Section 5. Geodesics and the exponential map

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1 Exponential map, the Gauss lemma and the Hopf-Rinow theorem.

Let (M^n, g) be a Riemannian manifold and $p \in M$. The exponential map $\exp_p : T_pM^n \to M^n$ is defined by

$$\exp_p(V) := \gamma_V(1),$$

where $\gamma_V : [0, \infty) \to M^n$ is the constant speed geodesic emanating from p with $\dot{\gamma}_V(0) = V$. Note that $\gamma_V(t) = \exp_p(tV)$. Suppose that $V \in T_pM$ and some $L > 0$ the constant speed geodesic $\gamma_{\widetilde{V}}$ with $\dot{\gamma}_{\widetilde{V}}(0) = \widetilde{V}$ is defined on $[0, L]$ for every $\widetilde{V}$ in some neighborhood of V. Given $u \in (0, L)$, let $d\exp_p$ denote the tangent map of $\exp_p$ at uV. Recall

Lemma 5.1 (Gauss). *If $W \in T_{uV}(T_pM) \cong T_pM$ is perpendicular to V, then the image $d\exp_p(W)$ of W is perpendicular to $d\exp_p(V) = \dot{\gamma}_V(u)$:*

$$\left\langle d\exp_p(W), d\exp_p(V)\right\rangle_{g\left(\exp_p(uV)\right)} = 0.$$

If the distance function $r(\cdot) := d(\cdot, p)$ is smooth at a point x, we then have

$$\nabla r = \frac{\partial \gamma_0(s)}{\partial s}(b) \tag{5.1}$$

where $\gamma_0 : [0, b] \to M^n$ is the unique unit speed minimal geodesic from p to x. Thus if $\gamma_0 = \gamma_{V_0}$ for some unit vector V_0, then $\nabla r = d\exp_p(V_0)$.

Proof. Given $V, W \in T_pM$, define the family of geodesics

$$\gamma_s(t) := \exp_p(t(uV + sW)), 0 \leq s \leq 1.$$

It is clear that $L(\gamma_s) = |uV + sW|_{g(p)}$. A simple calculation shows that

$$\frac{d}{ds}|_{s=0} L(\gamma_s) = \frac{1}{|V|} \langle V, W \rangle_{g(p)}.$$

On the other hand by the first variation formula we have

$$\begin{aligned} \frac{d}{ds}|_{s=0} L(\gamma_s) &= \frac{1}{u|V|} \left\langle \frac{\partial}{\partial s}|_{s=0} \gamma_s(1), \frac{\partial}{\partial t}\gamma_0(1) \right\rangle_{g\left(\exp_p(uV)\right)} \\ &= \frac{1}{|V|} \left\langle d\exp_p(W), d\exp_p(V) \right\rangle_{g\left(\exp_p(uV)\right)}, \end{aligned}$$

where the factor on the RHS of the first line is due to $\left|\frac{\partial \gamma_0}{\partial t}\right| = u|V|$. Hence

$$\left\langle d\exp_p(W), d\exp_p(V) \right\rangle_{g\left(\exp_p(uV)\right)} = \langle V, W \rangle_{g(p)}.$$

This proves the first part.

Let $\gamma_s : [0, b] \to M^n, s \in (-\varepsilon, \varepsilon) \subset \mathbb{R}$ be an arbitrary variation of γ_0 with $\gamma_s(0) = p$. Since γ_0 is a minimal geodesic, we have $L(\gamma_s) \geq d(p, \gamma_s(b))$ with $L(\gamma_0) = d(p, \gamma_0(b))$. Hence, since ∇r exists at x,

$$\langle \nabla r, X \rangle = \frac{d}{ds}|_{r=0} L(\gamma_s),$$

where $X := \frac{d}{ds}|_{s=0}\gamma_s(b)$. On the other hand, by the first variation formula

$$\frac{d}{ds}|_{s=0} L(\gamma_s) = \left\langle \frac{\partial \gamma_0}{\partial t}(b), X \right\rangle.$$

Therefore $\langle \nabla r, X \rangle = \left\langle \frac{\partial \gamma_0}{\partial t}(b), X \right\rangle$ for all X, and (5.1) follows. QED

