Section 6. Laplacian, volume and Hessian comparison theorems

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Two fundamental results in Riemannian geometry are the **Laplacian and Hessian comparison theorems for the distance function**. They are directly related to the volume comparison theorem and a special case of the Rauch comparison theorem. The Hessian comparison theorem may also be used to prove the Toponogov triangle comparison theorem.

1 Laplacian comparison theorem.

The idea of comparison theorems is to compare a geometric quantity on a Riemannian manifold with the corresponding quantity on a model space. Typically, in Riemannian geometry, model spaces have constant sectional curvature.

Theorem 6.1 (Laplacian comparison). *If (M^n, g) is a complete Riemannian manifold with $Rc \geq (n-1)K$, where $K \in R$, and if $p \in M^n$, then for any $x \in M^n$ where $d_p(x)$ is smooth, we have*

$$\Delta d_p(x) \leq \begin{cases} (n-1)\sqrt{K}\cot\left(\sqrt{K}d_p(x)\right) & \text{if } K > 0 \\ \frac{n-1}{d_p(x)} & \text{if } K = 0 \\ (n-1)\sqrt{|K|}\coth\left(\sqrt{|K|}d_p(x)\right) & \text{if } K < 0. \end{cases} \tag{6.1}$$

On the whole manifold, the Laplacian comparison theorem holds ***in the sense of distributions****.*

In general, we say that $\Delta f \leq F$ in the sense of distributioons if for any nonnegative C^∞ function φ on M^n with compact support, we have

$$\int_{M^n} f \Delta \varphi d\mu \leq \int_{M^n} F \varphi d\mu.$$

Form Theorem 6.1 we can derive the following

Corollary 6.1. *If $K \leq 0$, then*

$$\Delta d_p \leq \frac{n-1}{d_p} + (n-1)\sqrt{|K|} \tag{6.2}$$

in the sense of distributions. In particular, as above, if (M^n, g) is a complete Riemannian manifold with $Ric \geq 0$, then for any $p \in M^n$

$$\Delta d_p \leq \frac{n-1}{d_p} \tag{6.3}$$

in the sense of distributions.

Remark. Estimate (6.1) is sharp as can be seen from considering space forms of constant curvature $-K$. If $K = 0$, then (6.3) is sharp since on Euclidean space $\Delta |x| = \frac{n-1}{|x|}$.

2 Volume comparison theorem.

A consequence of the Laplacian comparison theorem is the following

Theorem 6.2 (Bishop volume comparison). *If (M^n, g) is a complete Riemannian manifold with $Rc \geq (n-1) K$, where $K \in \mathbb{R}$, then for any $p \in M^n$, the volume ratio*

$$\frac{Vol(B(p,r))}{Vol_K(B(p_K,r))}$$

is a nonincreasing function of r, where p_K is a point in the n-dimensional simply connected space form of constant curvature K and Vol_K denotes the volume in the space form. In particular

$$Vol(B(p,r)) \leq Vol_K(B(p_K,r)) \tag{6.4}$$

for all $r > 0$. Given p and $r > 0$, equality holds in (6.4) if and only if $B(p,r)$ is isometric to $B(p_K, r)$.

In the case of nonnegative Ricci curvature we have the following

Corollary 6.2. *If (M^n, g) is a complete Riemannian manifold with $Ric \geq 0$, then for any $p \in M^n$, the volume ratio $\frac{Vol(B(p,r))}{r^n}$ is a nonincreasing function of r. Since $\lim_{r\to 0} \frac{Vol(B(p,r))}{r^n} = \omega_n$, we have $\frac{Vol(B(p,r))}{r^n} \leq \omega_n$ for all $r > 0$, where ω_n is the volume of the Euclidean unit n-ball.*

One of the many useful consequences of this is the following characterization of Euclidean space.

Corollary 6.3. (Volume characterization of R^n). *If (M^n, g) is a complete noncompact Riemannian manifold with $Rc \geq 0$ and if for some $p \in M^n$*

$$\lim_{r\to\infty} \frac{Vol\left(B\left(p, r\right)\right)}{r^n} = \omega_n,$$

then (M^n, g) is isometric to Euclidean space.

