conformal class of g) such that the scalar curvature of $\tilde{g}$ equals to a constant $c \in \mathbb{R}$. This problem is equivalent to solve the equation

$$ce^{2f} = R - 2\,(n-1) \left(\Delta f + \frac{n-2}{2} \left| \nabla f \right|^2 \right).$$

We call

$$S := \frac{1}{(n-2)} \left(Rc - \frac{1}{2\,(n-2)} Rg \right)$$

the **Schouten tensor**. By (3.1), one may easily show that

$$Rm = Weyl + S \odot g. \tag{3.9}$$

We may compute that

$$\begin{aligned} \tilde{S}_{ij} &= S_{ij} - \nabla_i \nabla_j f + \nabla_i f \nabla_j f - \frac{1}{2} \left| \nabla f \right|^2 g_{ij} \\ &= S_{ij} - a_{ij}. \end{aligned} \tag{3.10}$$

By (3.9), (3.10) and (3.6), we can conclude that

Proposition 3.2. *If $\tilde{g} = e^{2f} g$, then the (1,3)-Weyl tensor*

$$\tilde{W}^{\,i}_{\ jkl} = W^i_{jkl} \tag{3.11}$$

and (0,4)-Weyl tensor satisfies

$$\tilde{W}_{ijkl} = e^{2f} W_{ijkl}. \tag{3.12}$$

Proposition 3.3. *If* $n \geq 3$, *then*

$$\nabla^l W_{lijk} = \frac{n-3}{n-2} C_{ijk}, \tag{3.13}$$

where

$$C_{ijk} := S_{ij,k} - S_{ik,j}$$

is the ***Cotton tensor***.

From proposition 3.3, we see that for $n \geq 4$, if the weyl tensor vanishes, then the Cotton tensor always vanishes. We also see that when $n = 3$, the Weyl tensor always vanishes but the Cotton tensor does not vanish in general.

Proposition 3.4. *When* $n = 3$, *if* $\tilde{g} = e^{2f} g$, *then*

$$\tilde{C}_{ijk} = C_{ijk}. \tag{3.14}$$

3. Locally conformally flat manifolds.

We say that a Riemannian manifold (M^n, g) is **locally conformally flat** if for every point $p \in M^n$, there exists a local coordinates $\{x^i\}$ in a neighborhood U of p such that

$$g_{ij} = v \cdot \delta_{ij}$$

for some function v defined on U, e.g., $v^{-1} g$ is a flat metric. When $n = 2$, every Riemannian manifold is locally conformally flat. Indeed, if (M^2, g) is a Riemannian surface and u is a function on M, then by (3.8) we have

$$\tilde{R}(e^u g) = e^{-u} \left(R(g) - \Delta_g u \right).$$

Thus to find u locally so that $\tilde{R}(e^u g) = 0$, we just need to solve the Poisson equation

$$\Delta_g u = R(g)$$

which is certainly possible.

Theorem 3.1 (Weyl, Schouten). *A Riemannian manifold* (M^n, g) *is locally conformally flat if and only if*

(1) for $n \geq 4$ *the Weyl tensor vanishes,*

(2) for $n = 3$ *the Cotton tensor vanishes.*

Proof. By the conformal invariance of the Weyl tensor, it is clear that if (M^n, g) is locally conformally flat, then the Weyl tensor vanishes. For $n = 3$, the Ricci tensor vanishes and therefore the Cotton tensor vanishes also.

Conversely, if the Weyl tensor vanishes, then by (3.6) and (3.9), the equation that the metric $\tilde{g} = e^{2f} g$ is flat:

$$\widetilde{Rm} = 0$$

is equivalent to

$$\begin{aligned} 0 &= Rm - a \odot g \\ &= \left(\frac{1}{(n-2)} \left(Rc - \frac{1}{2(n-2)} Rg \right) - a \right) \odot g. \end{aligned} \tag{3.15}$$

Since the map $\odot : S^2 M \to CM$ defined by $\odot(h) := h \odot g$ is injective, (3.15) is equivalent to

$$\frac{1}{(n-2)} \left(Rc - \frac{1}{2(n-2)} Rg \right) = a.$$

That is

$$\nabla_i \nabla_j f = S_{ij} + \nabla_i f \nabla_j f - \frac{1}{2} |\nabla f|^2 g_{ij}, \tag{3.16}$$

where

$$S_{ij} = \frac{1}{(n-2)} \left(R_{ij} - \frac{1}{2(n-2)} R g_{ij} \right).$$

Theorem 3.1 is now a consequence of the following, which gives the condition for when the flat metric equation for $\tilde{g}$ is locally solvable.