Let $\partial/\partial s$ denote both the radial unit outward pointing vector field on $T_pM - \{O\}$ and its image under the derivative of the exponential map: $\partial/\partial s := d\exp_p(\partial/\partial s)$. For the latter to be well-defined, we restrict $\exp_p$ to a punctured ball $B(O, \varepsilon) - \{O\} \subset T_pM, \varepsilon > 0$, where it is an embedding (since at O the map $d\exp_p = id_{T_pM}$ is invertible, there exists such an ε). We also denote

$$r(x) := \left|\exp_p^{-1}(x)\right| \text{ for } x \in \widetilde{B}(p, \varepsilon) := \exp_p(B(O, \varepsilon)).$$

Note that we have not yet shown that $r(x)$ equals $d(x,p)$ for $x \in \widetilde{B}(p,\varepsilon)$.

The Gauss lemma implies that at any point $x \in \widetilde{B}(p,\varepsilon) - \{p\}$, we have

$$\frac{\partial}{\partial r} = grad\ r;$$

that is, for every $X \in T_x M$, $\langle \frac{\partial}{\partial r}, X \rangle = X(r)$. Indeed, one sees this from writing X as the sum of its radial component and perpendicular vector and applying the Gauss lemma: if

$$X = a\frac{\partial}{\partial r} + \sum_{i=1}^{n-1} b^i \frac{\partial}{\partial \theta^i},$$

where $\{r, \theta^1, ..., \theta^{n-1}\}$ are spherical coordinates, then

$$\left\langle \frac{\partial}{\partial r}, X \right\rangle = a = X(r),$$

where the first equality follows from the Gauss lemma: $\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta^i} \rangle = 0$.

Lemma 5.2.

(1) *For every* $V \in B(O,\varepsilon), \gamma_V : [0,1] \to M$ *is the unique path, up to reparametrization, joining* p *and* $\gamma_V(1) = \exp_p(V)$ *whose length realizes the distance* $d(p, \exp_p(V)) = |V|$. *In particular, short geodesics are minimal and*

$$r(x) = d(x,p) \qquad \text{for } x \in \widetilde{B}(p,\varepsilon).$$

(2) *For every* q *outside of the ball* $\widetilde{B}(p,\varepsilon)$, *there exists a point* $q' \in \partial\widetilde{B}(p,\varepsilon)$ *such that*

$$d(p,q) = d(p,q') + d(q',q) = \varepsilon + d(q',q). \tag{5.2}$$

Proof. (1) This follows from the Gauss lemma, in particular on integrating the fact that for any path β we have

$$\left|\dot{\beta}(u)\right| \geq \left\langle \dot{\beta}, \partial/\partial r \right\rangle = \frac{d}{du} r(\beta(u))$$

as long as β stays inside $\widetilde{B}(p,\varepsilon)$.

(2). Proof see Bai Zhengguo and Shen Yibing's book.

QED

Part (2) is useful in proving the Hopf-Rinew theorem below. It also implies that for all $q \notin \widetilde{B}(p,\varepsilon)$, we have $\mathrm{d}(p,q) \geq \varepsilon$. On the other hand, if $q \in \widetilde{B}(p,\varepsilon)$, then there exists $V \in B(O,\varepsilon)$ such that $\exp_p(V) = q$ so that $d(p,q) = |V| < \varepsilon$. Hence

$$\widetilde{B}(p,\varepsilon) = B(p,\varepsilon) := \{x \in M : d(x,p) < \varepsilon\}.$$

Remark. If there exists a (unit speed) minimal geodesic γ from p to

q; then it is easy to show that there exists $q' \in \partial\widetilde{B}(p,\varepsilon)$ satisfiying (5.2); namely we define $q' := \gamma(\varepsilon)$. Part of the Hopf-Rinow Theorem is to prove the converse under the assumption (M,d) where d is the Riemannian distance, is complete as a metric space.

Now we recall the following well-know result.

Theorem 5.1 (Hopf-Rinow). *Let (M^n,g) be a Riemannian manifold. Then the following are equivalent:*

(1) (M,d) is a complete metric space.

