

On Basic Constraint Qualifications for Infinite System of Convex Inequalities in Banach Spaces

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Abstract The BCQ and the Abadie CQ for infinite systems of convex inequalities in Banach spaces are characterized in terms of the upper semi-continuity of the convex cones generated by the subdifferentials of active convex functions. Some relationships with other constraint qualifications such as the CPLV and the Slate condition are also studied. Applications in best approximation theory are provided.

Keywords system of convex inequalities, basic constraint qualification, Abadie constraint qualification, best approximation

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1 Introduction

Let X be a Banach space over the real field \mathbb{R} , and let C be a closed convex subset of X . Let I be a given (finite or infinite) index-set and $\{g_i : X \mapsto \mathbb{R}, i \in I\}$ a family of continuous convex functions on X . Consider the system of convex inequalities defined as follows:

$$g_i(x) \leq 0, \quad i \in I. \quad (\text{SCI})$$

Let S denote the set of all x satisfying the system (SCI). Throughout the whole paper, we always assume that the solution set S is not empty. Let $G(\cdot)$ denote the sup-function (cf. [1]) of (SCI) defined by

$$G(x) := \sup_{i \in I} g_i(x), \quad \forall x \in X.$$

Then S is also the solution set of the following convex inequality:

$$G(x) \leq 0. \quad (\text{SSCI})$$

In what follows, we also assume that the sup-function $G(x)$ is continuous on X . It is clear that if X is of finite dimension and $G(x)$ is finite for each $x \in X$, or if $\{g_i : i \in I\}$ is locally uniformly bounded, then the condition that $G(x)$ is continuous on X is automatically satisfied.

Recent interests are focused on the study of the notion of the BCQ (relative to a closed convex set) for a (finite or infinite) system of convex inequalities in a Banach space. The reason is that this notion plays an important role in constrained best approximation theory and optimization theory; for details, the readers are referred to [1–9]. One important piece of work for a infinite system of convex inequalities in a finite-dimensional space by Li, Nahak and

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Singer in [9] is to characterize the BCQ and the Abadie CQ in terms of the upper semi-continuity of the convex cone generated by the subdifferential of the active members of $G(x)$. Thus one question is presented naturally and interestingly: Can the characterizations of the BCQ and the Abadie CQ obtained in [9] be extended to a system of convex inequalities (SCI) associated with a convex closed set in an infinite-dimensional Banach space? We will see that such an extension for the BCQ and the Abadie CQ relative to a convex closed set is not trivial in Section 2 of the present paper, where similar characterizations for the BCQ and the Abadie CQ relative to a convex closed set in an infinite-dimensional reflexive Banach space are provided. As an application, in the last section, we will consider the problem of characterizing the optimality condition for constrained best approximation in a Banach space in view of the notion of the BCQ. Under the assumptions that the subspace spanned by C is reflexive and smooth, we will show that the BCQ relative to C is a sufficient and necessary condition such that the considered characterization of the best constrained approximation is satisfied for each $x \in X$. This result extends and improves the corresponding result given in [6].

2 Preliminaries

Let X be a Banach space over the real field \mathbb{R} and X^* the dual of X . Let $\langle \cdot, \cdot \rangle$ denote the inner product between X and X^* . For a set Z in X (or in X^*), the interior (resp. relative interior, closure, convex hull, convex cone hull, linear space, boundary, weak* closure) of Z is denoted by $\text{int}Z$ (resp. $\text{ri}Z$, \bar{Z} , $\text{conv}Z$, $\text{cone}Z$, $\text{span}Z$, $\text{bd}Z$, $\text{weak}^*\bar{Z}$); the negative polar of Z is defined by $Z^\circ := \{x^* \in X^* : \langle x^*, z \rangle \leq 0 \text{ for each } z \in Z\}$. Thus, the normal cone of Z at \bar{z} , denoted by $N_Z(\bar{z})$, is defined by $N_Z(\bar{z}) = (Z - \bar{z})^\circ$. \mathbb{R}_- denotes the subset of \mathbb{R} consisting of all non-positive real numbers.

For a proper convex function on X , the subdifferential of f at x is defined by

$$\partial f(x) := \{z^* \in X^* : f(x) + \langle z^*, y - x \rangle \leq f(y), \forall y \in X\}.$$

In particular, $N_Z(\bar{z}) = \partial \mathbf{I}_Z(\bar{z})$. Here and throughout \mathbf{I}_Z denotes the indicator function of Z defined by $\mathbf{I}_Z(z) = 0$ if $z \in Z$ and $\mathbf{I}_Z(z) = +\infty$ if $z \notin Z$.

Recall that the directional derivative $f'_+(x, d)$ of the function f at x in the direction d is defined by

$$f'_+(x, d) = \lim_{t \rightarrow 0^+} \frac{f(x + td) - f(x)}{t}. \quad (2.1)$$

Then, if f is a continuous convex function on X , we have by [10, Proposition 2.2.7] that

$$\partial f(x) = \{z^* \in X^* : \langle z^*, d \rangle \leq f'_+(x, d), \forall d \in X\} \quad (2.2)$$

and

$$f'_+(x, d) = \max\{\langle z^*, d \rangle : z^* \in \partial f(x)\}. \quad (2.3)$$

Remark 2.1 For a continuous convex function f on X , it is easy to show that $\text{cone} \partial f(x) \subseteq N_{f^{-1}(\mathbb{R}_-)}(x)$ if $f(x) = 0$ and the equality holds if x is not a minimizer of f ; see [10, Corollary 1, p. 56].