Proof. By the Bishop-Gromov volume comp[arison theorem, we actually have $\frac{Vol(B(p,r))}{r^n} \equiv \omega_n$ for all $r > 0$. The result now follows from the equality case. QED

The Bishop-Gromov volume comparison theorem has been generalized to the **relative volume comparison theorem**. Let (M^n, g) be a complete Riemannian manifold and $p \in M^n$. Given a measurable subset Γ of the unit sphere $S_p^{n-1} \subset T_pM$ and $0 < r \leq R < \infty$, define the annular-type region:

$$A_{r,R}^{\Gamma}(p) := \left\{ x \in M^n : \begin{array}{c} r \leq d(x,p) \leq R \text{ \& there exists a unit speed minimal} \\ \text{geodesic } \gamma \text{ from } \gamma(0) = p \text{ to } x \text{ satisfying } \gamma'(0) \in \Gamma \end{array} \right\}$$
$$\subset B(p,R) \backslash B(p,r).$$

Note that if $\Gamma = S_p^{n-1}$, then $A_{r,R}^{\Gamma}(p) = B(p,R) \backslash B(p,r)$. Given $K \in \mathbb{R}$ and a point p_K in the n-dimensional simply connected space form of constant curvature K, let $A_{r,R}^{\Gamma}(p_K)$ denote the corresponding set in the space form.

Theorem 6.3. *Suppose that (M^n, g) is a complete Riemannian manifold with $Rc(g) \geq (n-1)K$. If $0 \leq r \leq R \leq S$, $r \leq s \leq S$ and if $\Gamma \subset S_p^{n-1}$ is a measurable subset, then*

$$\frac{Vol\left(A_{s,S}^{\Gamma}(p)\right)}{Vol_K\left(A_{s,S}^{\Gamma}(p_K)\right)} \leq \frac{Vol\left(A_{r,R}^{\Gamma}(p)\right)}{Vol_K\left(A_{r,R}^{\Gamma}(p_K)\right)}.$$

Taking $r = s = 0$ and $\Gamma = S_p^{n-1}$ yields Theorem 6.2. In particular, taking the limit as $R \to 0$ gives (6.4).

As a consequence, we have the following result of Yau about the volume growth of a complete noncompact manifold with nonnegative Ricci curvature.

Corollary 6.4 ($Rc \geq 0$ has at least linear volume growth). *Let (M^n, g) be a complete noncompact Riemannian manifold with nonnegative Ricci curvature. For any point $p \in M^n$, there exists a constant $C > 0$ such that for any $r \geq 1$*

$$Vol\,(B\,(p,r)) \geq Cr.$$

Proof. Let $x \in M^n$ be a point with $d\,(x,p) = r \geq 2$. By the Bishop-Gromov relative volume comparison theorem, we have

$$\begin{aligned} &\frac{Vol\,(B\,(x,r+1)) - Vol\,(B\,(x,r-1))}{Vol\,(B\,(x,r-1))} \\ &\leq \frac{(r+1)^n - (r-1)^n}{(r-1)^n} \leq \frac{C\,(n)}{r}. \end{aligned} \tag{6.5}$$

Since $B\,(p,1) \subset B\,(x,r+1) \setminus B\,(x,r-1)$ and $B\,(x,r-1) \subset B\,(p,2r-1)$ by (6.5) we have

$$Vol\,(B\,(p,2r-1)) \geq Vol\,(B\,(x,r-1)) \geq \frac{Vol\,(B\,(p,1))}{C\,(n)} r.$$

We have proved the corollary for $r \geq 3$. Clearly it is then true for $r \geq 1$ (or any other positive constant). QED

3 Hessian comparison theorem.

The following roughly says that the larger the curvature, the smaller the Hessian of the distance function.