(2) There exists $p \in M$ such that $\exp_p$ is defined on all of T_pM:

(3) For all $p \in M, \exp_p$ is defined on all of T_pM,

Any one of these conditions implies

(4). For any $p,q \in M$ there exists a smooth minimal geodesic from p to q.

We omit its proof. See Bai Zhengguo and Shen Yibing's book.

2 Cut locus and injectivity radius.

Recall the following

Definition 5.1. *A function $f : M^n \to R$ is a (globally)* ***Lipschitz function*** *with Lipschitz constant equal to C if for all $x,y \in M$ we have*

$$|f(x) - f(y)| \leq C \cdot d(x,y).$$

If for every $z \in M$ there exists a neighborhood U_z of z and a constant C_z such that

$$|f(x) - f(y)| \leq C_z \cdot d(x,y)$$

for all $x,y \in U_z$, then we say that f is a ***locally Lipschitz function****.*

From the triangle inequality, it is easy to see that distance d_p is a Lipschitz function with Lipschitz constant equal to 1:

$$|d_p(x) - d_p(y)| \leq d(x, y).$$

Given a point $p \in M^n$ and a unit speed geodesic $\gamma : [0, \infty) \to M^n$ with $\gamma(0) = p$, either γ is a **geodesi ray** (i.e. minimal on each finite subinterval) or there exists a unique $r_\gamma \in (0, \infty)$ such that $d(\gamma(r), p) = r$ for $r \leq r_\gamma$ and $d(\gamma(r), p) < r$ for $r > r_\gamma$. We say that $\gamma(r_\gamma)$ is a **cut point to** p **along** γ. Note that if $\gamma(r)$ is a conjugate point to p along γ, then $r \geq r_\gamma$. The **cut locus** Cut(p) of p in M^n is the set of all cut points of p. Now let

$$D_p := \left\{V \in T_pM^n : d\left(\exp_p(V), p\right) = |V|\right\},$$

which is a closed subset of T_pM. We define $C_p := \partial D_p$ to be the cut locus of p in the tangent space. We have $\mathrm{Cut}(p) = \exp_p(C_p)$. We have $\exp_p : int(D_p) \to M^n - Cut(p)$ is a diffeomorphism. (We call $int(D_p)$ the interior to the cut locus in the tangent space $T_p(M)$.)

Lemma 5.3. *A point $\gamma(r)$ is a cut point to p along γ if and only if r is the smallest positive number such that either $\gamma(r)$ is a conjugate point to p along γ or there exist two distinct minimal geodesics joining p and $\gamma(r)$.*

Proof see Cheeger-Ebin's book.

Given $V \in T_pM^n$ and $r > 0$, we have $\gamma_V(r) = \exp_p(rV)$. For each unit vector $V \in T_pM^n$ there exists at most a unique $r_V \in (0, \infty)$ such that $\gamma_V(r_V)$ is a cut point of p along γ_V. Furthermore, if we set $r_V := \infty$ when γ_V is a ray, then the map from the unit tangent space at p to $(0, \infty]$ given by $V \mapsto r_V$ is a continuous function (See Bishop-Crittenden's book). Hence we have

$$C_p = \partial D_p = \{r_V V : V \in T_pM^n, |V| = 1, \gamma_V \text{ is not a ray}\}$$

has measure zero with respect to the Euclidean measure on $(T_pM^n, g(p))$. Since $\exp_p$ is a smooth function, we conclude that

Lemma 5.4. *$Cut(p) = \exp_p(C_p)$ has measure zero with respect to the* ***Riemannian measure*** *on (M^n, g).*

Now if $x \notin Cup(p)$ and $x \neq p$, then d_p is smooth at x and $|\nabla d_p(x)| = 1$. Since $Cut(p)$ has measure zero, we have $|\nabla d_p| = 1$ a.e. on M^n.

An alternate proof of Lemma 5.4 can be given as follows. Since d_p is locally Lipschitz, we have that d_p is differentiable a.e.. On the other hand it is easy to see that d_p is not C^1 at those points x in $Cut(p)$ for which there are two distinct minimal geodesics joining p to x. Thus, this set of points in $Cut\,(p)$ has measure zero. We also know that by Sard's theorem, the points in $Cut\,(p)$ which are conjugate points form a maesure zero set (since these points are singular values of $\exp_p$). We conclude that $Cut\,(p)$ has measure zero.