Let $z \in S$ and let $\mathcal{S}(z)$ denote the set of all active indices $i : \mathcal{S}(z) = \{i \in I : g_i(z) = G(z)\}$ and let $I(z)$ denote the subset of $\mathcal{S}(z)$ such that $i \in I(z)$ if and only if $g_i(z) = 0$. Throughout the whole paper, we shall use two kinds of convex cones and convex hulls of the subdifferentials of active constraints at x corresponding to the active index sets $\mathcal{S}(z)$ and $I(z)$, respectively. The first is given in [3, 9] while the second is in [6], and they coincide for each $z \in \text{bd}S$:

$$\begin{cases} \mathcal{N}'(z) := \text{cone} \left(\bigcup_{i \in \mathcal{S}(z)} \partial g_i(z) \right), & z \in X, \\ \mathcal{D}'(z) := \text{conv} \left(\bigcup_{i \in \mathcal{S}(z)} \partial g_i(z) \right), & z \in X; \end{cases} \quad (2.4)$$

$$\begin{cases} N'(z) := \text{cone} \left(\bigcup_{i \in I(z)} \partial g_i(z) \right), & z \in X, \\ D'(z) := \text{conv} \left(\bigcup_{i \in I(z)} \partial g_i(z) \right), & z \in X. \end{cases} \quad (2.5)$$

Note that

$$D'(z) \subseteq \mathcal{D}'(z) \subseteq \partial G(z) \quad \text{and} \quad N'(z) \subseteq \mathcal{N}'(z) \subseteq \text{cone } \partial G(z). \quad (2.6)$$

Let C be a closed convex subset of X . Let K consist of all $x \in C$ satisfying the system (SCI), that is,

$$K := C \cap S.$$

We introduce the following definitions.

Definition 2.1 *Let $x \in K$. The system (SCI) is said to satisfy*

- (i) *the BCQ (basic constraint qualification) relative to C at x if*

$$N_K(x) = N_C(x) + N'(x); \quad (2.7)$$

- (ii) *the Abadie CQ (Abadie constraint qualification) relative to C at x if*

$$N_K(x) = \text{weak}^* - \overline{N_C(x) + N'(x)}, \quad (2.8)$$

where we adopt the convention that $N_C(x) + A = N_C(x)$ if $A = \emptyset$;

- (iii) *the BCQ (resp. Abadie CQ) at x if it satisfies the BCQ (resp. Abadie CQ) relative to $C = X$ at x .*

Definition 2.2 *The system (SCI) is said to satisfy the BCQ (resp. Abadie CQ) relative to C if it satisfies the BCQ (resp. Abadie CQ) relative to C at each $x \in K$. Similarly, the system (SCI) is said to satisfy the BCQ (resp. Abadie CQ) if it satisfies the BCQ (resp. Abadie CQ) relative to $C = X$.*

Remark 2.2 (a) The notion of the BCQ relative to C is given in [4] when I is finite and in [6] when I is infinite (see also [9] in the case when I is infinite and $C = X = \mathbb{R}^n$).

- (b) Noting that, for each $x \in C \cap \text{bd}S$,

$$N_K(x) \supseteq \text{weak}^* - \overline{N_C(x) + N'(x)} \supseteq N_C(x) + N'(x) \quad (2.9)$$

automatically holds, the system (SCI) satisfies the BCQ (resp. Abadie CQ) relative to C at x if and only if

$$N_K(x) \subseteq N_C(x) + N'(x) \quad (\text{resp. } N_K(x) \subseteq \text{weak}^* - \overline{N_C(x) + N'(x)}). \quad (2.10)$$

In this paper we will use the following well-known characterization theorem of the best approximation from a closed convex set in a Banach space, which was established independently by Deutsch [11] and Rubenstein [12] (see also [13]). Let $x \in X$ and $G \subseteq X$. Let $P_G(x)$ denote the set of all best approximation to x from G , i.e., an element $g_0 \in P_G(x)$ if and only if $\|x - g_0\| \leq \|x - g\|$ holds for each $g \in G$.

Recall that the normalized duality map J from X to 2^{X^*} is defined by

$$J(x) := \{x^* \in X^* : \langle x^*, x \rangle = \|x\|^2, \|x^*\| = \|x\|\}. \quad (2.11)$$

In fact, $J(x) = \partial\phi(x)$, where $\phi(x) := \frac{1}{2}\|x\|^2$. Then X is smooth if and only if for each $x \in X$ the normalized duality map is a singleton.

Proposition 2.1 *Let G be a closed convex set in X . Then for any $x \in X$, $g_0 \in P_G(x)$ if and only if $g_0 \in G$ and there exists $x^* \in J(x - g_0)$ such that $\langle x^*, g - g_0 \rangle \leq 0$ for any $g \in G$, that is, $J(x - g_0) \cap N_G(g_0) \neq \emptyset$. In particular, if X is smooth, $g_0 \in P_G(x)$ if and only if $g_0 \in G$ and $J(x - g_0) \in N_G(g_0)$.*

The following renorm result will be useful (cf. [14, p. 185]). Recall that a Banach space is locally uniformly convex if for any $x_0 \in X$ with $\|x_0\| = 1$ and any $\epsilon > 0$ there exists $\delta > 0$ such that, for any $x \in X$ with $\|x\| \leq 1$, $\|x + x_0\| > 2 - \delta$ implies $\|x - x_0\| < \epsilon$.

Proposition 2.2 *Suppose that X is a reflexive Banach space. Then there exists an equivalent norm $\|\cdot\|_1$ such that, under the new norm $\|\cdot\|_1$, X is locally uniformly convex and Frechet differentiable.*

3 Characterizations of the BCQ and the Abadie CQ

Before proving the main theorem of this section, we first give a lemma, which was proved in [9, Lemma 2.2] when X is of finite dimension. Let Z be a nonempty convex closed subset of X

and let $Q : X \rightarrow 2^{X^*}$ (the collection of subsets of X^*) be a multifunction. Then Q is said to be (W)-upper Kuratowski semi-continuous at $x \in Z$ restricted to Z if, for any sequences $\{x_k\} \subset Z$ and $\{y_k^*\} \subset X^*$, the relations $\lim_{k \rightarrow \infty} x_k = x$ strongly, $\lim_{k \rightarrow \infty} y_k^* = y^*$ weakly and $y_k^* \in Q(x_k)$ for each k imply $y^* \in Q(x)$. In particular, Q is (W)-upper Kuratowski semi-continuous at $x \in X$ if it is (W)-upper Kuratowski semi-continuous at x restricted to $Z = X$. Note that the (W)-upper Kuratowski semi-continuity is a generalization of the upper Kuratowski semi-continuity, see, for example, [15, p. 55]. Moreover, in the remainder of this section, a neighbourhood of a point in X is understood in the strong topology, unless specially stated.