Proposition 6.1 (Hessian comparison theorem-General version). *Let $i = 1, 2$. Let (M_i^n, g_i) be complete Riemannian n-manifolds, let $\gamma_i : [0, L] \to M_i^n$ be geodesics parametrized by arc length such that γ_i does not inetrsect the cut locus of $\gamma_i\,(0)$, and let $d_i := d\,(\cdot, \gamma_i\,(0))$. If for all $t \in [0, L]$ we have*

$$K_{g_1}\left(V_1 \wedge \dot{\gamma}_1\,(t)\right) \geq K_{g_2}\left(V_2 \wedge \dot{\gamma}_2\,(t)\right)$$

for all unit vectors $V_i \in T_{\gamma_i(t)} M_i^n$ perpendicular to $\dot{\gamma}_i\,(t)$, then

$$\nabla^2 d_1\,(X_1, X_1) \leq \nabla^2 d_2\,(X_2, X_2)$$

for all $X_i \in T_{\gamma_i(t)}M_i^n$ *perpendicular to* $\dot{\gamma}_i(t)$ *and* $t \in (0, L]$.

Following theorem is the special case of the above result, namely comparing to constant curvature spaces.

Theorem 6.4 (Hessian comparison theorem –special case). *Let* (M^n, g) *be a complete Riemannian manifold with* $Sect \geq K$. *For any point* $p \in M$ *the distance function* $r(x) := d(x, p)$ *satisfies*

$$\nabla_i \nabla_j r = h_{ij} \leq \frac{1}{n-1} H_K(r) g_{ij}$$

at all points where r *is smooth (i.e. away from* p *and the cut locus). On all of* M *the above inequality holds in the sense of support functions.*

4 Mean value inequalities.

The following mean value inequality, which follows from the Laplacian comparison theorem, has an application in the proof of the splitting theorem.

Proposition 6.1 (Mean value inequality for $Ric \geq 0$). *If* (M^n, g) *is a complete Riemannian manifold with* $Ric \geq 0$ *and if* $f \leq 0$ *is a Lipschitz function with* $\Delta f \geq 0$ *in the sense of distributions (subharmonic), then for any* $x \in M^n$ *and* $0 < r < inj(x)$

$$f(x) \leq \frac{1}{\omega_n r^n} \int_{B(x,r)} f d\mu,$$

where ω_n *is the volume of the unit Euclidean* n*-ball.*

Proof. By the divergence theorem, we have

$$0 \leq \frac{1}{r^{n-1}} \int_{B(x,r)} \Delta f d\mu = \int_{\partial B(x,r)} \frac{\partial f}{\partial r} \frac{\sqrt{\det(g)}}{r^{n-1}} d\Theta,$$

where $d\Theta := d\theta^1 \wedge ... \wedge d\theta^{n-1}$. Since $\frac{\partial}{\partial r} \frac{\sqrt{\det(g)}}{r^{n-1}} \leq 0$ from $H = \frac{\partial}{\partial r} \log J \leq \frac{n-1}{r}$ and $f \leq 0$, we have

$$\begin{aligned} 0 &\leq \int_{\partial B(x,r)} \left(\frac{\partial f}{\partial r} \frac{\sqrt{\det(g)}}{r^{n-1}} + f \frac{\partial}{\partial r} \frac{\sqrt{\det(g)}}{r^{n-1}} \right) d\Theta \\ &= \frac{d}{dr} \left(\frac{1}{r^{n-1}} \int_{\partial B(x,r)} f d\sigma \right) \end{aligned}$$

where we used $d\sigma = \sqrt{\det(g)}d\Theta$. Since $\lim_{r\to 0} \frac{1}{r^{n-1}} \int_{\partial B(x,r)} f d\sigma = n\omega_n f(x)$, integrating the above inequality over $[0, s]$ yields

$$s^{n-1} f(x) \leq \frac{1}{n\omega_n} \int_{\partial B(x,s)} f d\sigma.$$

Integrating this again, now over $[0, r]$ implies

$$f(x) \leq \frac{1}{\omega_n r^n} \int_{B(x,r)} f d\mu.$$

QED

In the case where the sectional curvature is bounded from above, we have

Proposition 6.2 (Mean value inequality for $Sect \leq H$). *Suppose that (M^n, g) is a complete Riemannian manifold with $Sect(g) \leq H$ in a ball $B(x,r)$ where $r < inj(g)$. If $f \in C^\infty(M^n)$ is subharmonic, i.e., if $\Delta f \geq 0$, and if $f \geq 0$ on M^n, then*

$$f(x) \leq \frac{1}{V_H(r)} \int_{B(x,r)} f d\mu,$$

where $V_H(r)$ is the volume of a ball of radius r in the complete simply connected manifold of constant sectional curvature H.