Definition 5.2. *The* ***injectivity radius*** $inj\,(p)$ *of a point* $p \in M^n$ *is defined to be the supremum of all* $r > 0$ *such that* $\exp_p$ *is an embedding when restricted to* $B\,(O, r)$. *Equivalently,*

(1) $inj\,(p)$ *is the distance from* O *to* C_p *with respect to* $g\,(p)$,

(2) $inj\,(p)$ *is the Riemannian distance from* p *to* $Cut\,(p)$.

The injectivity radius of a Riemannian manifold is defined to be

$$inj\,(M^n, g) := \inf \{inj\,(p) : p \in M^n\}.$$

When M^n is closed, the injectivity radius is always positive. A basic result in Riemannian geometry is the following

Theorem 5.2 (Klingenberg).

(1) *If* (M^n, g) *is a closed Riemannian manifold with* $Sect\,(g) \leq H$, *then*

$$inj\,(M^n, g) \geq \min \left\{ \frac{\pi}{\sqrt{H}}, \frac{1}{2}\text{Length of shortest closed geodesic} \right\}.$$

(2) *If* (M^n, g) *is a complete simply connected Riemannian manifold with* $0 < \frac{1}{4}H < sect(g) \leq H$, *then*

$$inj\,(M^n, g) \geq \frac{\pi}{\sqrt{H}}.$$

(3) *If* (M^n, g) *is a closed, even-dimensional, orientable Riemannian manifold with* $0 < sect(g) \leq H$, *then*

$$inj\,(M^n, g) \geq \frac{\pi}{\sqrt{H}}.$$

Remark By Synge's Theorem 4.1, in part (3) it follows that M^n is simply-connected.

3 Geodesic coordinate expansion of the metric and volume form.

Recall that the exponential map $\exp_p : T_pM \to M^n$ is defined by $\exp_p(V) := \gamma_V(1)$ where $\gamma_V : [0,\infty) \to M^n$ is the constant speed geodesic emanating from p with $\dot{\gamma}_V(0) = V$. Given an orthonormal frame $\{e_i\}_{i=1}^n$ at p, Let $\{X^i\}_{i=1}^n$ denote the standard Euclidean coordinates on T_pM defined by $V = V^i e_i$. Geodesic coordinates are defined by

$$x^i := X^i \circ \exp_p^{-1} : M^n - Cut(p) \to \mathbb{R}.$$

In geodesic coordinates, we have

$$\begin{aligned} g_{ij} &= \delta_{ij} - \frac{1}{3} R_{ipjq} x^p x^q - \frac{1}{6} \nabla_r R_{ipjq} x^p x^q x^r \\ &\quad + \left(-\frac{1}{20} \nabla_r \nabla_s R_{ipjq} + \frac{2}{45} R_{ipmq} R_{jrms} \right) x^p x^q x^r x^s + O\left(r^5\right) \end{aligned}$$

so that $g_{ij} = \delta_{ij} + O(r^2)$, and

$$\begin{aligned} \det(g_{ij}) &= 1 - \frac{1}{3} R_{ij} x^i x^j - \frac{1}{6} \nabla_k R_{ij} x^i x^j x^k \\ &\quad - \left(\frac{1}{20} \nabla_l \nabla_k R_{ij} + \frac{1}{90} R_{ipjq} R_{kplq} - \frac{1}{18} R_{ij} R_{lk} \right) x^i x^j x^k x^l + O\left(r^5\right). \end{aligned} \tag{5.3}$$

These formulas exhibits how the curvature and its derivatives locally affect the metric and volume form

$$d\mu = \sqrt{\det(g_{ij})} dx^1 \wedge \cdots \wedge dx^n,$$

where $\{x^i\}$ is a positively oriented local coordinate system. In particular we see that the leading order term on the RHS of (5.3) is expressed interms of the Ricci curvature and that positive Ricci curvature yields a negative contribution.