Lemma 3.1 *Suppose that X is reflexive. Let $x \in X$ and let $Q : X \rightarrow 2^{X^*}$ be a multifunction such that, for all z in a neighbourhood of x , $Q(z)$ is a convex set, $\mathcal{I}(z) \neq \emptyset$ and $\mathcal{D}'(z) \subseteq Q(z)$. If Q is (W)-upper Kuratowski semi-continuous at x , then $\partial G(x) \subseteq Q(x)$.*

Proof The proof is similar to that of [9, Lemma 2.2] and so omitted.

Let Z^* be a subset of X^* and $Z \subseteq X$. Let $z^*|_Z$ denote the restriction of z^* on Z , i.e., $z^*|_Z$ is viewed as a functional defined on Z instead of X . Set

$$Z^*|_Z = \{z^*|_Z : z^* \in Z^*\}. \quad (3.1)$$

Now we are ready to prove the main theorem of this section.

Theorem 3.1 *Suppose that X is reflexive and let C be a nonempty closed convex subset of X . Let $x \in C \cap \text{bd}S$ such that $\mathcal{I}(z) \neq \emptyset$ for all z in a neighbourhood of x . Consider the following statements.*

- (i) *Both $N_C(\cdot) + \text{cone}(\partial G(\cdot))$ and $N_C(\cdot) + \mathcal{N}'(\cdot)$ are (W)-upper Kuratowski semi-continuous at x .*
- (ii) *$N_C(\cdot) + \mathcal{N}'(\cdot)$ is (W)-upper Kuratowski semi-continuous at x while $N_C(\cdot) + \text{cone}(\partial G(\cdot))$ is (W)-upper Kuratowski semi-continuous at x restricted to C .*
- (iii) *The system (SCI) satisfies the BCQ relative to C at x .*
- (iv) *Both $N_C(\cdot) + \text{cone}(\partial G(\cdot))$ and $N_C(\cdot) + \mathcal{N}'(\cdot)$ are (W)-upper Kuratowski semi-continuous at x restricted to C .*

Then

- (1) (i) \implies (ii) \implies (iii) \implies (iv);
- (2) (i) \iff (ii) \iff (iii) \iff (iv) if $\text{int}S = \emptyset$ or $\text{int}S \cap C \neq \emptyset$;
- (3) (i) \implies (ii) \implies (iii) \iff (iv) if C is a subspace of X ;
- (4) (i) \iff (ii) \iff (iii) \iff (iv) if $x \in \text{int}C$.

Proof (1) It is trivial that (i) \implies (ii). We next show (ii) \implies (iii). Since X is reflexive, by Proposition 2.2, there exists an equivalent $\|\cdot\|_1$ that is both locally uniformly convex and Frechet differentiable. Let $y^* \in N_K(x)$, $y^* \neq 0$. By reflexivity, we can take $\bar{x} \in X$ such that $\|\bar{x}\|_1 = \|y^*\|_1$ and $\langle y^*, \bar{x} \rangle_1 = \|y^*\|_1^2$, where $\|y^*\|_1$ denotes the dual norm corresponding to the norm $\|\cdot\|_1$ while $\langle \cdot, \cdot \rangle_1$ is the inner product between X and X^* under the new norm $\|\cdot\|_1$. Let $x_0 = x + \bar{x}$. Then $x_0 \notin K$ since $y^* \in N_K(x)$ and $\langle x_0 - x, y^* \rangle_1 = \|y^*\|_1^2 > 0$. In addition, under the new norm, the normalized duality map $J_1(x_0 - x) = y^*$. Set

$$S^k = \left\{ z \in X : G(z) \leq \frac{1}{k} \right\}$$

and

$$C_k = C \cap S^k.$$

Then, for $k > 0$ large enough, $x_0 \notin C_k$. In fact, since $x_0 \notin K$, $x_0 \notin C$ or $x_0 \notin S$. It is trivial in the first case while, in the second case, $G(x_0) > 0$ so that $\frac{1}{k} < G(x_0)$ and hence $x_0 \notin C_k$ for $k > 0$ large enough. Note that C_k is a closed convex set in X and X is reflexive. It follows that $P_{C_k}(x_0)$ is nonempty under the new norm $\|\cdot\|_1$. Take $x_k \in P_{C_k}(x_0)$. Then

$$\|x_0 - x_k\|_1 \leq \|x_0 - x\|_1, \quad (3.3)$$

since $x \in C_k$. Thus, $\{x_k\}$ is bounded. Without loss of generality, we may assume that there exists \hat{x} such that $x_k \rightarrow \hat{x}$ weakly. Since, for each $i \in I$, g_i is convex, it is lower semi-continuous with respect to the weak topology in X . This means that $g_i(\hat{x}) \leq 0$ for all $i \in I$. Hence, $\hat{x} \in K$. Letting $k \rightarrow +\infty$ in (3.3), we obtain

$$\|x_0 - \hat{x}\|_1 \leq \liminf_k \|x_0 - x_k\|_1 \leq \|x_0 - x\|_1. \quad (3.4)$$

Since $x \in K$ and $y^* = J_1(x_0 - x) \in N_K(x)$, by Proposition 2.1, we have $x \in P_K(x_0)$. Hence, (3.4) implies that $\hat{x} = x$ because the new norm $\|\cdot\|_1$ is locally uniformly convex. Note that

$$\|x_0 - x\|_1 \leq \liminf_k \|x_0 - x_k\|_1 \leq \limsup_k \|x_0 - x_k\|_1 \leq \|x_0 - x\|_1 \quad (3.5)$$

and

$$2\|x_0 - x\|_1 \leq \liminf_k \|x_0 - x_k + x_0 - x\|_1. \quad (3.6)$$

We get

$$\lim_k \|x_k - x_0\|_1 = \|x - x_0\|_1 \quad \text{and} \quad \lim_k \|x_0 - x_k + x_0 - x\|_1 = 2\|x_0 - x\|_1. \quad (3.7)$$

By locally uniform convexity, (3.7) implies that $\{x_k\}$ converges to x strongly.