Lemma 3.1 *Provided the Weyl tensor vanishes, equation (3.16) is locally solvable if and only if the following integrability condition is satisfied*

$$\nabla_k S_{ij} = \nabla_i S_{kj}, \tag{3.17}$$

That is, if and only if the Cotton tensor vanishes. Recall that when $n \geq 4$, (3.17) follows from the Weyl tensor vanishes. On the other hand, when $n = 3$, the Weyl tensor vanishes for any metric.

Proof. To solve (3.16) it is necessary and sufficient to find a 1-form X locally such that

$$\nabla_i X_j = c_{ij} := S_{ij} + X_i X_j - \frac{1}{2} |X|^2 g_{ij}, \tag{3.18}$$

where $c = c(X, g)$ is a symmetric 2-tensor depending only on X and g. Clearly if f is a solution of (3.16), then $X = df$ is a solution of (3.18). On the other hand, if X is a solution of (3.18), by the symmetry of the RHS, we have

$$\nabla_i X_j = \nabla_j X_i,$$

which implies $dX = 0$. Thus locally X is the exterior derivative of some functiuon f, which then solves (3.16). Now rewrite (3.18) as

$$\frac{\partial}{\partial x^i} X_j = \widetilde{c}_{ij}, \tag{3.19}$$

where

$$\begin{aligned} \widetilde{c}_{ij} = \widetilde{c}\,(X,g)_{ij} &:= c\,(X,g)_{ij} + \Gamma_{ij}^k X_k \\ &= S_{ij} + X_i X_j - \frac{1}{2}\,|X|^2\, g_{ij} + \Gamma_{ij}^k X_k. \end{aligned}$$

Suppose $p \in M$ and that the coordinates$\{x^i\}$ is defined in a neighborhood of p. The Frobenius theorem says a necessary and sufficient condition to locally solve (3.19) with $X\,(p) = X_0$ for any $X_0 \in T_p M$ is the following integrability condition arising from $\frac{\partial^2}{\partial x^k \partial x^i} X_j = \frac{\partial^2}{\partial x^i \partial x^k} X_j$:

$$\frac{\partial}{\partial x^k} \widetilde{c}_{ij} = \frac{\partial}{\partial x^j} \widetilde{c}_{ik}.$$

More invariantly, the integrability condition arises from

$$\nabla_k \nabla_i X_j = \nabla_i \nabla_k X_j + R_{jik}^l X_l$$

and is

$$\nabla_k c_{ij} - \nabla_j c_{ik} = R_{jik}^l X_l = \left(S_i^l g_{jk} + S_{jk} \delta_i^l - S_k^l g_{ji} - S_{ji} \delta_k^l\right) X_l \tag{3.20}$$

where for the last equality we used $W_{jik}^l = 0$. From the definition of c_{ij} (3.18), we have

$$\nabla_k c_{ij} = \nabla_k S_{ij} + X_j \nabla_k X_i + X_i \nabla_k X_j - X^l \nabla_k X_l g_{ij}.$$

Therefore by (3.20) we have

$$C_{ijk} = \nabla_k S_{ij} - \nabla_j S_{ik} = 0.$$

QED

Corollary 3.1. *If a Riemannian manifold* (M^n, g) *has constant sectional curvature, then* (M^n, g) *is locally conformally flat.*

We say that two Riemannian manifolds (M_1^n, g_1) and (M_2^n, g_2) are **conformally equivalent** if there exist a diffeomorphism $\varphi : M_1 \to M_2$ and a function $f : M_1^n \to \mathbb{R}$ such that $g_1 = e^f \varphi^* g_2$.