The following gives a geometric inetrpretation of the scalar curvature.

Lemma 5.5. (expansion for volume of balls)

$$Vol(B(p,r)) = \omega_n r^n \left(1 - \frac{R(p)}{6(n+2)} r^2 + O\left(r^3\right) \right).$$

Lemma 5.6. *In geodesic coodinates centered at a point $p \in M$, we have $g_{ij}(p) = \delta_{ij}$ and $\frac{\partial}{\partial x^i} g_{jk}(p) = 0$.*

4 Geodesic spherical coordinates and the Jacobian.

Typically we say that the geometry is controlled if there is a curvature bound and an injectivity radius lower bound. Since in the presence of a curvature bound, a lower bound on the volume gives a lower bound on the injectivity radius we are interested in the volume of balls. To understand the volumes of balls and their boundary spheres, it convenient to consider geodesic spherical coordinates. Given a point $p \in M^n$, let $\{x^i\}_{i=1}^n$ be local spherical coordinates on $T_pM^n - \{p\}$. That is,

$$x^n(v) = r(v) = |v| \quad \text{and} \quad x^i(v) = \theta^i\left(\frac{v}{|v|}\right) \text{ for } 1 \le i \le n-1,$$

where $\{\theta^i\}_{i=1}^{n-1}$ are local coordinates on $S_p^{n-1} := \{v \in T_pM^n : |v| = 1\}$. Let $\exp_p : T_pM^n \to M^n$ be the exponential map. We call the coordinate system

$$x = \left\{x^i \circ \exp_p^{-1}\right\} : B(p, inj(p)) - \{p\} \to \mathbb{R}^n$$

a **geodesic spherical coordinate system**. Abusing notation, we let $r := x^n \circ \exp_p^{-1}$ and $\theta^i := x^i \circ \exp_p^{-1}$ for $i = 1, ..., n-1$, so that

$$\frac{\partial}{\partial r} = \left(x^{-1}\right)_* \frac{\partial}{\partial x^n} \quad \text{and} \quad \frac{\partial}{\partial \theta^i} = \left(x^{-1}\right)_* \frac{\partial}{\partial x^i},$$

which form a basis of vector fields on $B(p, inj(p)) - \{p\}$. recall from the Gauss lemma that $grad\ r = \frac{\partial}{\partial r}$ at all points outside the cut locus of p, so that

$$|grad\ r|^2 = \left|\frac{\partial}{\partial r}\right|^2 = \left\langle grad\ r, \frac{\partial}{\partial r}\right\rangle = 1 \tag{5.4}$$

and

$$g_{in} := g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta^i}\right) = \frac{\partial r}{\partial \theta^i} = 0$$

for $i = 1, ..., n-1$. We may then write the metric as

$$g = dr \otimes dr + g_{ij} d\theta^i \otimes d\theta^j,$$

where $g_{ij} = g\left(\frac{\partial}{\partial \theta^i}, \frac{\partial}{\partial \theta^j}\right)$.

Along each geodesic ray emanating from p,

$$\frac{\partial}{\partial \theta^i} \text{ is a Jacobi field} \tag{5.5}$$

before the first conjugate point for each $i \leq n-1$. We call

$$J := \sqrt{\det (g_{ij})} = \sqrt{\det \left\langle \frac{\partial}{\partial \theta^i}, \frac{\partial}{\partial \theta^j} \right\rangle_g} \tag{5.6}$$

the **Jacobian of the exponential map.** The Jacobian of the exponential map is the volume density in spherical coordinates. The volume form of g is

$$d\mu = \sqrt{\det (g_{ij})} d\theta^1 \wedge \cdots \wedge d\theta^{n-1} \wedge dr = J d\Theta \wedge dr$$

in a positively oriented spherical coordinate system, where

$$d\Theta := d\theta^1 \wedge \cdots \wedge d\theta^{n-1}$$

and we have used $\det(g_{ij})_{i,j=1}^{n} = \det (g_{ij})_{i,j=1}^{n-1}$. If $\gamma(\overline{r})$ is a conjugate point to p along γ, then $J(r) \to 0$ as $r \to \overline{r}$. We have