Let $y_k^* = J_1(x_0 - x_k)$. Then y_k^* converges to y^* weakly, see for example [16, Proposition 2.8]. To apply the (W)-upper semi-continuity, we need to show

$$y_k^* \in N_C(x_k) + \text{cone}\partial G(x_k). \quad (3.8)$$

In fact, since $x_k \in C_k$ and $C \cap \text{int}S^k \neq \emptyset$, by [17, Proposition 2.3]. (The proof is still valid for reflexive Banach space setting although it is for Hilbert Space setting), $\{C, S^k\}$ has the strong CHIP at x_k ; that is,

$$N_{C_k}(x_k) = N_C(x_k) + N_{S^k}(x_k). \quad (3.9)$$

Because the system $G(z) \leq \frac{1}{k}$ satisfies the Slater condition, it follows from Remark 2.1 that

$$N_{C_k}(x_k) = N_C(x_k) + N_{S^k}(x_k) \subseteq N_C(x_k) + \text{cone}\partial G(x_k). \quad (3.10)$$

By Proposition 2.1, it follows from $x_k \in P_{C_k}(x_0)$ that $y_k^* = J_1(x_0 - x_k) \in N_{C_k}(x_k)$. This, with (3.10), implies (3.8). Note that $x_k \in P_{C_k}(x_0) \subseteq C$. We have $y^* \in N_C(x) + \text{cone}\partial G(x)$ because $N_C(z) + \text{cone}\partial G(z)$ is (W)-upper Kuratowski semi-continuous at x restricted to C . Consequently, $N_K(x) \subseteq N_C(x) + \text{cone}\partial G(x)$. Since $N_C(z) + \mathcal{N}'(z)$ is (W)-upper Kuratowski semi-continuous at x and $\mathcal{D}'(x) \subseteq N_C(x) + \mathcal{N}'(x)$, one can apply Lemma 3.1 (letting $Q(z) = N_C(z) + \mathcal{N}'(z)$ in Lemma 3.1) to conclude that $\text{cone}\partial G(x) \subseteq N_C(x) + \mathcal{N}'(x)$; hence

$$N_K(x) \subseteq N_C(x) + \text{cone}\partial G(x) \subseteq N_C(x) + \mathcal{N}'(x).$$

This shows that the system (SCI) satisfies the BCQ relative to C at x . The proof of (ii) \implies (iii) is complete.

Now let us verify (iii) \implies (iv). Suppose that the system (SCI) satisfies the BCQ relative to C at x . Let $x_k \in C$, $x_k \rightarrow x$, $y_k^* \in N_C(x_k) + \mathcal{N}'(x_k)$ (or $y_k^* \in N_C(x_k) + \text{cone}(\partial G(x_k))$) and $y_k^* \rightarrow y^*$ weakly. Define

$$S_k = \{z \in X : G(z) - G(x_k) \leq 0\}. \quad (3.11)$$

Then

$$\partial G(x_k) = \partial(G - G(x_k))(x_k) \subseteq N_{S_k}(x_k). \quad (3.12)$$

By (3.12), since $S_k \cap C \neq \emptyset$, we have

$$y_k^* \in N_C(x_k) + \mathcal{N}'(x_k) \subseteq N_C(x_k) + \text{cone}(\partial G(x_k)) \subseteq N_{C \cap S_k}(x_k). \quad (3.13)$$

We claim that $y^* \in N_K(x)$. In fact, if there exists a subsequence of $\{x_k\}$, denoted by itself, such that $G(x_k) \geq 0$, we have $K \subseteq C \cap S_k$; hence, for each $z \in K$,

$$\langle y_k^*, z - x_k \rangle \leq 0 \quad (3.14)$$

thanks to (3.13). As $\{y_k^*\}$ is bounded by the fact that $y_k^* \rightarrow y^*$ weakly and $x_k \rightarrow x$, we have $\lim_k |\langle y_k^*, x - x_k \rangle| = 0$. This implies that

$$\langle y^*, z - x \rangle = \lim_k (\langle y_k^*, z - x_k \rangle + \langle y_k^*, x_k - x \rangle) \leq 0 \quad (3.15)$$

for all $z \in K$; hence $y^* \in N_K(x)$ in this case. Otherwise, without loss of generality, we may assume that $G(x_k) < 0$ for each k . This implies that $\text{int}S = \{x \in X : G(x) < 0\}$; hence $\text{int}S \cap C \neq \emptyset$ (since $x_k \in \text{int}S \cap C$). Noting that $G(x_k) \rightarrow 0$ as $k \rightarrow \infty$, we have that, for each $z \in \text{int}S \cap C$, $z \in S_k \cap C$ for all k large enough. By (3.12) and (3.14), (3.15) holds for all $z \in \text{int}S \cap C$. Consequently, by $K = \overline{\text{int}S \cap C}$, (3.15) holds for all $z \in K$. Again we have $y^* \in N_K(x)$ and so the claim stands. Consequently, $y^* \in N_K(x) = N_C(x) + \mathcal{N}'(x) = N_C(x) + \text{cone}(\partial G(x))$ (noting that $N_C(x) + \mathcal{N}'(x) \subseteq N_C(x) + \text{cone}(\partial G(x)) \subseteq N_K(x)$) because the system (SCI) satisfies the BCQ relative to C at x . Therefore, $N_C(x) + \text{cone}(\partial G(x))$ and $N_C(x) + \mathcal{N}'(x)$ are (W)-upper Kuratowski semi-continuous at x restricted to C and so (iii) \implies (iv) is proved.

(2) Suppose that the condition (3.2) holds. Via almost the same arguments as in the proof of the implication (iii) \implies (iv), we can also show implication (iii) \implies (i). Hence (2) is true.