Theorem 3.2 (Kuiper). *If* (M^n, g) *is a simply connected, locally conformally flat, closed Riemannian manifold, then* (M^n, g) *is conformally equivalent to the standard sphere* S^n.

A map ψ from one manifold (M_1^n, g_1) to another (M_2^n, g_2) is said to be **conformal** if there exists a function $f : M_1 \to \mathbb{R}$ such that $g_1 = e^f \varphi^* g_2$.

Theorem 3.3 (Schoen and Yau). *If* (M^n, g) *is a simply connected locally conformally flat, complete Riemannian manifold in the conformal class of a metric with nonnegative scalar curvature, then there exists a one-to-one conformal map of* (M^n, g) *into the standard sphere* S^n.

When M^n is not simply connected, it is useful to apply the above results to the universal cover $\left(\widetilde{M}^n, \widetilde{g}\right)$.

4. Cartan structure equations.

We shall often find it convenient to compute curvatures in a moving (orthonormal) frame. The method of moving frames, which we describe below, was primarily developed first by Elie Cartan and then by S.-S. Chern. Let $\{e_i\}_{i=1}^n$ be a local orthonormal frame in an open set $U \subset M^n$. The dual orthonormal basis (or coframe field) $\{\omega^i\}_{i=1}^n$ of T^*M is defined by $\omega^i(e_j) = \delta_j^i$. We may write the metric as

$$g = \Sigma_{i=1}^n \omega^i \otimes \omega^i.$$

The connection 1-forms ω_i^j are the components of the Levi-Civita connection with respect to $\{e_i\}_{i=1}^n$:

$$\nabla_X e_i = \omega_i^j(X) e_j,$$

for all $i, j = 1, ..., n$ and all vector field X on U. The connection 1-forms are antisymmetric:

$$\omega_j^i = -\omega_i^j$$

since for all X

$$
\begin{aligned}
0 = X\left\langle e_i, e_j\right\rangle &= \left\langle \nabla_X e_i, e_j\right\rangle + \left\langle e_i, \nabla_X e_j\right\rangle \\
&= \omega_i^k\left(X\right)\delta_{kj} + \omega_j^k\left(X\right)\delta_{ki} \\
&= \omega_i^j + \omega_j^i.
\end{aligned}
$$

The curvature 2-forms Ω_j^i on U are defined by:

$$
\Omega_i^j\left(X,Y\right)e_j = R\left(X,Y\right)e_i
$$

so that $\Omega_i^j\left(X,Y\right) = \left\langle R\left(X,Y\right)e_i, e_j\right\rangle$.

Theorem 3.4 (Cartan structure equations). *The first and second Cartan structure equations are:*

$$
d\omega^i = -\omega_j^i \wedge \omega^j, \tag{3.21}
$$

$$
d\omega_j^i = -\omega_k^i \wedge \omega_j^k + \Omega_j^i. \tag{3.22}
$$

Proof. We may compute

$$
\begin{aligned}
d\omega^i\left(X,Y\right) &= \left(\nabla_X\omega^i\right)\left(Y\right) - \left(\nabla_Y\omega^i\right)\left(X\right) \\
&= -\omega_j^i\left(X\right)\omega^j\left(Y\right) + \omega_j^i\left(Y\right)\omega^j\left(X\right),
\end{aligned}
$$

and (3.21) follows. We also have

$$
\begin{aligned}
\Omega_i^j\left(X,Y\right)e_j &= R\left(X,Y\right)e_i \\
&= \nabla_X\nabla_Y e_i - \nabla_Y\nabla_X e_i - \nabla_{[X,Y]}e_i \\
&= \ldots\ldots \\
&= d\omega_i^j\left(X,Y\right)e_j + \left(\omega_k^j\left(X\right)\omega_i^k\left(Y\right) - \omega_k^j\left(Y\right)\omega_i^k\left(X\right)\right)e_j
\end{aligned}
$$

and (3.22) follows. QED

By the Cartan structure equations, we may easily prove the 1-st and 2-rd Bianchi identities.