Lemma 5.7. *Along a geodesic ray emanating from p we have that*

$$\lim_{x \to p} \left(\nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial \theta^i} \right)(x) = \lim_{r \to 0} \left(\frac{1}{r} \frac{\partial}{\partial \theta^i} \right) := E_i \in T_p M \tag{5.7}$$

exists. Suppose $\{E_i\}_{i=1}^{n-1}$ is orthonormal (one can always choose such geodesic spherical coordinates and we shall often make this assumption in the sequel). Then we have

$$\lim_{r \to 0} \frac{J}{r^{n-1}} = 1. \tag{5.8}$$

5 The second fundamental form of distance spheres and the Ricatti equation.

Now we consider the distance spheres

$$S(p, r) := \{x \in M^n : d(x, p) = r\}.$$

Let h denote the second fundamental form of $S(p,r)$. We have

$$\begin{aligned} h_{ij} &:= h\left(\frac{\partial}{\partial\theta^i}, \frac{\partial}{\partial\theta^j}\right) = \left\langle \nabla_{\frac{\partial}{\partial\theta^i}} \frac{\partial}{\partial r}, \frac{\partial}{\partial\theta^j} \right\rangle \\ &= -\left\langle \frac{\partial}{\partial r}, \nabla_{\frac{\partial}{\partial\theta^i}} \frac{\partial}{\partial\theta^j} \right\rangle = -\Gamma^n_{ij} = \frac{1}{2}\frac{\partial}{\partial r} g_{ij} \end{aligned}$$

since $\frac{\partial}{\partial r}$ is the unit normal to $S(p,r)$ and $g_{in} = g_{jn} = 0$. The mean curvature H of $S(p,r)$ is

$$\begin{aligned} H &= -g^{ij}\Gamma^n_{ij} = \frac{1}{2} g^{ij} \frac{\partial}{\partial r} g_{ij} \\ &= \frac{\partial}{\partial r} \log\sqrt{\det(g_{ij})} = \frac{\partial}{\partial r}\log J. \end{aligned} \tag{5.9}$$

Lemma 5.8. *For r small enough, we have*

$$h_{ij} = \frac{1}{r} g_{ij} + O(r), \tag{5.10}$$

and

$$H = \frac{n-1}{r} + O(r). \tag{5.11}$$

In spherical coordinates, the Laplacian is

$$\begin{aligned} \Delta &= g^{ab}\left(\frac{\partial^2}{\partial x^a \partial x^b} - \Gamma^c_{ab}\frac{\partial}{\partial x^c}\right) \\ &= \frac{\partial^2}{\partial r^2} + H\frac{\partial}{\partial r} + \Delta_{S(p,r)} \\ &= \frac{\partial^2}{\partial r^2} + \frac{\partial}{\partial r}\log\sqrt{\det g}\frac{\partial}{\partial r} + \Delta_{S(p,r)} \end{aligned}$$

since $\Gamma^a_{nn} = 0$ for $a = 1, 2, ..., n$ and where $\Delta_{S(p,r)}$ is the Laplacian with respect to the induced metric on $S(p,r)$.

We compute

$$\begin{aligned} \frac{\partial}{\partial r} h_{ij} &= -\left\langle \frac{\partial}{\partial r}, \nabla_{\frac{\partial}{\partial r}} \nabla_{\frac{\partial}{\partial\theta^i}} \frac{\partial}{\partial\theta^j} \right\rangle \\ &= -\left\langle R\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial\theta^i}\right)\frac{\partial}{\partial\theta^j}, \frac{\partial}{\partial r} \right\rangle - \left\langle \frac{\partial}{\partial r}, \nabla_{\frac{\partial}{\partial\theta^i}} \nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial\theta^j} \right\rangle. \end{aligned}$$