(3) Suppose that C is a subspace of X . From (1), it suffices to show that (iv) \implies (iii). To this end, suppose that (vi) holds; that is, both $N_C(\cdot) + \text{cone}(\partial G(\cdot))$ and $N_C(\cdot) + \mathcal{N}'(\cdot)$ are (W)-upper Kuratowski semi-continuous at x restricted to C . Let \tilde{G} and \tilde{g}_i (for each $i \in I$) denote the restrictions of G and g_i on the subspace C , respectively. Consider the convex inequality system on C defined by

$$\tilde{g}_i(z) \leq 0, \quad i \in I, z \in C. \quad (3.16)$$

Applying (2) to the system (3.16) (on the space C), we get that (3.16) satisfies the BCQ at x (on the space C); hence

$$N_K(x)|_C = \text{cone}\{\partial \tilde{g}_i(x) : i \in I(x)\}. \quad (3.17)$$

Since any $y^* \in N_K(x)$ can be viewed as an element of $N_K(x)|_C$, (3.17) implies that there exist $\tilde{y}_j^* \in \partial \tilde{g}_{i_j}(x)$, $\lambda_j \geq 0$, $i_j \in I(x)$, $j = 1, 2, \dots, m$, such that

$$\langle y^*, z \rangle = \left\langle \sum_{j=1}^m \lambda_j \tilde{y}_j^*, z \right\rangle, \quad \forall z \in C. \quad (3.18)$$

As

$$\langle \tilde{y}_j^*, z \rangle \leq \tilde{g}'_{i_j+}(x, z) = g'_{i_j+}(x, z), \quad \forall z \in C, \quad (3.19)$$

the Hahn–Banach Extension Theorem is applied to show that \tilde{y}_j^* can be extended to $y_j^* \in \partial g_{i_j}(x)$ satisfying

$$\langle y_j^*, z \rangle = \langle \tilde{y}_j^*, z \rangle, \quad \forall z \in C. \quad (3.20)$$

Consequently, $\sum_{j=1}^m \lambda_j y_j^* \in \text{cone}\{\partial g_i(x) : i \in I(x)\}$. Let $y_0^* = y^* - \sum_{j=1}^m \lambda_j y_j^*$. Then, $y_0^* \in N_C(x)$ since, by (3.18)

$$\langle y_0^*, z \rangle = \left\langle y^* - \sum_{j=1}^m \lambda_j y_j^*, z \right\rangle = 0, \quad \forall z \in C - x.$$

This implies that $y^* = y_0^* + \sum_{j=1}^m \lambda_j y_j^* \in N_C(x) + \text{cone}\{\partial g_i(x) : i \in I(x)\}$ and the proof is complete.

(4) It is trivial that (iv) \iff (i) in the case when $x \in \text{int}C$, and hence (4) holds.

Similarly, we have the following theorem for the Abadie CQ.

Theorem 3.2 *Suppose that X is reflexive and let C be a nonempty closed convex subset of X . Let $x \in C \cap \text{bd}S$ such that $\mathcal{I}(z) \neq \emptyset$ for all z in a neighbourhood of x . Consider the following statements.*

(i) *Both weak $-\overline{N_C(\cdot) + \text{cone}(\partial G(\cdot))}$ and weak $-\overline{N_C(\cdot) + \mathcal{N}'(\cdot)}$ are (W)-upper Kuratowski semi-continuous at x .*

(ii) *Weak $-\overline{N_C(\cdot) + \mathcal{N}'(\cdot)}$ is (W)-upper Kuratowski semi-continuous at x while weak $-\overline{N_C(\cdot) + \text{cone}(\partial G(\cdot))}$ is (W)-upper Kuratowski semi-continuous at x restricted to C .*

(iii) *The system (SCI) satisfies the Abadie CQ relative to C at x .*

(iv) *Both weak $-\overline{N_C(\cdot) + \text{cone}(\partial G(\cdot))}$ and weak $-\overline{N_C(\cdot) + \mathcal{N}'(\cdot)}$ are (W)-upper Kuratowski semi-continuous at x restricted to C .*

Then

- (1) (i) \implies (ii) \implies (iii) \implies (iv);
- (2) (i) \iff (ii) \iff (iii) \implies (iv) if (3.2) holds;
- (3) (i) \implies (ii) \implies (iii) \iff (iv) if C is a subspace of X ;
- (4) (i) \iff (ii) \iff (iii) \iff (iv) if $x \in \text{int}C$.

In particular, noting that the condition (3.2) holds automatically for $C = X$, we have the following corollary, which is one of the main results established by Li, Nahak and Singer in [9] for finitely dimensional Banach space X .

Corollary 3.1 *Suppose that X is reflexive. Let $x \in \text{bd}S$ such that $\mathcal{I}(z) \neq \emptyset$ for all z in a neighbourhood of x . Then*

- (i) *the system (SCI) satisfies the BCQ at x if and only if both $\text{cone}(\partial G(\cdot))$ and $\mathcal{N}'(\cdot)$ are (W) -upper Kuratowski semi-continuous at x ;*
- (ii) *the system (SCI) satisfies the Abadie CQ at x if and only if both $\text{weak} - \overline{\text{cone}(\partial G(\cdot))}$ and $\text{weak} - \mathcal{N}'(\cdot)$ are (W) -upper Kuratowski semi-continuous at x .*

For further results we need some more concepts introduced in the following definition, where (b) and (d) are new while the others are referred to [7] for details.

Definition 3.1 *Let $x \in X$. The system (SCI) is said to satisfy*

- (a) *the PLV property relative to C at x if*

$$N_C(x) + \partial G(x) \subseteq N_C(x) + \mathcal{D}'(x); \tag{3.21}$$

- (b) *the CPLV property relative to C at x if*

$$N_C(x) + \partial G(x) \subseteq N_C(x) + \mathcal{N}'(x); \tag{3.22}$$

- (c) *the weak PLV property relative to C at x if*

$$N_C(x) + \partial G(x) \subseteq \text{weak}^* - \overline{N_C(x) + \mathcal{D}'(x)}; \tag{3.23}$$

- (d) *the weak CPLV property relative to C at x if*

$$N_C(x) + \partial G(x) \subseteq \text{weak}^* - \overline{N_C(x) + \mathcal{N}'(x)}; \tag{3.24}$$

- (e) *the PLV (resp. the CPLV, the weak PLV, the weak CPLV) property at x_0 if it satisfies the PLV (resp. the CPLV, the weak PLV, the weak CPLV) property relative to $C = X$ at x_0 .*

Remark 3.1 (a) Clearly, the PLV (resp. the weak PLV) property relative to C at $x \implies$ the CPLV (resp. the weak CPLV) property relative to C at x ; the BCQ (resp. the Abadie CQ) relative to C at $x \implies$ the CPLV (resp. the weak CPLV) property relative to C at x . But the converses are not true, in general. Indeed, the system of convex inequalities in [7, Example 3.1(a)] satisfies the CPLV and the weak PLV but not the PLV relative to C at x_0 , while the system in [7, Example 3.1(b)] satisfies the CPLV and the weak PLV but not the Abadie CQ relative to C at x_0 .

(b) The PLV (resp. the CPLV, the weak PLV, the weak CPLV) property implies the PLV (resp. the CPLV, the weak PLV, the weak CPLV) property relative to C but, obviously, the converse may not be true.

The following proposition, which was verified in [7] for the general case and [9, 18–20] for some special cases, provides a sufficient condition for a system of convex inequalities to satisfy the PLV property and/or the weak PLV property.

Proposition 3.1 *Let C be a closed convex nonempty set in X . Assume that I is a compact set in a metric space, on which the function: $i \mapsto g_i(z)$ is upper semicontinuous for each $z \in \text{span}C$. Then*

- (i) *the system (SCI) satisfies the weak PLV property relative to C at each $x \in C \cap \text{bd}S$;*
- (ii) *the system (SCI) satisfies the PLV property relative to C at each $x \in C \cap \text{bd}S$ if C is of finite dimension.*

The relationships between the CPLV property (resp. the weak CPLV property), the BCQ (resp. the Abadie CQ) and the Slater condition (resp. the weak Slater condition) are described

in Propositions 3.2 and 3.3 below, which improve respectively Theorems 2.1 and 3.1 in [7]. The proofs of Propositions 3.2 and 3.3 are the same as those of Theorems 2.1 and 3.1 in [7]. Recall from [7, Definition 3.4] that the system (SCI) satisfies

- (a) the Slater condition relative to C if there exists $\bar{x} \in \text{ri } C$ such that $G(\bar{x}) < 0$;
- (b) the weak Slater condition relative to C if there exists $\bar{x} \in \text{ri } C \cap S$ such that $I(\bar{x})$ is finite, g_i is affine for each $i \in I(\bar{x})$ and G_0 is a continuous function with $G_0(\bar{x}) < 0$, where G_0 is the sup-function of the system $\{g_i : i \in I \setminus I(\bar{x})\}$:

$$G_0(z) = \sup_{i \in I \setminus I(\bar{x})} g_i(z), \quad z \in X.$$

Proposition 3.2 *Let $x \in C \cap \text{bd } S$. Then the system (SCI) satisfies the BCQ (resp. the Abadie CQ) relative to C at x if and only if (SCI) satisfies the CPLV (resp. the weak CPLV) property relative to C at x and (SSCI) satisfies the BCQ (resp. the Abadie CQ) relative to C at x .*

Proposition 3.3 *Suppose that the system (SCI) satisfies the weak Slater condition relative to C , and let $x \in C \cap \text{bd } S$. Then (SCI) satisfies BCQ (resp. the Abadie CQ) relative to C at x if and only if (SCI) satisfies the CPLV (resp. the weak CPLV) relative to C at x .*

Remark 3.2 In general, the wording ‘‘CPLV’’ in Propositions 3.2 and 3.3 cannot be replaced by ‘‘PLV’’.

Theorem 3.3 *Suppose that X is reflexive. Let $x \in C \cap \text{bd } S$. Suppose that $\mathcal{I}(z) \neq \emptyset$ in a neighbourhood of x and that the condition (3.2) holds. Then*

- (i) *the system (SCI) satisfies the BCQ relative to C at x if and only if $N_C(z) + \mathcal{N}'(z)$ is (W)-upper Kuratowski semi-continuous at x and the system (SCI) satisfies the CPLV property relative to C at each point in a neighbourhood of x ;*
- (ii) *the system (SCI) satisfies the Abadie CQ relative to C at x if and only if weak $-N_C(z) + \mathcal{N}'(z)$ is (W)-upper Kuratowski semi-continuous at x and the system (SCI) satisfies the weak CPLV property relative to C at each point in a neighbourhood of x .*

Proof We only prove the assertion (i) since the assertion (ii) can be dealt with similarly.

The necessity part (i) follows from Theorem 3.1 and Remark 3.1(a). Now let us show the sufficiency part. We claim that, for each z in a neighbourhood of x ,

$$N_C(z) + \text{cone}(\partial G(z)) = N_C(z) + \mathcal{N}'(z). \quad (3.25)$$

Indeed, first we have, by the CPLV property,

$$N_C(z) + \text{cone}(\partial G(z)) \subseteq \text{cone}(N_C(z) + \partial G(z)) \subseteq N_C(z) + \mathcal{N}'(z).$$

Next, it follows from (2.6) that

$$N_C(z) + \mathcal{N}'(z) \subseteq N_C(z) + \text{cone}(\partial G(z)).$$

Hence, (3.25) holds. By Theorem 3.1 and the assumption that $N_C(z) + \mathcal{N}'(z)$ is (W)-upper Kuratowski semi-continuous at x , the system (SCI) satisfies the BCQ relative to C at x . The proof is complete.

Similarly, using Theorem 3.1(3), we immediately get the following corollary.

Corollary 3.2 *Suppose that X is reflexive and that C is a subspace of X . Let $x \in C \cap \text{bd } S$ such that $\mathcal{I}(z) \neq \emptyset$ in a neighbourhood of x . Then*

- (i) *The system (SCI) satisfies the BCQ relative to C at x if and only if $N_C(z) + \mathcal{N}'(z)$ is (W)-upper Kuratowski semi-continuous at x restricted to C and the system (SCI) satisfies the CPLV property relative to C at x .*
- (ii) *The system (SCI) satisfies the Abadie CQ relative to C at x if and only if weak $-N_C(z) + \mathcal{N}'(z)$ is (W)-upper Kuratowski semi-continuous at x restricted to C and the system (SCI) satisfies the weak CPLV property relative to C at x .*

The following corollary, which was established in [9] for $C = X = \mathbb{R}^n$, is a direct consequence of Corollary 3.2.