Since

$$\left\langle \frac{\partial}{\partial r}, \nabla_{\frac{\partial}{\partial \theta^i}} \nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial \theta^j} \right\rangle = \frac{\partial}{\partial \theta^i} \left\langle \frac{\partial}{\partial r}, \nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial \theta^j} \right\rangle - \left\langle \nabla_{\frac{\partial}{\partial \theta^i}} \frac{\partial}{\partial r}, \nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial \theta^j} \right\rangle$$
$$= -h_{ik} g^{kl} h_{lj}$$

for $\left\langle \frac{\partial}{\partial r}, \nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial \theta^j} \right\rangle = 0$ and $\nabla_{\frac{\partial}{\partial \theta^i}} \frac{\partial}{\partial r} = \nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial \theta^i}$, we obtain the Ricatti equation

$$\frac{\partial}{\partial r} h_{ij} = -R_{injn} + h_{ik} g^{kl} h_{lj.} \tag{5.12}$$

Since $\frac{\partial}{\partial r} H = g^{ij} \frac{\partial}{\partial r} h_{ij} - \frac{\partial}{\partial r} g_{ij} \cdot h_{ij}$ and $\frac{\partial}{\partial r} g_{ij} = 2h_{ij}$, tracing this equation yields

$$\frac{\partial}{\partial r} H = -Rc\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) - |h|^{2.} \tag{5.13}$$

In particular (5.13) implies that if $Rc \geq (n-1)\,K$, then since $|h|^2 \geq \frac{1}{n-1} H^2$, we have

$$\frac{\partial}{\partial r}\left(\frac{H}{n-1}\right) \leq -K - \left(\frac{H}{n-1}\right)^2. \tag{5.14}$$

By (5.11) we have

$$\lim_{r \to 0+} \frac{rH}{n-1} = 1$$

Now returning to the second fundamental form, in terms of the radial covariant derivative

$$\left(\nabla_{\frac{\partial}{\partial r}} h\right)_{ij} = \frac{\partial}{\partial r} h_{ij} - \Gamma_{ni}^k h_{kj} - \Gamma_{nj}^k h_{ki}$$

and $\Gamma_{ni}^k = h_i^k$, we deduce from (5.13) that

$$\left(\nabla_{\frac{\partial}{\partial r}} h\right)_{ij} = -R_{injn} - h_{ik} g^{kl} h_{lj}. \tag{5.15}$$

Space forms and rotationally symmetric metrics.

It is useful to consider geodesic spheres in simply connected space forms (M_K^n, g_K). In this case the metric is given by

$$g_K = dr^2 + s_K\,(r)^2\, g_{S^{n-1}},$$

where

$$s_K(r) = \begin{cases} \frac{1}{\sqrt{K}} \sin\left(\sqrt{K} r\right) & \text{if } K > 0, \\ r & \text{if } K = 0, \\ \frac{1}{\sqrt{K}} \sinh\left(\sqrt{|K|} r\right) & \text{if } K < 0. \end{cases}$$

Recall that if

$$g = dr^2 + \varphi^2(r)\, g_{S^{n-1}}$$

for some function φ, which is called a rotationally symmetric metric, then the sectional curvatures are

$$K_{rad} = \frac{\varphi''}{\varphi} \quad \text{and} \quad K_{sph} = \frac{1 - (\varphi')^2}{\varphi^2}, \tag{5.16}$$

where K_{rad} (rad for radial) or K_{sph} (sph for spherical) is the sectional curvature of planes containing or perpendicular to respectively, the radial vector. In fact (5.16) can be derived by moving frames and the Cartan structure equations. Moreover, if $\varphi : [0, \rho) \to [0, \infty)$, with $\varphi(r) > 0$ for $r > 0$, then the metric g extends smoothly over the origin if and only if $\varphi(0) = 0$ and $\varphi'(0) = 1$. We easily check that for any $K \in \mathbb{R}$, if we take $\varphi(r) = s_K(r)$, then $g = g_K$ is a constant sectional curvature K metric. If $K \leq 0$, then g_K is defined on $\mathbb{R}^n$; and if $K > 0$,then the metric g_K defined on $B\left(0, \pi/\sqrt{K}\right)$ extends smoothly to a metric on S^n by taking the 1-point compactification.