Corollary 3.3 *Suppose that X is reflexive and that C is a subspace of X . Let $x \in C \cap \text{bd}S$. Suppose that $\mathcal{S}(z) \neq \emptyset$ in a neighbourhood of x . Then*

(i) *The system (SCI) satisfies the BCQ relative to C at x if and only if $N_C(z) + \mathcal{N}'(z)$ is (W) -upper Kuratowski semi-continuous at x restricted to C provided that the system (SCI) satisfies the PLV property relative to C at x .*

(ii) *The system (SCI) satisfies the Abadie CQ relative to C at x if and only if weak $-N_C(z) + \mathcal{N}'(z)$ is (W) -upper Kuratowski semi-continuous at x restricted to C provided that the system (SCI) satisfies the weak PLV property relative to C at x .*

4 Optimality Condition of Best Approximation with Constraints

We begin with the definition of notion of strong CHIP. This notion has been shown to play an important role in best approximation and optimization, and has been studied extensively. For details, see [2, 4, 5, 17, 21–27] for the case when I is finite and [6, 28] for the case when I is infinite.

Definition 4.1 *Let $\{C, C_i : i \in I\}$ be a family of nonempty closed convex subsets of X and let $x_0 \in C \cap (\bigcap_{i \in I} C_i)$. Then the family is said to have the strong CHIP at x_0 if*

$$N_{C \cap (\bigcap_{i \in I} C_i)}(x_0) = N_C(x_0) + \sum_{i \in I} N_{C_i}(x_0). \quad (4.1)$$

For later use, we still require the following definitions (cf. [6, 7]). Recall that $K = C \cap S$.

Definition 4.2 *Let $x \in C \cap \text{bd}S$. An element $d \in X$ is called*

(a) *a linearized feasible direction of (SCI) at x if*

$$\langle z^*, d \rangle \leq 0, \quad \text{for } \forall z^* \in \text{ext} \partial g_i(x), \forall i \in I(x), \quad (4.2)$$

where $\text{ext}Z$ denotes the set of all extreme points from the closed convex set Z ;

(b) *a sequentially feasible direction of K at x if there exist a sequence $\{d_k\} \subset K$ with $d_k \rightarrow d$ and a sequence of positive real numbers $\{\delta_k\}$ with $\delta_k \rightarrow 0$ such that $\{x + \delta_k d_k\} \subseteq K$.*

Let $\text{LFD}(x)$ (resp. $\text{SFD}(x)$) denote the set of all d satisfying (a) (resp. (b)). Note that both $\text{LFD}(x)$ and $\text{SFD}(x)$ are closed convex cones. We use $K_S(x)$ and $K_L(x)$ to denote two closed convex sets in X defined respectively by

$$K_S(x) = (x + \text{SFD}(x)) \cap C \quad (4.3)$$

and

$$K_L(x) = (x + \text{LFD}(x)) \cap C. \quad (4.4)$$

It is clear that the two sets are closed convex sets. We have the following inclusion relationship; see for example [6, 7].

Proposition 4.1 *Let $x \in C \cap S$. Then $\text{SFD}(x) \subseteq \text{LFD}(x)$, and hence*

$$K \subseteq K_S(x) \subseteq K_L(x). \quad (4.5)$$

Before proving the main theorem of this section, recall that $\hat{S}_{z^*}(x_0)$ is defined by

$$\hat{S}_{z^*}(x_0) = \{x \in X : \langle z^*, x - x_0 \rangle \leq 0\} \quad \text{for each } z^* \in X^*. \quad (4.6)$$

Let $x_0 \in K$. Consider the following statements.

(i) *The system (SCI) satisfies the BCQ relative to C at x_0 .*

(ii) *$K_S(x_0) = K_L(x_0)$ and the family $\{C, \hat{S}_{z^*}(x_0) : z^* \in \bigcup_{i \in I(x_0)} \partial g_i(x_0)\}$ has the strong CHIP at x_0 .*

(iii) *For each $x \in X$, $x_0 \in P_K(x)$ if and only if*

$$J(x - x_0) \cap (N_C(x_0) + N'(x_0)) \neq \emptyset. \quad (4.7)$$

(iv) *For each $x \in X$, $x_0 \in P_K(x)$ if and only if*

$$J(x - x_0)|_{C - x_0} \cap (N_C(x_0)|_{C - x_0} + N'(x_0)|_{C - x_0}) \neq \emptyset. \quad (4.8)$$

The following theorem about their relationships is known in [6].

Theorem LN *Let $x_0 \in K$. Then*

(1) (i) \implies (iii) \implies (iv); (ii) \implies (iii) \implies (iv);

- (2) (i) \iff (ii) \implies (iii) \implies (iv), if X is reflexive;
 (3) (i) \iff (ii) \iff (iii) \implies (iv), if X is both reflexive and smooth.

Now we are ready to give the main theorem of this section, which is clearly a generalization and improvement of Theorem LN.

Theorem 4.1 *Let $x_0 \in K$. Then*

- (1) (i) \implies (iii) \iff (iv); (ii) \implies (iii) \iff (iv);
 (2) (i) \iff (ii) \implies (iii) \iff (iv), if $\text{span}(C - x_0)$ is reflexive;
 (3) (i) \iff (ii) \iff (iii) \iff (iv), if $\text{span}(C - x_0)$ is both reflexive and smooth.

Proof The assertions are trivial when $x_0 \in C \cap \text{int} S$ since each of (i)–(iv) in Theorem 3.1 holds automatically. Hence we assume that $x_0 \in C \cap \text{bd} S$.