Note that the mean curvature H of $S(p, r)$ is $H = (n-1)\frac{\varphi'}{\varphi}$. In particular for the constant curvature K metric given by $g = dr^2 + s_K(r)^2 g_{S^{n-1}}$, the mean curvature $H_k(r)$ of the distance sphere $S_K(p, r)$ is

$$H_K(r) = \begin{cases} (n-1)\sqrt{K} \cot\left(\sqrt{K} r\right) & \text{if } K > 0, \\ \frac{n-1}{r} & \text{if } K = 0, \\ (n-1)\sqrt{|K|} \coth\left(\sqrt{|K|} r\right) & \text{if } K < 0. \end{cases}$$

Note that $H_K(r)$ is a solution to the equality case of (5.14), that is

$$\frac{\partial}{\partial r}\left(\frac{H_K}{n-1}\right) = -K - \left(\frac{H_K}{n-1}\right)^2. \tag{5.17}$$

and $\lim_{r \to 0+} \frac{r H_K}{n-1} = 1$. An easy calculus shows $H_K = \frac{n-1}{r} + O(r)$.

6 Mean curvature of geodesic spheres and the Bonnet-Myers theorem

By the ODE comparison theorem, we have

Lemma 5.9. (Mean curvature of distance spheres comparison). *If the Ricci curvature of* (M^n, g) *satisfies the lower bound* $Ric \geq (n-1)K$ *for some* $K \in R$*, then the mean curvatures of the distance spheres* $S(p,r)$ *satisfy*

$$H(r,\theta) \leq H_K(r) \tag{5.18}$$

at points where the distance function is smooth.

Proof. To see (5.18) more clearly, we compute from (5.14) and (5.17) that

$$\frac{\partial}{\partial r}(H - H_K) \leq -\frac{(H_K + H)}{n-1}(H - H_K). \tag{5.19}$$

Note from (5.11) that $(H - H_K)(r) = O(r)$. Integrating (5.19) we get that for any $r \geq \varepsilon > 0$,

$$(H - H_K)(r) \leq (H - H_K)(\varepsilon) \cdot \exp\left\{-\int_\varepsilon^r \frac{(H_K + H)}{n-1}(s)\,ds\right\}. \tag{5.20}$$

Clearly for all $r > 0$ we have

$$\lim_{\varepsilon \to 0}\left((H - H_K)(\varepsilon) \cdot \exp\left\{-\int_\varepsilon^r \frac{(H_K + H)}{n-1}(s)\,ds\right\}\right) = 0.$$

Hence (5.20) implies $(H - H_K)(r) \leq 0$ for all $r > 0$. QED

We take this opportunity to give a proof of the Bonnet-Myers theorem.

Theorem 5.3 (Bonnet-Myers). *If* (M^n, g) *is a complete Riemannian manifold with* $Ric \geq (n-1)K$*, where* $K > 0$*, then* $diam(g) \leq \pi/\sqrt{K}$*. In particular,* M^n *is compact and* $\pi_1(M) < \infty$*.*

Proof. Choose any point $p \in M$ and suppose $\gamma : [0, L] \to M$ is a unit speed minimal geodesic emanating from p. Then d_p is smooth on $\gamma((0, L))$ and for every $r \in (0, L)$, the distance sphere $S(p, r)$ is smooth in a neighborhood of $\gamma(r)$. By Lemma 5.9, we have

$$H(r,\theta) \leq (n-1)\sqrt{K}\cot\left(\sqrt{K}r\right)$$

along $\gamma|_{(0,L)}$. Since $\lim_{r \to \pi/\sqrt{K}+}\cot\left(\sqrt{K}r\right) = -\infty$, we conclude that $L \leq \pi/\sqrt{K}$. Thus $diam(g) \leq \pi/\sqrt{K}$. Now a complete Riemannian manifold with finite diameter is compact. Furthermore, we may apply the diameter bound to the universal covering Riemannian manifold $\left(\widetilde{M}^n, \widetilde{g}\right)$, where $\widetilde{g}$ is the lift metric. (By definition, the covering map $\pi : \left(\widetilde{M}^n, \widetilde{g}\right) \to (M^n, g)$ is a local isometric.) Indeed, $\widetilde{g}$ satisfies the same Ricci curvature lower bound as g. this implies $\widetilde{M}^n$ is compact and we conclude $\pi_1(M) < \infty$. QED