(1) By Theorem LN above, we only need to show (iii) \iff (iv). To do this, it suffices to verify that (4.7) holds if and only if (4.8) holds. It is trivial that (4.7) implies (4.8). Now let us show the converse implication. Suppose that (4.8) holds and let $x^* \in J(x - x_0)$ such that $x^*|_{C-x_0} \in J(x - x_0)|_{C-x_0} \cap (N_C(x_0)|_{C-x_0} + N'(x_0)|_{C-x_0})$. Then by (2.5) there exist a positive integer m and $x_0^* \in N_C(x_0)$, $x_i^* \in \text{cone } \partial g_i(x_0)$, $i = 1, 2, \dots, m$, such that $x^*|_{C-x_0} = \sum_{i=0}^m x_i^*|_{C-x_0}$. Write $y^* = x^* - \sum_{i=0}^m x_i^*$. Then $y^* \in N_C(x_0)$ and so $x^* = y^* + \sum_{i=0}^m x_i^* \in N_C(x_0) + \sum_{i \in I(x_0)} \text{cone } \partial g_i(x_0)$. Hence, $x^* \in J(x - x_0) \cap (N_C(x_0) + \sum_{i \in I(x_0)} \text{cone } \partial g_i(x_0))$ and (4.7) holds. (1) is proved.

(2) Let $Z = \text{span}(C - x_0)$ and define

$$\tilde{g}_i(z) = g_i(x_0 + z) \quad \text{for each } z \in Z \text{ and } i \in I. \quad (4.9)$$

Consider the system of convex inequalities (Z-SCI) on Z defined by

$$\tilde{g}_i(z) \leq 0 \quad \text{for each } i \in I. \quad (\text{Z-SCI})$$

Let \tilde{S} denote the solution set of (Z-SCI) on Z . Set $\tilde{C} = (C - x_0)$ and $\tilde{K} = \tilde{C} \cap \tilde{S}$. Then $0 \in \tilde{K}$ and $\tilde{I}(0) = I(x_0)$, where $\tilde{I}(0)$ is defined by $\tilde{I}(0) = \{i \in I : \tilde{g}_i(0) = 0\}$. We first note the following fact:

$$\partial \tilde{g}_i(0) = \{x^*|_Z : x^* \in \partial g_i(x_0)\} \quad \text{for each } i \in I, \quad (4.10)$$

that is, $\tilde{x}^* \in \partial \tilde{g}_i(0)$ if and only if \tilde{x}^* can be extended as an element $x^* \in \partial g_i(x_0)$. In fact, it is clear that the set on the right-hand side of (4.10) is contained in the set on the left-hand side. To verify the converse inclusion, let $\tilde{x}^* \in \partial \tilde{g}_i(0)$. By (2.2),

$$\langle \tilde{x}^*, d \rangle \leq \tilde{g}'_{i+}(0, d) = g'_{i+}(x_0, d) \quad \text{for each } d \in Z. \quad (4.11)$$

By the Hahn–Banach Extension Theorem, \tilde{x}^* can be extended as an element $x^* \in X^*$ such that

$$\langle x^*, d \rangle \leq g'_{i+}(x_0, d) \quad \text{for each } d \in X. \quad (4.12)$$

Again, by (2.2), one sees that $x^* \in \partial g_i(x_0)$ and so (4.10) is proved.

Next we verify the following two assertions:

- (a) (i) holds if and only if the system (Z-SCI) satisfies the BCQ relative to \tilde{C} at 0;
 (b) The family $\{C, \tilde{S}_{z^*}(x_0) : z^* \in \bigcup_{i \in I(x_0)} \partial g_i(x_0)\}$ has the strong CHIP at x_0 if and only if the family $\{\tilde{C}, \tilde{S}_{z^*}(0) : z^* \in \bigcup_{i \in \tilde{I}(0)} \partial \tilde{g}_i(0)\}$ has the strong CHIP at 0, where

$$\tilde{S}_{z^*}(0) = \{z \in Z : \langle z^*, z \rangle \leq 0\} \quad \text{for each } z^* \in Z^*. \quad (4.13)$$

As the proof of (b) is similar, we only prove (a). It suffices to show the sufficiency part since the necessity is trivial. Assume that the system (Z-SCI) satisfies the BCQ relative to \tilde{C} at 0 and let $x^* \in N_K(x_0)$. Then $x^*|_Z \in N_{\tilde{K}}^Z(0)$, where N^Z denotes the normal cone taken in Z . Consequently, there exist finite many, say $m + 1$, functionals $\tilde{x}_0^* \in N_{\tilde{C}}^Z(0)$, $\tilde{x}_i^* \in \partial \tilde{g}_i(0)$ and m positive numbers λ_i , $i = 1, \dots, m$, such that

$$\tilde{x}^*|_Z = \tilde{x}_0^* + \sum_{i=1}^m \lambda_i \tilde{x}_i^* \quad \text{on } Z. \quad (4.14)$$

Let $x_0^* \in X^*$ be a norm-preserved extension of \tilde{x}_0^* . Then $x_0^* \in N_C(x_0)$ since $\tilde{C} \subseteq Z$. By (4.10), for each $i = 1, \dots, m$, there exists $x_i^* \in \partial g_i(x_0)$ such that $x_i^*|_Z = \tilde{x}_i^*$. Let $y^* = x^* - x_0^* -$

$\sum_{i=1}^m \lambda_i x_i^*$. By (4.14), $y^* \in N_Z(0) \subseteq N_C(x_0)$ and so $x^* = y^* + x_0^* + \sum_{i=1}^m x_i^* \in N_C(x_0) + N'(x_0)$. This means that (i) holds and (a) is proved.

Now let us prove the equivalence of (i) and (ii). Let $\tilde{K}_S(0)$ and $\tilde{K}_L(0)$ be defined respectively by (4.3) and (4.4) with replacement of X , C and the system (SCI) by Z , \tilde{C} and (Z-SCI), respectively. Since Z is reflexive, by Theorem LN (2), the following two statements are equivalent.

(i*) The system (Z-SCI) satisfies the BCQ relative to \tilde{C} at 0.

(ii*) $\tilde{K}_S(0) = \tilde{K}_L(0)$ and the family $\{\tilde{C}, \tilde{S}_{z^*}(0) : z^* \in \bigcup_{i \in \tilde{I}(0)} \partial \tilde{g}_i(0)\}$ has the strong CHIP at 0.

Note that, by (4.10), $\tilde{K}_S(0) = K_S(x_0)$ and $\tilde{K}_L(0) = K_L(x_0)$. It follows from the assertions

(a) and (b) that (i) \iff (i*) \iff (ii*) \iff (ii). Hence (2) holds.

(3) This can be proved similarly and so omitted.

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